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Positive and asymptotically stable realizations for descriptor discrete-time linear systems

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Abstract. Conditions for the existence of positive and asymptotically stable realizations for descriptor discrete-time linear systems are established. Procedures for computation of positive and asymptotically stable realizations for improper transfer matrices are proposed. The effectiveness of the methods is demonstrated on numerical examples.

Key words: positive, stable realization, procedure, descriptor, linear system.

1. Introduction

A dynamical system is called positive if its trajectory starting from any nonnegative initial state remains forever in the positive orthant for all nonnegative inputs. An overview of state of the art in positive theory is given in the monographs [1, 2]. Variety of models having positive behavior can be found in engineering, economics, social sciences, biology and medicine, etc. The positive fractional linear systems have been addressed in [3–5].

An overview on the positive realization problem is given in [1, 2, 6, 7]. The realization problem for positive continuous-time and discrete-time linear systems has been considered in [8-15] and the positive minimal realization problem for singular discrete-time systems with delays in [12]. The realization problem for fractional linear systems has been analyzed in [5, 16, 17] and for positive 2D hybrid systems in [18]. A method based on the similarity transformation of the standard realizations to the desired form has been proposed in [13].

Positive stable realizations problem for continuous-time standard and fractional linear systems has been addressed in [9, 16] and computation of realizations of discrete-time cone systems in [19]. Necessary and sufficient conditions for the existence of a set of positive asymptotically stable realizations of a proper transfer function has been established in [11].

In this paper a method for computation of positive asymptotically stable realizations of descriptor discrete-time linear systems will be proposed.

The paper is organized as follows. In Sec. 2 the positive and asymptotically stable realization problem for standard discrete-time linear systems is recalled. The positive realization problem for descriptor discrete-time linear systems is formulated and solved in Sec. 3. An extension of this problem for asymptotically stable linear systems is given in Sec. 4, where two methods for computation of positive asymptotically stable realizations of improper transfer matrices are presented. Concluding remarks are given in Sec. 5.

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The following notation will be used: $\Re$ – the set of real numbers, $\Re^{n\times m}$ – the set of $n \times m$ real matrices, $\Re_+^{n\times m}$ – the set of $n \times m$ matrices with nonnegative entries and $\Re_+^n = \Re_+^{n\times 1}$, $\Re^{p\times m}(z)$ – the set of $p \times m$ rational matrices in z with real coefficients, $\Re^{p\times m}[z]$ – the set of $p \times m$ polynomial matrices in z with real coefficients, I_n – the $n \times n$ identity matrix

2. Preliminaries and positive realization problem for standard systems

Consider the standard discrete-time linear system

$$x_{i+1} = Ax_i + Bu_i, \quad i \in Z_+ = \{0, 1, ...\}, \tag{1a}$$

$$y_i = Cx_i + Du_i, \tag{1b}$$

where $x_i \in \Re^n$, $u_i \in \Re^m$, $y_i \in \Re^p$ are the state, input and output vectors and $A \in \Re^{n\times n}$, $B \in \Re^{n\times m}$, $C \in \Re^{p\times n}$, $D \in \Re^{p\times m}$.

Definition 1. The system (1) is called (internally) positive if $x_i \in \Re_+^n$, $y_i \in \Re_+^p$, $i \in Z_+$ for any initial conditions $x_0 \in \Re_+^n$ and all inputs $u_i \in \Re_+^m$, $i \in Z_+$.

Theorem 1. [1, 2] The system (1) is positive if and only if

$$A \in \Re_+^{n\times n}, \quad B \in \Re_+^{n\times m}, \quad C \in \Re_+^{p\times n}, \quad D \in \Re_+^{p\times m}. \tag{2}$$

The transfer matrix of the system (1) is given by

$$T(z) = C[I_n z - A]^{-1}B + D. \tag{3}$$

The transfer matrix $T(z) \in \Re^{p\times m}(z)$is called proper if and only if

$$\lim_{z\to\infty} T(z) = K \in \Re^{p\times m} \tag{4}$$

and it is called strictly proper if $K = 0$. Otherwise the transfer matrix is called improper.

The positive system (1) (matrix A) is asymptotically stable if

$$\lim_{i\to\infty} x_i = 0 \quad \text{for all} \quad x_0 \in \Re_+^n. \tag{5}$$

Theorem 2. [2] The positive system (1) is asymptotically stable if and only if all coefficients of the polynomial

$$p_A(z) = \det[I_n(z+1) - A] = z^n + a_{n-1}z^{n-1} + ... + a_1 z + a_0 \tag{6}$$

are positive, i.e. $a_k > 0$ for $k = 0, 1, \ldots, n-1$.

Definition 2. Matrices (2) are called a positive realization of transfer matrix $T(z)$ if they satisfy the equality (3).

Definition 3. A positive realization (2) is called asymptotically stable if the matrix $A \in \Re_+^{n\times n}$ is asymptotically stable.

Different methods for computation of a positive realization (2) for a given proper transfer matrix $T(z)$ have been proposed in [2, 7, 12, 14, 15, 18] and for positive asymptotically stable realizations in [9–11, 13, 16].

