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# ANALYTICAL SOLUTION TO DETERMINE DISPLACEMENT OF NONLINEAR OSCILLATIONS WITH PARAMETRIC EXCITATION BY DIFFERENTIAL TRANSFORMATION METHOD

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**Abstract-** In this study, sub-harmonic displacement of nonlinear oscillations with parametric excitation is solved using a simulation method called the Differential Transformation Method (DTM). We employed this method to derive solutions of nonlinear oscillations with parametric excitation equation. Also Runge-Kutta as numerical method is exerted to this equation too. The obtained results from DTM are compared with those from the numerical solution to verify the accuracy of the proposed method. The results specify that the technique introduced here is accurate and achieve suitable results in predicting the solution of such problems.

**Keywords-** Differential Transformation Method (DTM); Numerical Solution (NS); Runge-Kutta; Sub-harmonic; Nonlinear oscillations; parametric excitation

## 1. INTRODUCTION

An ordinary rigid pendulum whose axis is driven periodically in the vertical direction is a paradigm of contemporary nonlinear dynamics [1]. This rather simple mechanical system is also interesting because the differential equation of the pendulum is frequently encountered in various problems of modern physics [2]. In this study, DTM used to solve Sub-harmonic resonances of nonlinear oscillation systems with parametric excitations, governed by:

$$\frac{d^2x(t)}{dt^2} + (1-\varepsilon\cos(\phi t))(\lambda x(t) + \beta x(t)^3) = 0 \qquad x(0) = A,\ \dot{x}(0) = 0. \tag{1}$$

where $\varepsilon, \phi, \beta, \lambda$ are known as physical parameters. [3]

The DTM is based on the Taylor's series expansion, and offer an effective numerical means of solving linear and non-linear problems. A study of the related literature makes known that Chiou [4] Make use of the intrinsic ability of differential transforms to solve non-linear problems. The differential transformation method may be used to solve both ODE and PDE. For example, Chen and Ho [5, 6], solved the free vibration problems using differential transforms. In their study of 2001, Jang, Chen and Liu [7], successfully applied the two-dimensional differential transformation method to the solution of partial differential equations. Yu and Chen [8, 9], applied the differential transformation method to optimize the rectangular fin with variable thermal parameters.

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Finally, Hassan [10] adopted the differential transformation method to solve some eigenvalue problems. In general, the previous studies have verified that the differential transformation method is an efficient technique for solving non-linear or parameter-varying systems as Numerical method. In most of prior articles, used DTM technique for problems without harmonic terms but in current study uses the differential transformation method to investigate the behavior of the vibrations of a non-linear oscillator and illustrates how the corresponding non-linear equations with harmonic terms may be converted into differential transforms and then solved by a process of inverse transformation. Something new in this research is using sub domain technique in DTM program code to achieve more accuracy toward ordinary domain that will be explained below. Also for solving these problems other analytic techniques like Homotopy Perturbation method and Modified Homotopy Perturbation is exist [11-19].
A comparison of the present results with those yielded by the established Runge–Kutta method confirms the accuracy of the proposed method.

## 2. APPLICATION OF DIFFRENTIAL TRANSFORMATION METHOD

The fundamental mathematical operations performed by differential transformation method are listed in Ref. [10]. Now we apply Differential Transformation Method into Eq. (1) with respect to $t$ gives:

$$(k+2)(k+1)X_{k+2}+\lambda X_k-\lambda\varepsilon\left(\sum_{l=0}^{k}\frac{X_{k-l}\phi^{l}\cos(\frac{1}{2}\pi l)}{l!}\right)+\beta\left(\sum_{l=0}^{k}X_{k-l}\left(\sum_{p=0}^{l}X_{l-p}X_p\right)\right)$$
$$-\beta\varepsilon\left(\sum_{l=0}^{k}\frac{1}{(k-l)!}\left(\phi^{(k-l)}\cos(\frac{1}{2}(k-l)\pi\right)\left(\sum_{p=0}^{l}X_{l-p}\left(\sum_{q=0}^{p}X_{p-q}X_q\right)\right)\right)=0 \tag{2}$$

