



AMPLITUDE NOISE REDUCTION IN A NANO-MECHANICAL OSCILLATOR

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Abstract- We study the quantum properties of a nano-mechanical oscillator via the squeezing of the oscillator amplitude. The static longitudinal compressive force F_0 close to a critical value at the Euler buckling instability leads to an anharmonic term in the Hamiltonian and thus the squeezing properties of the nano-mechanical oscillator are to be obtained from the Hamiltonian of the form $H = a^\dagger a + \beta(a^\dagger + a)/4$. This Hamiltonian has no exact solution unlike the other known models of nonlinear interactions of the forms $a^\dagger a^2$, $(a^\dagger a)^2$ and $a^\dagger a + a^4 - (a^\dagger a^2 + a^2 a^\dagger)$ previously employed in quantum optics to study squeezing. Here we solve the Schrodinger equation numerically and show that in-phase quadrature gets squeezed for both vacuum and coherent states. The squeezing can be controlled by bringing F_0 close to or far from the critical value F_c . We further study the effect of the transverse driving force on the squeezing in nano-mechanical oscillator.

Key Words- Squeezing, Nano-mechanical Systems, Quantum Noise

1.INTRODUCTION

There is currently a wide effort to observe quantum behavior in nanoscale devices [1-4]. In the limit of high resonator frequency with high mechanical quality factors and long coherence lifetimes, the nanomechanical oscillator (NMO) phonons will be analogous to photons in an electromagnetic cavity. With current technology it is possible to reach resonator frequency of GHz order [5]. At a temperature of around 50mK, one can principally prepare the resonator into the ground state. These sub-Kelvin temperatures are well within the range of today's dilution refrigerators. However cooling the resonator down to these temperatures requires some other techniques [6,7].

With the assumption that quantum mechanics should apply to these mesoscopic systems, variety of methods and techniques have been proposed to observe some quantum optical effects including solid-state laser cooling [6, 7], quantum nondemolition measurement [8, 9], phonon lasing [10] and squeezed state generation [11, 12]. There are also proposals analyzing macroscopic quantum tunneling [13], resonant multi-phonon excitations [14] and a variety of methods to entangle mechanical resonators with other quantum systems [15-18].

The next question is what could be the best way to study quantum properties of a NMO. In line with the work in quantum optics on squeezing, we can consider studying the squeezed states of the NMO. One proposal considers modulating the spring constant to produce squeezing [11] as it is known from the earlier work [19] that any modulation of the frequency of the oscillator can result in squeezing. Here we adopt a different

model. We consider the situation shown schematically in the Fig. 1. We show various forces acting on a nanobeam structure which is clamped at both ends vibrating in the transverse direction. There is a static mechanical force, F_0 , acting in the longitudinal direction and an ac-driving force to excite vibrations in the transverse direction. The longitudinal force F_0 which is close to a critical value at the Euler instability gives rise to an additional term in the potential energy which is quartic in the fundamental mode amplitude x . The effective Hamiltonian that describes the system would be in the form $H = p^2/2m + mw^2x^2/2 + \beta x^4/4$. The derivation of this

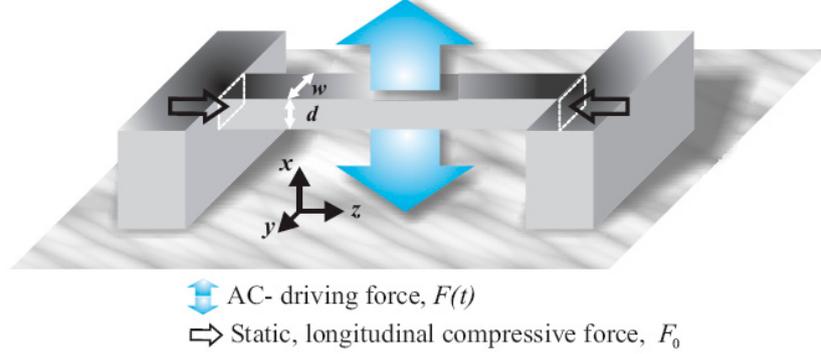


FIG. 1: (Color online) Schematic diagram of the freely suspended nanomechanical beam of total length L , width w and thickness d . The beam is clamped at both ends. A static, mechanical force F_0 compresses the beam in longitudinal direction controlling the nonlinearity. An ac-driving force can be used to excite the beam to transverse vibrations.

nonlinearity in x is given in the next section. Unlike the previous work on squeezing [11] in a NMO we would consider the effect of the nonlinearity in x . Note that the nonlinearity can be switched on and off by controlling F_0 . We would thus study the quantized behavior of a NMO subject to the force F_0 .