3. Positive realization problem for descriptor systems

3.1. Necessary and sufficient conditions for the existence of positive realizations. Consider the descriptor discrete-time linear system

$$Ex_{i+1} = Ax_i + Bu_i, \quad i \in Z_+ = \{0, 1, ...\} \tag{7a}$$

$$y_i = Cx_i \tag{7b}$$

where $x_i \in \Re^n$, $u_i \in \Re^m$, $y_i \in \Re^p$ are the state, input and output vectors and $E, A \in \Re^{n\times n}$, $B \in \Re^{n\times m}$, $C \in \Re^{p\times n}$.

It is assumed that det E = 0 and the pencil of (E, A) is regular, i.e.

$$\det[Ez - A] \neq 0 \quad \text{for some } z \in \mathrm{C} \quad \text{(the field of complex numbers).} \tag{8}$$

Definition 4. The descriptor system (7) is called (internally) positive if $x_i \in \Re_+^n$, $y_i \in \Re_+^p$, $i \in Z_+$ for any consistent initial conditions $x_0 \in \Re_+^n$ and all inputs $u_i \in \Re_+^m$, $i \in Z_+$.

If the nilpotency index μ of the matrix E is greater or equal to 1 [9] then the transfer matrix of (7) is improper and given by

$$T(z) = C[Ez - A]^{-1}B \in \Re^{p\times m}(z). \tag{9}$$

The improper matrix (9) can be always written as the sum of strictly proper part $T_{sp}(z)$ and the polynomial part $P(z)$, i.e.

$$T(z) = T_{sp}(z) + P(z), \tag{10a}$$

where

$$P(z) = D_0 + D_1 z + ... + D_q z^q \in \Re^{p\times m}[z], \quad q \in N = \{1, 2, ...\} \tag{10b}$$

and $q = \mu - 1$.

Theorem 3. Let the matrices

$$A \in \Re_+^{n\times n}, \qquad B \in \Re_+^{n\times m}, \qquad C \in \Re_+^{p\times n} \tag{11}$$

be a positive realization of the strictly proper transfer matrix $T_{sp}(z)$. Then there exists a positive realization of $T(z) \in \Re^{p\times m}(z)$ of the form

$$\overline{E} = \begin{bmatrix} I_n & 0 & 0 & ... & 0 & 0 \\ 0 & 0 & 0 & ... & 0 & 0 \\ 0 & I_m & 0 & ... & 0 & 0 \\ \vdots & \vdots & \vdots & ... & \vdots & \vdots \\ 0 & 0 & 0 & ... & I_m & 0 \end{bmatrix} \in \Re_+^{\overline{n}\times\overline{n}},$$

$$\overline{A} = \begin{bmatrix} A & B & 0 & ... & 0 & 0 \\ 0 & I_m & 0 & ... & 0 & 0 \\ 0 & 0 & I_m & ... & 0 & 0 \\ \vdots & \vdots & \vdots & ... & \vdots & \vdots \\ 0 & 0 & 0 & ... & 0 & I_m \end{bmatrix} \in \Re_+^{\overline{n}\times\overline{n}},$$

$$\overline{B} = -\begin{bmatrix} 0 \\ I_m \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in \Re_+^{\overline{n}\times m}, \tag{12}$$

$$\overline{C} = [\; C \quad D_0 \quad D_1 \quad ... \quad D_q \;] \in \Re_+^{p\times\overline{n}},$$

$$\overline{n} = n + (q+1)m$$

if and only if

$$D_k \in \Re_+^{p\times m} \quad \text{for} \quad k = 0, 1, \ldots, q. \tag{13}$$

Proof. If the matrices (11) are a positive realization of $T_{sp}(z)$ then the standard system

$$x_{i+1} = Ax_i + Bu_i, \tag{14a}$$

$$y_i = Cx_i \tag{14b}$$

is positive and $x_i \in \Re_+^n$, $i \in Z_+$ for any initial conditions $x_0 \in \Re_+^n$ and all inputs $u_i \in \Re_+^m$, $i \in Z_+$. Defining the new state vector

$$\overline{x}_i = \begin{bmatrix} x_i \\ u_i \\ u_{i+1} \\ \vdots \\ u_{i+q} \end{bmatrix} \in \Re^{\overline{n}} \tag{15}$$

and using (12) we obtain

$$\overline{E}\overline{x}_{i+1} = \overline{A}\overline{x}_i + \overline{B}\overline{u}_i, \tag{16a}$$

$$\overline{y}_i = \overline{C}\overline{x}_i. \tag{16b}$$

From (16) if follows that $\overline{x}_i \in \Re_+^{\overline{n}}$ and $\overline{y}_i \in \Re_+^p$ for $i \in Z_+$ if and only if (13) holds since $x_i \in \Re_+^n$ and $u_i \in \Re_+^m$ for $i \in Z_+$. Using (12), (9) and (10) it is easy to verify that

$$\overline{C}[\overline{E}z - \overline{A}]^{-1}\overline{B} = \begin{bmatrix} C & D_0 & D_1 & \dots & D_q \end{bmatrix}$$
$$\begin{bmatrix} I_n z - A & -B & 0 & \dots & 0 & 0 \\ 0 & -I_m & 0 & \dots & 0 & 0 \\ 0 & I_m z & -I_m & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & I_m z & -I_m \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ -I_m \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$
$$= \begin{bmatrix} C & D_0 & D_1 & \dots & D_q \end{bmatrix} \begin{bmatrix} [I_n z - A]^{-1}B \\ I_m \\ I_m z \\ \vdots \\ I_m z^q \end{bmatrix}$$
$$= C[I_n z - A]^{-1}B + D_0 + D_1 z + \dots + D_q z^q$$
$$= T_{sp}(z) + P(z) = T(z). \tag{17}$$

3.2. Determination of positive realizations. The positive realization problem for the descriptor system can be stated as follows. Given an improper rational matrix $T(z) \in \Re^{p\times m}(z)$, find its positive realization (12).