By suppose $X_0$ and $X_1$ are apparent from boundary conditions by solving Eq. (20) respect $X_{k+2}$, we will have:

$$X_2=\frac{1}{2}\beta\varepsilon X_0^3-\frac{1}{2}\lambda X_0+\frac{1}{2}\lambda\varepsilon X_0-\frac{1}{2}\beta X_0^3 \tag{3}$$

$$X_3=-\frac{1}{6}\lambda X_1+\frac{1}{6}\lambda\varepsilon X_1+\frac{1}{6}\lambda\varepsilon X_0\phi\cos(\frac{1}{2}\pi)$$
$$-\frac{1}{2}\beta X_1X_0^2+\frac{1}{6}\beta\varepsilon\phi\cos(\frac{1}{2}\pi)X_0^3+\frac{1}{2}\beta\varepsilon X_1X_0^2 \tag{4}$$

$$X_4=-\frac{1}{3}\lambda\beta\varepsilon X_0^3+\frac{1}{24}\lambda^2X_0-\frac{1}{12}\lambda^2\varepsilon X_0+\frac{1}{6}\lambda\beta X_0^3$$
$$+\frac{1}{6}\lambda\beta\varepsilon^2X_0^3+\frac{1}{24}\lambda^2\varepsilon^2X_0+\frac{1}{12}\lambda\varepsilon X_1\phi\cos(\frac{1}{2}\pi)+\ldots \tag{5}$$

$$X_5=\frac{1}{20}\beta\varepsilon X_1^3+\frac{9}{40}X_0^4\beta^2X_1+\frac{1}{120}\lambda^2\varepsilon^2X_1-\frac{1}{60}\lambda^2\varepsilon X_1-\frac{1}{30}\lambda^2\varepsilon X_0\phi\cos(\frac{1}{2}\pi)$$
$$+\frac{1}{120}\lambda^2X_1+\frac{1}{120}\lambda\varepsilon X_0\phi^3\cos(\frac{3}{2}\pi)+\frac{1}{40}\lambda\varepsilon X_1\phi^2\cos(\pi)+\ldots \tag{6}$$

$\vdots$

The above process is continuous. Substituting Eq. (3-6) into the main equation based on DTM, it can be obtained the closed form of the solutions.

$$x(t) = X_0 + tX_1 + \frac{t^2}{2!}\left(\frac{1}{2}\beta\varepsilon X_0^3 - \frac{1}{2}\lambda X_0 + \frac{1}{2}\lambda\varepsilon X_0 - \frac{1}{2}\beta X_0^3\right) +$$
$$\frac{t^3}{3!}\left(-\frac{1}{6}\lambda X_1 + \frac{1}{6}\lambda\varepsilon X_1 + \frac{1}{6}\lambda\varepsilon X_0\phi\cos(\frac{1}{2}\pi) - \frac{1}{2}\beta X_1 X_0^2 + \frac{1}{6}\beta\varepsilon\phi\cos(\frac{1}{2}\pi)X_0^3 + \frac{1}{2}\beta\varepsilon X_1 X_0^2\right) + \cdots \quad (7)$$

In this stage for achieve higher accuracy we use sub-domain technique, i.e. the domain of $t$ should be divided into some adequate intervals and the values at the end of each interval will be the initial values of next one. For example for first sub-domain assume that distance of each interval is $0.2$. For first interval, $0 \rightarrow 0.2$, boundary conditions are from boundary conditions given in Eq. (1) at point $t = 0$. By exerting transformation we will have:

$$X_0 = A \quad (8)$$

And the other boundary conditions are considered as follow:

$$X_1 = 0 \quad (9)$$

As mentioned above for next interval, $0.2 \rightarrow 0.4$, new boundary conditions are:

$$X_0 = x(0.2) \quad (10)$$

The next boundary condition is considered as follow:

$$X_1 = \frac{dx}{dt}(0.2) \quad (11)$$

For this interval function $x(t)$ is represented by power series whose center is located at $0.2$, by means that in this power series $t$ convert to $(t - 0.2)$.

## 3. RUNGE-KUTTA METHOD

The Runge-Kutta methods are an important iterative method for the approximation solutions of linear and nonlinear differential equations. In order to verify the effectiveness of the proposed Differential Transformation Method, by using package of Maple 10, the fourth-order Runge–Kutta as numerical method is used to compute the displacement response of the non-linear oscillator for a set of initial amplitudes and different physical parameters. These results are then compared with the DTM corresponding to the same set of amplitudes.