The organization of the paper is as follows. The model is described in section 2 and the effective Hamiltonian is derived briefly by referring to the previous works for the doubly clamped elastic rectangular beam. We discuss the previous works on the squeezing in nonlinear oscillators in section 3. Then, we study the quantum Dynamics and analyze the squeezing properties in section 4. The conclusion and future perspective are given in section 5.

2. THE MODEL

We start with an elastic rectangular beam of length L , width w and thickness d as shown in Fig. 1. The beam is freely suspended and clamped at both ends. The transverse motion in the direction of d is allowed. The dimensions are such that ($L \gg w > d$) there is no appreciable vibrations in other directions. A static mechanical force F_0 acts on the beam in the longitudinal direction ($F_0 > 0$ for compression). An ac-driving field, $F(t) = \tilde{e}f \cos(\omega_{ex} t)$ can also be added to excite the vibrations. The dynamics of the beam can be completely described by the transverse deflection $\phi(s)$ parametrized by the arclength $s \in [0; L]$ in a classical picture. Assuming single transverse degree of freedom for simplicity the nonlinear Lagrangian of the system, for arbitrary strong deflections $\phi(s)$ is then [20, 21],

$$L(\phi, \dot{\phi}, t) = \int_0^L ds \left[\frac{\rho}{2} \dot{\phi}^2 - \frac{\mu}{2} \frac{\phi''^2}{1 - \phi'^2} - F_0 (\sqrt{1 - \phi'^2} - 1) + F(t)\phi \right] \quad (1)$$

Here $\rho = m/L$ is the mass density, $\mu = EI$ is the product of the elasticity modulus E and the moment of inertia I . In ϕ' , prime denotes partial derivative with respect to s , i.e. $\partial\phi/\partial s$. For small oscillations $|\phi'(s)| \ll 1$, the Lagrangian is quadratic and it leads to the linear equation of motion

$$\rho \ddot{\phi} + \mu \phi'''' + F_0 \phi'' = 0 \quad (2)$$

The equation of motion can be separated and transformed into an eigenvalue problem with boundary conditions applied to the endpoints. One can write the general solution as a superposition $\phi(s, t) = \sum_n \phi_n(s, t) = \sum_n A_n(t) g_n(s)$ where $g_n(s)$ are the normal modes

which follow as solution of the characteristic equation. For the doubly clamped nanobeam, we have $\phi(0) = \phi(L) = 0$ and $\phi'(0) = \phi'(L) = 0$. The expressions for ϕ_n 's are then given by a superposition of trigonometric and hyperbolic functions, and the eigenfrequencies w_n 's are the solutions of transcendental equations. To obtain simple expressions for the eigenfunctions we can use the free boundary conditions, $\phi'(0) = \phi'(L) = 0$ leaving the essential physics of the problem unchanged. In that case the normal-mode expansion becomes

$$\phi(s, t) = \sum_n A_n(t) \sin \frac{n\pi}{L} s \quad (3)$$

with the eigenfrequencies

$$w_n = \left(\frac{\mu(n\pi/L)^2 - F_0}{\rho} \right)^{1/2} \frac{n\pi}{L} \quad (4)$$