If the conditions of Theorem 3 are satisfied then the desired positive realization (12) of $T(z)$ can be computed by the use of the following procedure.

Procedure 1.

Step 1. Decompose the given matrix $T(z)$ into the strictly proper part $T_{sp}(z)$ and the polynomial part $P(z)$ satisfying (10).

Step 2. Using one of the well-known methods [2, 5, 7, 9, 11–15, 17, 18, 20] find the positive realization (11) of $T_{sp}(z)$.

Step 3. Knowing the realization (11) and the matrices $D_k \in \Re_+^{p\times m}$, $k = 0,1,\dots,q$ of (10b) find the desired realization (12).

Example 1. Find a positive realization (12) of the transfer matrix

$$T(z) = \begin{bmatrix} \dfrac{z^4 - 3z^3 + 3z^2 - 2z + 0.5}{z^2 - 3z + 2} & \dfrac{z^3 - 2z^2 - 4z + 4}{z^2 - 4z + 3} \\ \dfrac{3z^3 - 11z^2 + 6z + 0.5}{z^2 - 4z + 3} & \dfrac{2z^4 - 9z^3 + 8z^2 + 2z + 3.2}{z^2 - 5z + 6} \end{bmatrix}. \tag{18}$$

Using Procedure 1 we obtain the following.

Step 1. The transfer matrix (18) has the strictly proper part

$$T_{sp}(z) = \begin{bmatrix} \dfrac{z - 1.5}{z^2 - 3z + 2} & \dfrac{z - 2}{z^2 - 4z + 3} \\ \dfrac{z - 2.5}{z^2 - 4z + 3} & \dfrac{z - 2.8}{z^2 - 5z + 6} \end{bmatrix} \tag{19}$$

and the polynomial part

$$P(z) = \begin{bmatrix} z^2 + 1 & z + 2 \\ 3z + 1 & 2z^2 + z + 1 \end{bmatrix} = D_0 + D_1 z + D_2 z^2, \qquad (q = 2), \tag{20a}$$

where

$$D_0 = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}, \qquad D_1 = \begin{bmatrix} 0 & 1 \\ 3 & 1 \end{bmatrix}, \qquad D_2 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}. \tag{20b}$$

Step 2. The strictly proper transfer matrix (19) can be rewritten in the form

$$T_{sp}(z) = \frac{1}{(z-1)(z-2)(z-3)} \cdot \begin{bmatrix} (z-1.5)(z-3) & (z-2)^2 \\ (z-2.5)(z-2) & (z-2.8)(z-1) \end{bmatrix} \tag{21}$$

and the well-known Gilbert method can be applied to find its positive realization [2, 9, 21]. The poles of (21) are $z_1 = 1$, $z_2 = 2$, $z_3 = 3$. Following Gilbert method we compute the matrices

$$T_1 = \lim_{z\to z_1=1}(z - z_1)T_{sp}(z) = \begin{bmatrix} \dfrac{z-1.5}{z-2} & \dfrac{z-2}{z-3} \\ \dfrac{z-2.5}{z-3} & \dfrac{(z-1)(z-2.8)}{(z-2)(z-3)} \end{bmatrix}_{z=1} = \begin{bmatrix} 0.5 & 0.5 \\ 0.75 & 0 \end{bmatrix},$$
$$r_1 = \text{rank } T_1 = 2, \tag{22a}$$

$$T_2 = \lim_{z\to z_2=2}(z - z_2)T_{sp}(z) = \begin{bmatrix} \dfrac{z-1.5}{z-1} & \dfrac{(z-2)^2}{(z-1)(z-3)} \\ \dfrac{(z-2)(z-2.5)}{(z-1)(z-3)} & \dfrac{z-2.8}{z-3} \end{bmatrix}_{z=2}$$
$$= \begin{bmatrix} 0.5 & 0 \\ 0 & 0.8 \end{bmatrix}, \qquad r_2 = \text{rank } T_2 = 2, \tag{22b}$$

$$T_3 = \lim_{z\to z_3=3}(z - z_3)T_{sp}(z) = \begin{bmatrix} \dfrac{(z-1.5)(z-3)}{(z-1)(z-2)} & \dfrac{z-2}{z-1} \\ \dfrac{z-2.5}{z-1} & \dfrac{z-2.8}{z-2} \end{bmatrix}_{z=3}$$
$$= \begin{bmatrix} 0 & 0.5 \\ 0.25 & 0.2 \end{bmatrix}, \qquad r_3 = \text{rank } T_3 = 2 \tag{22c}$$