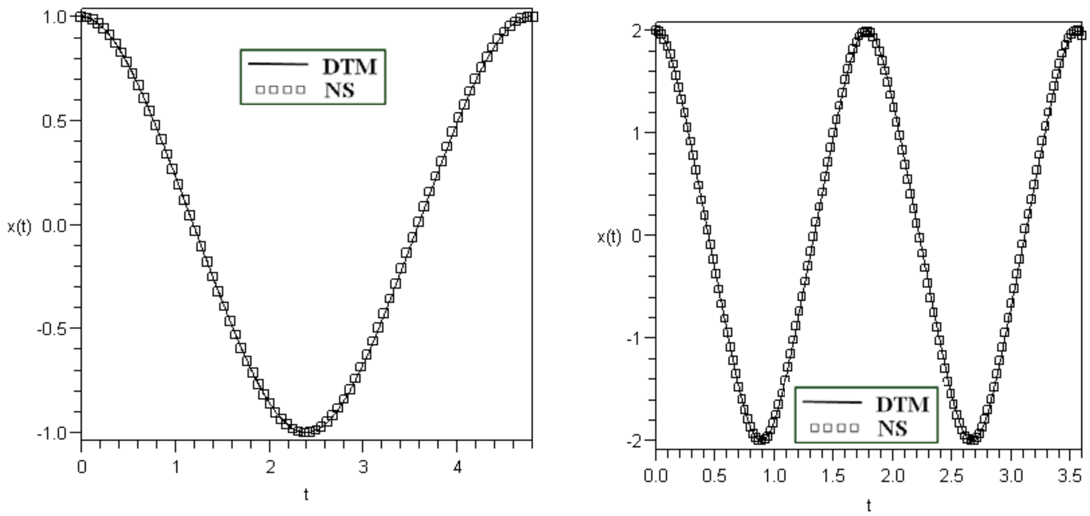
## 4. RESULTS AND DISGUSSION

In this study, the DTM was applied successfully to find the analytical solution of the nonlinear oscillations with parametric excitation. Comparison of the solutions between DTM and Runge–Kutta method for a set of initial amplitudes and different physical parameters are shown in Fig. (1,2). By comparison the results, the advantages and features of the DTM can be summarized as follows.

- The present methods reduce the computational difficulties of the other methods and all the calculations can be made simple manipulations.
- By using sub domain technique the accuracy of the method is very good.
- By attending the DTM results in this study and referenced papers, obtained that this method can be used for both linear and nonlinear differential equations.

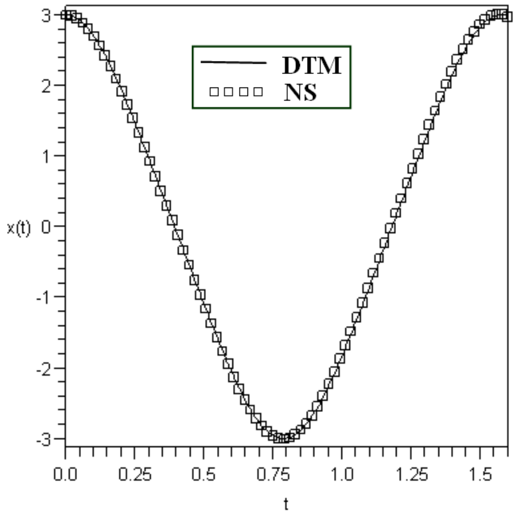
## 5. CONCLUSION

In the present work, we have applied Differential Transformation Method (DTM) to compute the sub-harmonic amplitude of nonlinear oscillations with parametric excitation. Consequently, this equation is solved by the fourth order Runge–Kutta as Numerical solution. Solutions are quite elegant and fully acceptable in accuracy. The results of the different methods of DTM and Runge–Kutta are compared in Fig. (1-4).The figures clearly show high accuracy of this method and show this approximate analytical solution is in an admirable agreement with the corresponding numerical solutions.

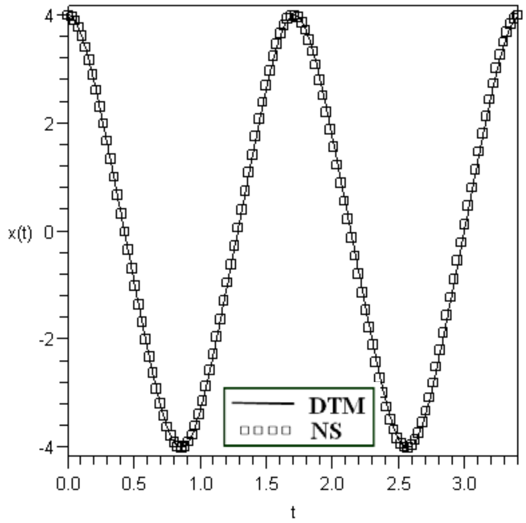



(a) (b)

Fig. 1 The comparison between DTM and numerical solutions of $x[m]$ to $t[s]$,a) for $A$ = 1, $\lambda$ = 1, $\beta$ = 1, $\varepsilon = 0.01$, $\phi = 10.$, b) for $A$ = 2, $\lambda$ = 1, $\beta$ =4, $\varepsilon = 0.01$, $\phi = 10$




(a)




(b)

Fig. 2 The comparison between DTM and numerical solutions of $x[m]$ to $t[s]$ ,a) for $A$ = 3, $\lambda$ = 3, $\beta$ = 2, $\varepsilon = 0.01$, $\phi = 10.$, b) for $A$ = 4, $\lambda$ = 2, $\beta$ = 2, $\varepsilon = 0.01$, $\phi = 10$

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