When F_0 has the critical value $F_c = \mu(\pi/L)^2$, the fundamental frequency ω_1 vanishes as $\sqrt{\epsilon}$ where $\epsilon = (F_c - F_0)/F_c$ is the distance to critical to the critical force and the system reaches to the well known Euler instability. Close to the Euler instability $F_0 \rightarrow F_c$, for the doubly clamped beam one can get the simplified fundamental frequency $w_1(F_0 - F_c) = \sqrt{\epsilon} w_0$ where $w_0 = (2\pi^2/3)\sqrt{E/\rho}(d/L^2)$ is the fundamental frequency of the relaxed beam ($F_0 = 0$). The frequencies of higher modes $n = 2, 3, \dots$ remain finite. The dynamics at low frequencies is determined by the fundamental mode alone. Since the fundamental frequency w_1 vanishes at the critical value F_c , one has to include the contributions beyond the quadratic terms ϕ'^2 and ϕ''^2 in the Lagrangian. The next higher order terms, $-(\mu/2)\phi''^2\phi'^2$ and $(F_0/4)\phi'^4$ are quartic in the Lagrangian. Inserting the normal mode expansion (3) in the Lagrangian and assuming that the fundamental mode $n = 1$ dominates the dynamics (by neglecting the higher modes $n = 2, 3, \dots$) one can quantize the theory by introducing the canonically conjugate momentum $p = -i\hbar\partial/\partial A_1$ with the ‘‘coordinate’’ $x \equiv A_1$. Note that when the driving frequency is close to the fundamental frequency of the beam, the fundamental mode will dominate also in absence of a static longitudinal compression force F_0 . However, a compression force close to a critical value helps to enhance the nonlinear effects which are of the importance of this paper.

By using the above definition of coordinate and the conjugate momentum, an effective quantum mechanical time-dependent Hamiltonian results describing the dynamics of a single quantum particle

$$\tilde{H}(t) = \frac{p^2}{2m^*} + \frac{m^* w_1^2}{2} x^2 + \frac{\tilde{\beta}}{4} x^4 + xF(t) \quad (5)$$

with the effective mass $m^* = 3\rho L/8$, the fundamental frequency $w_1 = \sqrt{\varepsilon}(2\pi^2/3)\sqrt{E/\rho}(d/L^2)$ and the nonlinearity parameter $\tilde{\beta} = (\pi/L)^4 F_c L(1+3\varepsilon)$ [22]. Now, Eq. (5) can be put in a second-quantized form by replacing x and p with the creation and annihilation operators a^+ and a ,

$$x = \sqrt{\frac{\hbar}{m^* w_1}} \frac{1}{\sqrt{2}} (a^+ + a) \quad (6)$$

$$p = \sqrt{m^* \hbar w_1} \frac{i}{\sqrt{2}} (a^+ - a) \quad (7)$$

Upon scaling the Hamiltonian by $\hbar w_0$ we obtain the dimensionless form,

$$H(t) = a^+ a + \frac{1}{2} + \beta \left(\frac{a^+ + a}{\sqrt{2}} \right)^4 + f \cos(w_{ex} t) \left(\frac{a^+ + a}{\sqrt{2}} \right) \quad (8)$$

with the redefined dimensionless parameters,

$$\beta = \frac{\tilde{\beta} x_0^4}{4\hbar w_1}, \quad f = \frac{\tilde{f} x_0}{\hbar w_1} \quad (9)$$

where $x_0 = \sqrt{\hbar/m^* w_1}$. One can obtain an expression for β that depends on the dimensions and the material properties of the beam. By substituting the parameters in Eq. (9) one finds

$$\beta = \frac{\hbar}{2} \frac{1}{L w d^2} \frac{1}{\sqrt{\rho E}} \frac{1+3\varepsilon}{\varepsilon^{3/2}} \quad (10)$$

Note that the equation (10) is valid for $\varepsilon \ll 1$ and β can be controlled by fine tuning the distance parameter ε at this regime. Table I lists the range of the nonlinearity parameter β as well as the relaxed fundamental frequencies and the critical compressions for three different sizes of Si nanobeams. Note that one should have extremely precise control over ε to increase the nonlinearity. In light of the measurements done in the experiment [23] ε was found to be of the size $\approx 10^{-5}$ for a 100 nm length carbon nanotube.

Si bar, $\rho = 2330 \text{ kg/m}^3$, $E = 169 \text{ GPa}$		
$L = 200 \text{ nm}$	$\omega_0/2\pi = 1.12 \text{ GHz}$	$\varepsilon = 10^{-6} \implies \beta = 0.05$
$d = 5 \text{ nm}$	$F_c = 4.34 \text{ nN}$	$\varepsilon = 10^{-7} \implies \beta = 1.68$
$w = 10 \text{ nm}$		
$L = 400 \text{ nm}$	$\omega_0/2\pi = 557 \text{ MHz}$	$\varepsilon = 10^{-7} \implies \beta = 0.11$
$d = 10 \text{ nm}$	$F_c = 17.3 \text{ nN}$	
$w = 20 \text{ nm}$		

TABLE I: Table shows calculated β values, the nonlinearity parameter, for two different beam size. For each size, the relaxed fundamental mode frequency w_0 and the critical force F_c defined in the text, are also shown. The parameter ε shows the distance to the critical force near the Euler instability.