$$T_1 = C_1 B_1, \quad C_1 = \begin{bmatrix} 0.5 & 0.5 \\ 0.75 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$T_2 = C_2 B_2, \quad C_2 = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.8 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$T_3 = C_3 B_3, \quad C_3 = \begin{bmatrix} 0 & 0.5 \\ 0.25 & 0.2 \end{bmatrix}, \quad B_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{23}$$

and

$$A = \text{blockdiag}[I_{r_1} z_1, I_{r_2} z_2, I_{r_3} z_3] = \text{diag}[1, 1, 2, 2, 3, 3],$$

$$B = \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \tag{24}$$

$$C = [\ C_1 \quad C_2 \quad C_3\ ] = \begin{bmatrix} 0.5 & 0.5 & 0.5 & 0 & 0 & 0.5 \\ 0.75 & 0 & 0 & 0.8 & 0.25 & 0.2 \end{bmatrix}.$$

Step 3. The desired positive realization of (18) has the form

$$\overline{E} = \begin{bmatrix} I_6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & I_2 & 0 & 0 \\ 0 & 0 & I_2 & 0 \end{bmatrix} \in \Re_+^{\overline{n} \times \overline{n}},$$

$$\overline{A} = \begin{bmatrix} A & B & 0 & 0 \\ 0 & I_2 & 0 & 0 \\ 0 & 0 & I_2 & 0 \\ 0 & 0 & 0 & I_2 \end{bmatrix} \in \Re_+^{\overline{n} \times \overline{n}}, \tag{25}$$

$$\overline{B} = \begin{bmatrix} 0 \\ -I_2 \\ 0 \\ 0 \end{bmatrix} \in \Re_+^{\overline{n} \times 2},$$

$$\overline{C} = [\ C \quad D_0 \quad D_1 \quad D_2\ ] \in \Re_+^{2 \times \overline{n}},$$

$$\overline{n} = n + (q+1)m = 6 + 3 * 2 = 12$$

and the matrices A, B, C, D_0, D_1, D_2 are given by (24) and (20b).

A matrix $A \in \Re_+^{n \times n}$ is called monomial if in its every row and every column only one entry is positive and remaining entries are zero.

Theorem 4. The matrices

$$\widehat{E} = PEP^{-1} \in \Re_+^{n \times n}, \qquad \widehat{A} = PAP^{-1} \in \Re_+^{n \times n},$$
$$\widehat{B} = PB \in \Re_+^{n \times m}, \qquad \widehat{C} = CP^{-1} \in \Re_+^{p \times n}, \tag{26}$$
$$\widehat{D}_k = D_k \in \Re_+^{p \times m} \qquad \text{for} \quad k = 0, 1, \dots, q$$

are a positive realization of $T(z)$ for any monomial matrix $P \in \Re_+^{n \times n}$ if and only if the matrices

$$E \in \Re_+^{n \times n}, \qquad A \in \Re_+^{n \times n},$$
$$B \in \Re_+^{n \times m}, \qquad C \in \Re_+^{p \times n}, \tag{27}$$
$$D_k \in \Re_+^{p \times m} \quad \text{for} \quad k = 0, 1, \dots, q$$

are its positive realizations.

Proof. It is well-known [2] that $P^{-1} \in \Re_+^{n \times n}$ if and only if $P \in \Re_+^{n \times n}$ is a monomial matrix. In this case (26) holds if and only if the conditions (26) are met. Using (26) we obtain

$$\begin{aligned} &\widehat{C}[\widehat{E}z - \widehat{A}]^{-1}\widehat{B} + \widehat{D}_0 + \widehat{D}_1 z + \dots + \widehat{D}_q z^q \\ &= CP^{-1}[PEP^{-1}z - PAP^{-1}]^{-1}PB + D_0 \\ &\quad + D_1 z + \dots + D_q z^q \\ &= CP^{-1}\{P[Ez - A]P^{-1}\}^{-1}PB + D_0 \\ &\quad + D_1 z + \dots + D_q z^q \\ &= CP^{-1}P[Ez - A]^{-1}P^{-1}PB + D_0 \\ &\quad + D_1 z + \dots + D_q z^q \\ &= C[Ez - A]^{-1}B + D_0 + D_1 z + \dots + D_q z^q. \end{aligned} \tag{28}$$

Therefore, the matrices (26) are a positive realization of $T(z)$ if and only if the matrices (27) are also its positive realization.

From Theorem 4 we have the following corollary.

Corollary 1. If there exists a positive realization (27) of $T(z)$ then there exists a set of positive realizations (26) for every monomial matrix $P \in \Re_+^{n \times n}$.

4. Positive asymptotically stable realizations

It is well-known [2, 20] that the positive realization (12) of the transfer matrix $T(z)$ is asymptotically stable if and only if the positive realization (11) of the strictly proper transfer matrix $T_{sp}(z)$ is asymptotically stable. Therefore, the computation of positive asymptotically stable realization of $T(z)$ has been reduced to computation of the positive asymptotically stable realization (11) of the $T_{sp}(z)$.

Note that Theorem 4 and Corollary 1 are also valid for any asymptotically stable matrix $A \in \Re_+^{n \times n}$.

The following two methods for computation of positive asymptotically stable realizations of improper transfer matrices will be presented.

Method 1.