3. PREVIOUS WORKS ON THE SQUEEZING IN NONLINEAR OSCILLATORS

The squeezing produced by the nonlinearity has been investigated in the past by using a number of approximations however none of these are suitable for the problem of the NAMO. Milburn dropped all phase sensitive terms from (8) and studied [24] the simplified Hamiltonian,

$$H = \hbar w \left[(1 + \beta) a^+ a + \beta (a^+ a)^2 \right] \quad (11)$$

By solving exactly the phase space distribution function he showed that squeezing can be obtained for a coherent state of amplitude $\alpha = 0.5$ for very short times. Buzek [25] and Tanas [26] studied Hamiltonian models of the form,

$$H = \hbar w a^+ a + \frac{1}{2} \hbar \beta a^{+2} a^2 \quad (12)$$

$$H = \hbar w a^+ a + \frac{1}{2} \hbar \beta (a^+ a)^2 \quad (13)$$

They solved the Heisenberg equations of motion exactly and showed periodic squeezing for coherent states in both quadratures. Tanas [26] showed also that maximum squeezing can be obtained in the limit of large mean number ($|\alpha|^2 \gg 1$) and small times ($t \ll 1$). No squeezing is allowed for the vacuum state in the above models. In this paper, we calculate the squeezing in an anharmonic oscillator for the anharmonicity quartic in x which is the amplitude of the fundamental mode of the oscillation. One could write x^4 in terms of creation and annihilation operators as follows:

$$x^4 \alpha (a^+ + a)^4 = a^{+4} + a^4 + f(a^+, a) \quad (14)$$

where $f(a^+, a)$ is a polynomial in a and a^+ of order four which is given by

$$f(a^+, a) = 6a^{+2} a^2 + 4(a^{+2} a^+ a + a^+ a a^2) + 6(a^{+2} + a^2) + 12a^+ a + 3 \quad (15)$$

The Hamiltonian containing the second and third terms given in Eq. (14) give rise to two-photon (or phonons in quantum mechanical descriptions of solid systems) transitions. It is known that two-photon transitions are necessary for producing squeezed states and thus the terms $a^+ a a^2$, a^2 and their hermitian conjugate should be important. In the literature, multi-photon processes have also been analyzed to study normal and higher order squeezing in the limit of small times [27-29]. In relation to this paper, Tombesi and Mecozzi [30] studied the harmonic oscillator model which has four-photon transitions in the interaction term,

$$H_I = \beta [a^{+4} + a^4 - (a^{+2} a^2 + a^2 a^{+2})] \quad (16)$$

This model was solved exactly. They showed that significant amount of normal and higher order squeezing is possible for initial coherent states of amplitude $|\alpha| > 1$ with certain phases and for short times. No squeezing is allowed for the vacuum and fluctuations diverge as time grows. In the description of NMO systems the anharmonic

Hamiltonian models given in Eqs. (11), (12), (13) and (15) are no good for vacuum squeezing. Moreover, the nonlinearity λ cannot be controlled externally since it is an intrinsic property of the medium. To observe the nonclassical (quantum) properties of a mesoscopic system in general, the control of the parameter that gives rise to the nonclassical behavior would be crucial for an experimentalist. The harmonic oscillator having the nonlinearity of Eq.(14). that we shall work in the next sections, gives important squeezing in the in-phase quadrature for both vacuum and coherent states. Furthermore, it will be shown in section III that the physical model of the NMO allows one to control the nonlinearity by the application of a static external force. The numerical solution of the Hamiltonian shows that the vacuum squeezing displays periodicity and it stays squeezed for the whole cycle of the period. In fact, the vacuum squeezing is important for the mesoscopic resonators because bringing the harmonic oscillator representing the nano-mechanical system to its vibrational ground state is a necessary prerequisite for quantum state engineering. The effect of driving term is also examined.