Consider the strictly proper transfer function

$$T_{sp}(z) = \frac{b_{n-1}z^{n-1} + \dots + b_1 z + b_0}{z^n - a_{n-1}z^{n-1} - \dots - a_1 z - a_0}. \tag{29}$$

Theorem 5. There exists a positive asymptotically stable realization of the form

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ a_0 & a_1 & a_2 & \dots & a_{n-1} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \tag{30}$$
$$C = \begin{bmatrix} b_0 & b_1 & b_2 & \dots & b_{n-1} \end{bmatrix}$$

of (29) if the following conditions are satisfied

1) $a_k \geq 0$ for $k = 0, 1, ..., n-1$, (31a)

2) $b_k \geq 0$ for $k = 0, 1, ..., n-1$, (31b)

3) $a_0 + a_1 + ... + a_{n-1} < 1$. (31c)

Proof is given in [10].

Remark 1. It is easy to show [10] that the following matrices are also the positive stable realizations of (29):

$$A = \begin{bmatrix} 0 & 0 & \dots & 0 & a_0 \\ 1 & 0 & \dots & 0 & a_1 \\ 0 & 1 & \dots & 0 & a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & a_{n-1} \end{bmatrix}, \quad B = \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ \vdots \\ b_{n-1} \end{bmatrix}, \tag{32}$$
$$C = \begin{bmatrix} 0 & 0 & \dots & 0 & 1 \end{bmatrix},$$

$$A = \begin{bmatrix} a_{n-1} & a_{n-2} & \dots & a_1 & a_0 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \tag{33}$$
$$C = \begin{bmatrix} b_{n-1} & \dots & b_2 & b_1 & b_0 \end{bmatrix},$$

$$A = \begin{bmatrix} a_{n-1} & 1 & 0 & \dots & 0 \\ a_{n-2} & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_1 & 0 & 0 & \dots & 1 \\ a_0 & 0 & 0 & \dots & 0 \end{bmatrix}, \quad B = \begin{bmatrix} b_{n-1} \\ b_{n-2} \\ \vdots \\ b_1 \\ b_0 \end{bmatrix}, \tag{34}$$
$$C = \begin{bmatrix} 1 & 0 & \dots & 0 & 0 \end{bmatrix}.$$

Knowing the positive asymptotically stable realization (30) of the strictly proper transfer function (29) a positive asymptotically stable realization (12) of an improper transfer function $T(z)$ can be computed by the use of Procedure 1 with slight modified Step 2 (a positive asymptotically stable realization instead of a positive realization should be found). The details will be demonstrated on the following example.

Example 2. Compute a positive asymptotically stable realization (12) of the transfer function

$$T(z) = \frac{z^5 + 0.3z^4 + 1.2z^3 + 2.82z^2 + 0.92z + 2}{z^3 - 0.7z^2 - 0.1z - 0.08}. \tag{35}$$

Using the Procedure 1 we obtain the following.

Step 1. The transfer function (35) has the strictly proper part

$$T_{sp}(z) = \frac{4.4z^2 + 1.2z + 2.16}{z^3 - 0.7z^2 - 0.1z - 0.08} \tag{36}$$

and the polynomial part

$$P(z) = D_0 + D_1 z + D_2 z^2 \tag{37}$$

where

$$D_0 = [2], \qquad D_1 = [1], \qquad D_2 = [1]. \tag{38}$$

Step 2. The positive realization (30) of (36) has the form

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0.08 & 0.1 & 0.7 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \tag{39}$$
$$C = [\ 2.16 \quad 1.2 \quad 4.4\ ].$$

The realization (39) is asymptotically stable since the condition (31c) is met. The poles of (36) are: $z_1 = 0.9074$, $z_2 = -0.1037 + j0.282$, $z_3 = -0.1037 - j0.282$.

Step 3. The desired positive asymptotically stable realization of (38) has the form

$$\overline{E} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix},$$
$$\overline{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0.08 & 0.1 & 0.7 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \tag{40}$$
$$\overline{B} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix},$$
$$\overline{C} = [\ 2.16 \quad 1.2 \quad 4.4 \quad 2 \quad 1 \quad 1\ ].$$

The method can be extended for m-inputs p-outputs systems as follows.

The strictly proper transfer matrix $T_{sp}(z) \in \Re^{p\times m}(z)$ can be written in the form

$$T_{sp}(z) = \begin{bmatrix} \frac{N_{11}(z)}{d_1(z)} & \dots & \frac{N_{1,m}(z)}{d_m(z)} \\ \vdots & \dots & \vdots \\ \frac{N_{p,1}(z)}{d_1(z)} & \dots & \frac{N_{p,m}(z)}{d_m(z)} \end{bmatrix} = N(z)D^{-1}(z) \tag{41a}$$

where

$$N(z) = \begin{bmatrix} N_{11}(z) & \dots & N_{1,m}(z) \\ \vdots & \dots & \vdots \\ N_{p,1}(z) & \dots & N_{p,m}(z) \end{bmatrix} \in \Re^{p\times m}[z],$$
$$D(z) = \mathrm{diag}[\; d_1(z) \quad \dots \quad d_m(z) \;] \in \Re^{m\times m}[z], \tag{41b}$$

$$N_{i,j}(s) = c_{i,j}^{d_j-1} z^{d_j-1} + \dots + c_{i,j}^1 z + c_{i,j}^0, \quad i = 1,\dots,p; \qquad j = 1,\dots,m \tag{41c}$$

$$d_j(z) = z^{d_j} - a_{j,d_j-1} z^{d_j-1} - \dots - a_{j,1} z - a_{j,0}, \quad j = 1,\dots,m. \tag{41d}$$

Theorem 6. There exists a positive asymptotically stable realization of the form

$$A = \mathrm{blockdiag}\,[\overline{A}_1, \overline{A}_2, \dots, \overline{A}_m],$$
$$\overline{A}_j = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ a_{j,0} & a_{j,1} & a_{j,2} & \dots & a_{j,d_j-1} \end{bmatrix}, \quad j = 1,\dots,m;$$

$$B = \mathrm{blockdiag}\,[\overline{b}_1, \overline{b}_2, \dots, \overline{b}_m], \quad \overline{b}_j = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \in \Re^{d_j}, \quad j = 1,\dots,m;$$

$$C = \begin{bmatrix} c_{11}^0 & c_{11}^1 & \dots & c_{11}^{d_1-1} & \dots & c_{1,m}^0 & c_{1,m}^1 & \dots & c_{1,m}^{d_m-1} \\ \vdots & \vdots & \dots & \vdots & \dots & \vdots & \vdots & \dots & \vdots \\ c_{p,1}^0 & c_{p,1}^1 & \dots & c_{p,1}^{d_1-1} & \dots & c_{p,m}^0 & c_{p,m}^1 & \dots & c_{p,m}^{d_m-1} \end{bmatrix} \tag{42}$$

of the transfer matrix $T_{sp}(z)$ if the following conditions are satisfied

1) $a_{j,k} \geq 0$ for $j = 1,\dots,mk = 0,1,\dots,d_j-1$, (43a)

2) $c_{i,j}^k \geq 0$ for $i = 1,\dots,p$, $j = 1,\dots,mk = 0,1,\dots,d_j-1$, (43b)

3) $a_{j0} + a_{j1} + \dots + a_{j,d_j-1} < 1$ for $j = 1,\dots,m$. (43c)

Proof is given in [10].

If the conditions of Theorem 6 are satisfied then the positive asymptotically stable realization of the transfer matrix (41) can be found by the use of the following procedure.

Procedure 2.

Step 1. Find the common denominators $d_j(z)$ $j = 1,\dots,m$ and write the strictly proper transfer matrix in the form (41a).

Step 2. Using

$$d_j(z) = z^{d_j} - [\; a_{j,0} \quad a_{j,1} \quad \dots \quad a_{j,d_j-1} \;] Z_j, \quad j = 1,\dots,m \tag{44a}$$

where

$$Z_j = [\; 1 \quad z \quad \dots \quad z^{d_j-1} \;] \tag{44b}$$

find the matrices $\overline{A}_1,\dots,\overline{A}_m$ and the matrix A.

Step 3. Using

$$N(z) = CZ \tag{45a}$$

where

$$Z = \mathrm{blockdiag}\,[Z_1, Z_2, \dots, Z_n] \tag{45b}$$

find the matrix C.

Example 3. Find a positive stable realization of the transfer matrix

$$T(z) = \begin{bmatrix} \frac{2z^3+0.6z^2+0.6z+0.2}{z^2-0.2z-0.1} & \frac{z^3+1.7z^2+0.2z+0.2}{z^2-0.3z-0.2} \\ \frac{z^3-0.2z^2+1.9z+0.2}{z^2-0.2z-0.1} & \frac{z^3+0.7z^2+0.5z+0.4}{z^2-0.3z-0.2} \end{bmatrix}. \tag{46}$$

Using Procedure 1 and 2 we obtain the following.

Step 1. The strictly proper part of (46) has the form

$$T_{sp}(s) = \begin{bmatrix} \frac{z+0.3}{z^2-0.2z-0.1} & \frac{z+0.6}{z^2-0.3z-0.2} \\ \frac{2z+0.2}{z^2-0.2z-0.1} & \frac{z+0.6}{z^2-0.3z-0.2} \end{bmatrix} \tag{47}$$

and the polynomial part

$$P(z) = D_0 + D_1 z \tag{48a}$$

where

$$D_0 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \qquad D_1 = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}. \tag{48b}$$

Step 2. The strictly proper transfer matrix (47) has the form

$$T_{sp}(s) = N(z)D^{-1}(z) \tag{49a}$$

where

$$N(z) = \begin{bmatrix} z+0.3 & z+0.6 \\ 2z+0.2 & z+0.6 \end{bmatrix},$$
$$D(z) = \begin{bmatrix} z^2-0.2z-0.1 & 0 \\ 0 & z^2-0.3z-0.2 \end{bmatrix}. \tag{49b}$$

Using (44a) and (49b) we obtain

$$\overline{A}_1 = \begin{bmatrix} 0 & 1 \\ 0.1 & 0.2 \end{bmatrix}, \quad \overline{A}_2 = \begin{bmatrix} 0 & 1 \\ 0.2 & 0.3 \end{bmatrix},$$

$$A = \begin{bmatrix} \overline{A}_1 & 0 \\ 0 & \overline{A}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0.1 & 0.2 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0.2 & 0.3 \end{bmatrix}. \tag{50}$$