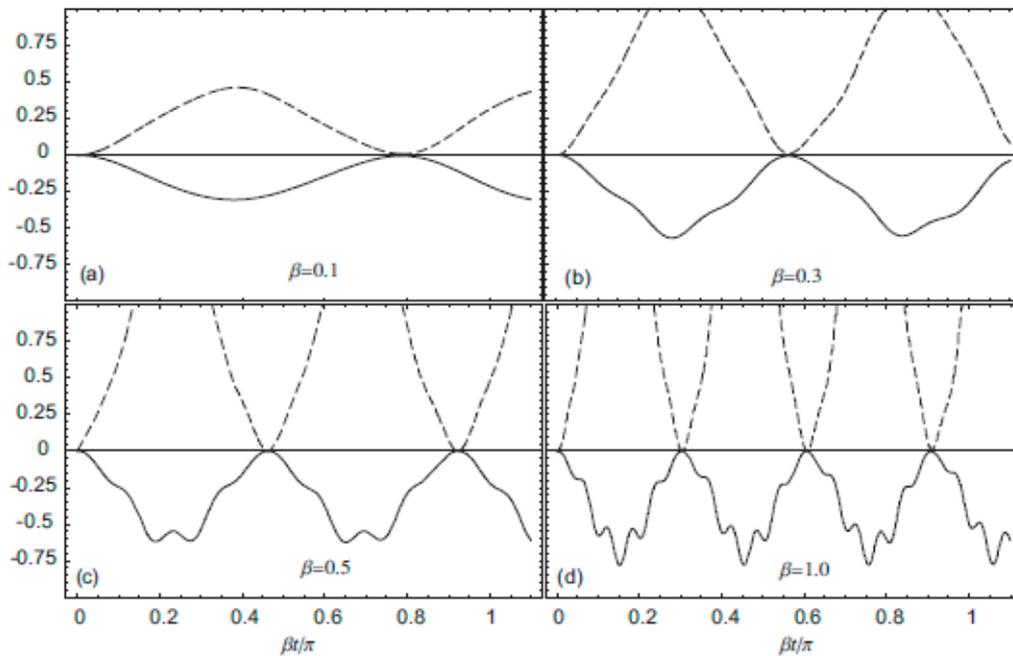


FIG. 2: Squeezing of the vacuum in the anharmonic oscillator model given by Eq. (8) changing from weak to strong nonlinearity β ; (a) $\beta=0.1$, (b) $\beta=0.3$, (c) $\beta=0.5$, (d) $\beta=1.0$. Solid line, $S_x(t)$, normally ordered normalized fluctuations in x and dashed line, $S_p(t)$, normally ordered normalized fluctuations in p as given by Eqs. (19) and (20) respectively.