From (45a) and (49b) we have

$$N(z) = \begin{bmatrix} z+0.3 & z+0.6 \\ 2z+0.2 & z+0.6 \end{bmatrix} = \begin{bmatrix} 0.3 & 1 & 0.6 & 1 \\ 0.2 & 2 & 0.6 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ z & 0 \\ 0 & 1 \\ 0 & z \end{bmatrix} = CZ \tag{51}$$

and

$$C = \begin{bmatrix} 0.3 & 1 & 0.6 & 1 \\ 0.2 & 2 & 0.6 & 1 \end{bmatrix}. \tag{52}$$

The matrix B defined by (42) has the form

$$B = \begin{bmatrix} \overline{b}_1 & 0 \\ 0 & \overline{b}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}. \tag{53}$$

Step 3. The desired positive asymptotically stable realization of the (47) has the form

$$\overline{E} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix},$$

$$\overline{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.1 & 0.2 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.2 & 0.3 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\overline{B} = -\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \; \overline{C} = \begin{bmatrix} 0.3 & 1 & 0.6 & 1 & 1 & 2 & 2 & 1 \\ 0.2 & 2 & 0.6 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}. \tag{54}$$

Method 2.

It is assumed that the distinct poles z_1, z_2,…, z_n of the transfer matrix $T_{sp}(z)$ are real positive and satisfy the condition

$$z_k < 1 \quad \text{for} \quad k = 1, 2, \ldots, n. \tag{55}$$

A positive asymptotically stable realization

$$A \in \Re_+^{N\times N}, \quad B \in \Re_+^{N\times m}, \quad C \in \Re_+^{p\times N} \tag{56}$$

of the transfer matrix $T_{sp}(z)$ can be computed by the use of the following procedure based on Gilbert method [2, 20, 21].

Procedure 3.

Step 1. Write the transfer matrix $T_{sp}(z) \in \Re^{p\times m}(z)$ in the form

$$T_{sp}(z) = \frac{N(z)}{d(z)}, \qquad N(z) \in \Re^{p\times m}[z] \tag{57}$$

and compute the poles $z_1, \ldots, z_n$ satisfying (55)

$$d(z) = (z - z_1)(z - z_2)...(z - z_n). \tag{58}$$

Step 2. Using the formula

$$T_i = \lim_{z\to z_i} (z - z_i)T_{sp}(z), \qquad i = 1, 2, ..., n \tag{59}$$

compute T_i, $i = 1, 2, ..., n$ where

$$r_i = \text{rank}\, T_i, \qquad i = 1, 2, ..., n \tag{60}$$

and

$$\begin{gathered} T_i = C_i B_i, \\ \text{rank} C_i = \text{rank} B_i = r_i, \\ C_i \in \Re_+^{p\times r_i}, \\ B_i \in \Re_+^{r_i\times m}, \\ i = 1, 2, ..., n. \end{gathered} \tag{61}$$

Step 3. Knowing z_1,…,z_n, r_i and C_i, B_i for $i = 1, 2, \ldots, n$ find the realization (56) of the form

$$\begin{gathered} A = \text{blockdiag}[I_{r_1}z_1, I_{r_2}z_2, ...I_{r_n}z_n] \in \Re_+^{N\times N}, \\ B = \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_n \end{bmatrix} \in \Re_+^{N\times m}, \\ C = [\; C_1 \quad C_2 \quad ... \quad C_n \;] \in \Re_+^{p\times N}, \\ N = \sum_{i=1}^{n} r_i. \end{gathered} \tag{62}$$

Theorem 7. There exists a positive asymptotically stable realization of the form (62) of (57) if the following conditions are satisfied:

1. the real positive poles $z_k \quad k = 1, 2, \ldots, n$ of the transfer matrix (57) satisfies the conditions (55),
2. the real matrices T_i defined by (59) have nonnegative entries $T_i \in \Re_+^{p\times m}$ for $i = 1, 2, \ldots, n$.

Proof. If the condition 1) is met then $A \in \Re_+^{N\times N}$ is asymptotically stable. If the assumption 2) is satisfied then (61) holds for $i = 1, 2, \dots, n$ and $B \in \Re_+^{N\times m}$, $C \in \Re_+^{p\times N}$. Using (62) it is easy to check that $C[I_N z - A]^{-1} B = T_{sp}(z)$.

Example 4. Find a positive asymptotically stable realization of the transfer matrix

$$T(z) = \begin{bmatrix} \frac{z^3+0.7z^2+0.72z-0.13}{(z-0.1)(z-0.2)} & \frac{z^3+0.6z^2+0.63z-0.17}{(z-0.1)(z-0.3)} \\ \frac{2z^3-z^2+1.12z-0.25}{(z-0.2)(z-0.3)} & \frac{3z^3-0.2z^2+0.69z-0.18}{(z-0.1)(z-0.3)} \end{bmatrix}. \tag{63}$$

Using the slight modified Procedure 1 we obtain the following.