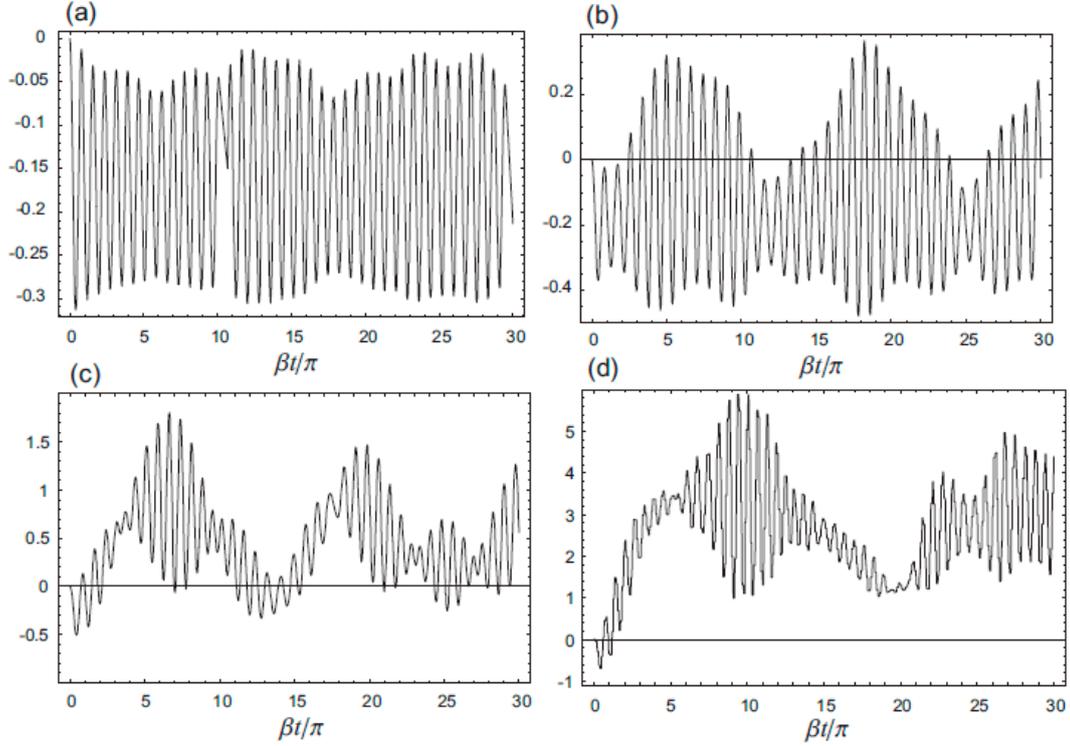


FIG. 3: The squeezing $S_x(t)$ for the coherent state of amplitudes, (a) $|\alpha| = 0.1$, (b) $|\alpha| = 0.5$, (c) $|\alpha| = 1.0$, (d) $|\alpha| = 2.0$. The phase of α is equal to $\pi/2$. The nonlinearity parameter β is taken as 0.1.

4. QUANTUM DYNAMICS AND SQUEEZING

We first consider the case in which there is no driving. The Hamiltonian

$$H = a^+ a + \frac{1}{2} + \beta \left(\frac{a^+ + a}{\sqrt{2}} \right)^4 \quad (17)$$

has no analytical solution. For the numerical calculations, we employ the split-operator method [31] for the time propagation of the initial state. In this method, one can split the propagator on a time step Δt as

$$U(t + \Delta t) = e^{-i(H_0 + V)\Delta t} \\ = e^{-iH_0\Delta t/2} e^{-iV\Delta t} e^{-iH_0\Delta t/2} + O[(\Delta t)^3] \quad (18)$$

where $H = a^+ a + 1/2$ and $V = \beta(a^+ + a)^4/4$. That means splitting the exponential of the operators which are not commuting is accurate to second order in the time step Δt . Therefore one can make the calculation as accurate as possible by taking the time step sufficiently small. Then we can calculate the normally ordered variances for x and p normalized over vacuum fluctuations,

$$S_x(t) = \frac{\langle :(\Delta x(t))^2 : \rangle}{\langle (\Delta x(0))^2 \rangle_{vac}}, \quad (19)$$

$$S_p(t) = \frac{\langle :(\Delta p(t))^2 : \rangle}{\langle (\Delta p(0))^2 \rangle_{vac}}, \tag{20}$$

where the sign $::$ means normal ordering with respect to the pair of creation operator a^+ and annihilation operator a . The denominators are the initial vacuum fluctuations of the of the fundamental mode amplitude x and its conjugate p . Negative values of $S_{x,p}(t)$ means squeezing and it gets -1 when the squeezing is perfect. Fig. 2 shows squeezing for the vacuum for different nonlinearity values $\beta=0.1, 0.3, 0.5$ and 1.0 . The maximum squeezing ranges from ~ -0.30 to ~ -0.80 . The value -0.80 means that the uncertainty in x is squeezed to $\sqrt{1-0.80} = \%45$ of its vacuum value.

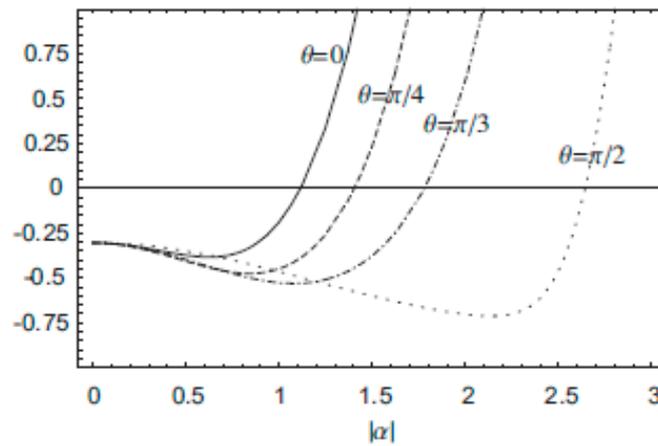


FIG. 4: Maximum squeezing S_{xmax} vs. coherent state amplitude $|\alpha|$. The dependence is shown for different values of the phase. The nonlinearity parameter β is equal to 0.1.

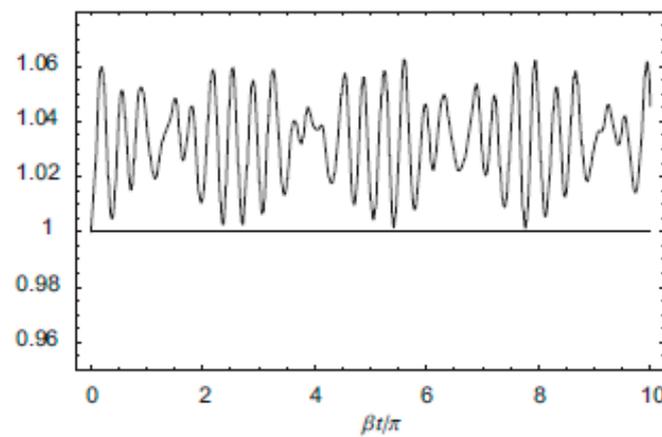


FIG. 5: Time evolution of the normalized uncertainty product $\Delta x(t)\Delta p(t) / \Delta x(0)\Delta p(0)$ for the vacuum.

One could also start with a coherent state to check squeezing. We analyzed squeezing for different initial coherent state amplitudes and phases, $\alpha = |\alpha|e^{i\theta}$ at the nonlinearity value of $\beta = 0.1$. Fig. 3 shows the time evolution of the squeezing for increasing coherent state amplitude α at the phase $\theta = \pi/2$. Fig. 4 shows the dependence of the maximum squeezing to the phase. It can be seen from the figure that the phase does not make much difference for low amplitudes $|\alpha| < 0.5$ but it changes squeezing behavior drastically for amplitudes larger than 0.5. Increasing the amplitude increases the maximum squeezing whereas the state never becomes squeezed after a short duration. On the other hand, low amplitudes show periodic squeezing all the time at a moderate value. One can also plot the time evolution of the normalized uncertainty product $\Delta x(t)\Delta p(t) / \Delta x(0)\Delta p(0)$ for the vacuum. Fig. 5 shows that the oscillator recovers its minimum uncertainty periodically and the fluctuation remains bounded.