Step 1. The transfer matrix (63) has the strictly proper part

$$T_{sp}(z) = \begin{bmatrix} \frac{z-0.15}{(z-0.1)(z-0.2)} & \frac{z-0.2}{(z-0.1)(z-0.3)} \\ \frac{z-0.25}{(z-0.2)(z-0.3)} & \frac{z-0.21}{(z-0.1)(z-0.3)} \end{bmatrix} \tag{64}$$

and the polynomial part

$$P(z) = D_0 + D_1 z \tag{65a}$$

where

$$D_0 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}. \tag{65b}$$

Step 2. To find the positive asymptotically stable realization of (64) we use Procedure 3. The transfer matrix (64) can be written in the form

$$T_{sp}(z) = \frac{1}{(z-0.1)(z-0.2)(z-0.3)} \cdot \begin{bmatrix} (z-0.15)(z-0.3) & (z-0.2)^2 \\ (z-0.25)(z-0.1) & (z-0.21)(z-0.2) \end{bmatrix} \tag{66}$$

and its poles are: $z_1 = 0.1$, $z_2 = 0.2$, $z_3 = 0.3$, which satisfy the condition 1) of Theorem 7. Using (59) and (66) we obtain

$$\begin{aligned}
T_1 &= \lim_{z\to z_1} (z - z_1) T_{sp}(z) = \begin{bmatrix} \frac{z-0.15}{z-0.2} & \frac{z-0.2}{z-0.3} \\ \frac{(z-0.25)(z-0.1)}{(z-0.2)(z-0.3)} & \frac{z-0.21}{z-0.3} \end{bmatrix}_{z=0.1} = \begin{bmatrix} 0.5 & 0.5 \\ 0 & 0.55 \end{bmatrix}, \\
&\operatorname{rank} T_1 = r_1 = 2, \\
C_1 &= \begin{bmatrix} 0.5 & 0.5 \\ 0 & 0.55 \end{bmatrix}, \\
B_1 &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \\
T_2 &= \lim_{z\to z_2} (z - z_2) T_{sp}(z) = \begin{bmatrix} \frac{z-0.15}{z-0.1} & \frac{(z-0.2)^2}{(z-0.1)(z-0.3)} \\ \frac{z-0.25}{z-0.3} & \frac{(z-0.21)(z-0.2)}{(z-0.1)(z-0.3)} \end{bmatrix}_{z=0.2} = \begin{bmatrix} 0.5 & 0 \\ 0.5 & 0 \end{bmatrix}, \\
&\operatorname{rank} T_2 = r_2 = 1, \\
C_2 &= \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}, \\
B_2 &= \begin{bmatrix} 1 & 0 \end{bmatrix}, \\
T_3 &= \lim_{z\to z_3} (z - z_3) T_{sp}(z) = \begin{bmatrix} \frac{(z-0.15)(z-0.3)}{(z-0.1)(z-0.2)} & \frac{z-0.2}{z-0.1} \\ \frac{z-0.25}{z-0.2} & \frac{z-0.21}{z-0.1} \end{bmatrix}_{z=0.3} = \begin{bmatrix} 0 & 0.5 \\ 0.5 & 0.45 \end{bmatrix}, \\
&\operatorname{rank} T_3 = r_3 = 2, \\
C_3 &= \begin{bmatrix} 0 & 0.5 \\ 0.5 & 0.45 \end{bmatrix}, \\
B_3 &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.
\end{aligned} \tag{67}$$

Using (62) and (67) we obtain the positive asymptotically stable realization of (64) in the form

$$A = \text{blockdiag}[I_{r_1}z_1, I_{r_2}z_2, I_{r_3}z_3]$$
$$= \text{diag}[0.1, 0.1, 0.2, 0.3, 0.3] \in \Re_+^{5\times 5},$$
$$B = \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \in \Re_+^{5\times 2}, \tag{68}$$
$$C = [\ C_1 \quad C_2 \quad C_3\ ]$$
$$= \begin{bmatrix} 0.5 & 0.5 & 0.5 & 0 & 0.5 \\ 0 & 0.55 & 0.5 & 0.5 & 0.45 \end{bmatrix} \in \Re_+^{2\times 5}.$$

Step 3. The desired positive asymptotically stable realization of the (63) is given by

$$\overline{E} = \begin{bmatrix} I_5 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & I_2 & 0 \end{bmatrix} \in \Re_+^{9\times 9},$$
$$\overline{A} = \begin{bmatrix} A & B & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_2 \end{bmatrix} \in \Re_+^{9\times 9}, \tag{69}$$
$$\overline{B} = -\begin{bmatrix} 0 \\ I_2 \\ 0 \end{bmatrix} \in \Re_+^{9\times 2},$$
$$\overline{C} = [\ C \quad D_0 \quad D_1\ ] \in \Re_+^{2\times 9},$$

where A, B, C, D_0, D_1 are defined by (68) and (65b).

5. Concluding remarks

Methods for computation of positive asymptotically stable realizations for descriptor discrete-time linear systems with improper transfer matrices have been proposed. Conditions for the existence of positive realizations for given improper transfer matrices have been established. Procedures for computation of positive asymptotically stable realizations have been proposed and illustrated by numerical examples. The proposed method can be easily extended to descriptor continuous-time linear systems. An open problem is an extension of this methods for fractional descriptor linear systems and descriptor continuous-discrete linear systems.

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