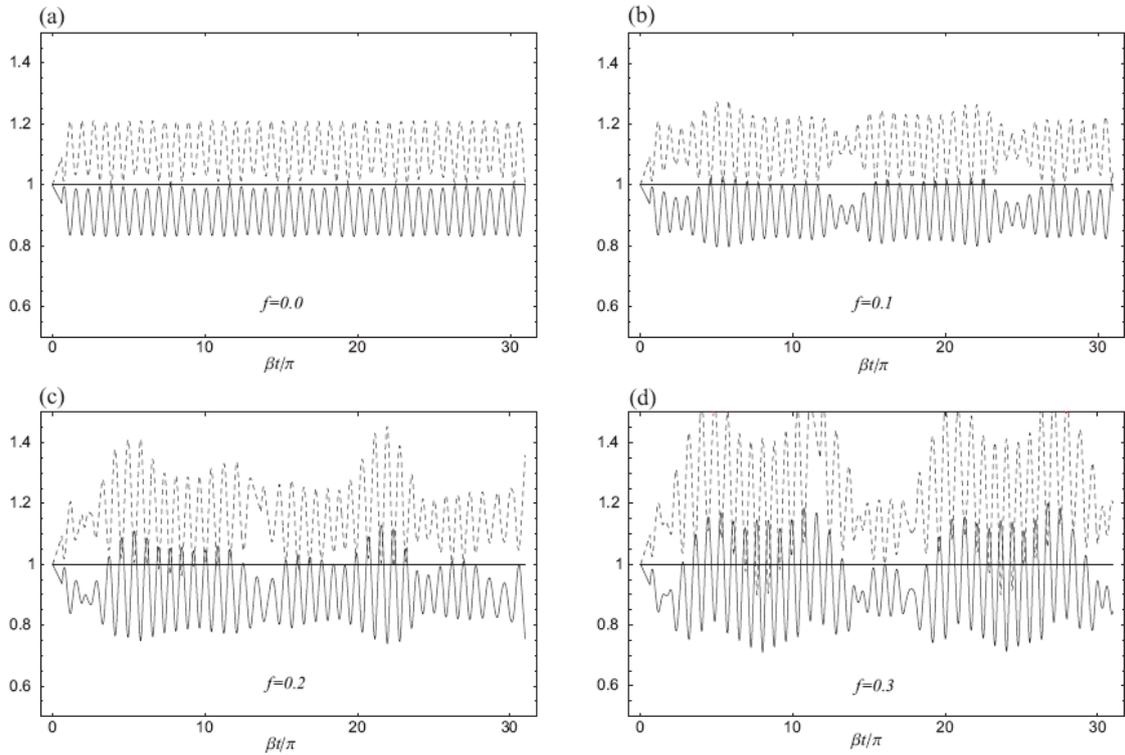


FIG. 6: The effect of the driving term on squeezing. (a) shows the squeezing without driving and (b), (c), (d) show the effect for the dimensionless driving parameter (f) values of 0.1, 0.2 and 0.3 respectively. The nonlinearity parameter β is equal to 0.1. Solid line is for $S_x(t)$ and the dashed line is for $S_p(t)$.

Next we analyze the effect of driving in the dynamics of the Hamiltonian given in Eq. (8). We calculate the propagator again by using the split-operator method. This time, we include the nonlinear term into H_0 and we take the time dependent driving term as $V(t)$ to employ the splitting given in Eq. (18). Fig. 6 shows the time evolution of the normally ordered variances for the in-phase and the out-of-phase quadratures for driving parameter values of $f = 0.0, 0.1, 0.2$ and 0.3 at the nonlinearity $\beta = 0.1$. We take the vacuum as the initial state. The frequency of the driving term, ω , is on resonance with

the frequency of the oscillator, ω_0 . As clearly seen, the effect of the driving is to enhance the squeezing in x periodically to a larger value. For the values of f in consideration, squeezing went from -0.30 (for $f = 0.0$) to -0.55 (for $f = 0.3$). Increasing the driving further does not enhance squeezing, rather it starts to enhance the fluctuations and the squeezing cycle becomes shorter. And the driving frequencies off the resonance do not help enhancing the squeezing at all. The upper limit \bar{f}_{\max} for the regime of validity of the fundamental mode description is given by the 1st harmonic threshold, that is $\bar{f}_{\max} < \hbar\omega_2 \approx 3\hbar\omega_1$ [13] which means for the dimensionless parameter, $f_{\max} < 3$.

4. CONCLUSION

In this paper we show that squeezed states can be obtained in the amplitude of the fundamental mode of a nanomechanical oscillator which has quartic nonlinearity in its effective single particle quantum mechanical Hamiltonian. The quantum dynamics of the system is solved numerically both with and without external acdriving. For various strength of nonlinearity and driving, the squeezing dynamics is investigated for both initial vacuum and coherent states of different amplitudes. The terms that lead to multilevel transitions in the quartic nonlinearity is compared to similar models and proved advantageous results. It should be noted that working closer to Euler buckling instability produces larger squeezing due to large nonlinearity. This is reminiscent of similar result for quantum optical systems [32].

Finally, we comment on the possibility of observing squeezing in nano-mechanical beams. Remarkably, quantum squeezing may be controlled externally in these systems just by controlling strain provided by a classical compressive force. However, one has to cool down the resonator to mili-Kelvin temperatures. Moreover, controlling the nonlinearity at the desired values, one would have to apply compression with extreme delicacy, $F_c - F_0 \sim 10^{-6} F_c$. Controlling the strain to this precision for sufficient time to identify squeezing may be difficult. Thus, while observing squeezing will be challenging, the prospect of exploring tunable quantum squeezing in nano-mechanical beams and the connection to Euler buckling instability, are intriguing.

5. REFERENCES

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