



## SECURE COMMUNICATION USING PRACTICAL SYNCHRONIZATION BETWEEN TWO DIFFERENT CHAOTIC SYSTEMS WITH UNCERTAINTIES

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**Abstract-** Adaptive control technique is used to design suitable controllers to synchronize two different coupled chaotic systems with uncertainties. The information signal hidden in chaotic signal is transmitted and then successfully recovered at the receiver once synchronization is achieved. Different chaotic systems with uncertainties are chosen respectively as the transmitter and receiver systems to ensure higher security in communication. Numerical simulation examples verify the effectiveness of the proposed method.

**Keywords-** chaos synchronization, secure communication, adaptive control, different chaotic systems

### 1. INTRODUCTION

Chaos synchronization has been widely investigated for its advantages in practical applications, particularly in secure communication [1-3]. The process of secure communication via chaos synchronization is outlined as follows. Firstly, the information signal is added to a much stronger chaotic signal in order to hide the information in transmitter. Secondly, the generated chaotic signal containing the information is transmitted to receiver. Finally, the information signal can be recovered successfully if synchronization is achieved between the transmitter and receiver under certain conditions.

Chaotic systems can be used in signal transmission in various ways, for examples, driving-response scheme [4], adaptive synchronization strategy [5], parameter modulation [6], observer-based synchronization [7]. Adaptive control is an effective method to synchronize mostly systems with unknown parameters or disturbances, which can improve the security of information and be more suitable for communication [8-10]. However, most of the mentioned literatures were designed to synchronize two identical chaotic systems and recover single signal hidden in the transmitter. Studies on secure communication with several different signals and different chaotic structures between the transmitter and receiver are limited. Adaptive synchronization between different chaotic systems with unknown parameters was investigated [11,12]. However, the effects of noise and parameter mismatch were not taken into account, and the controller contains all the information appeared in the error dynamical system or the transmitter-receiver systems, which is hard to realize in practical application.

In this paper adaptive control technique is used to design suitable controllers to synchronize different coupled chaotic systems with uncertainties, including unknown parameters, internal or external perturbations. Such transmitter-receiver systems have higher security. Because of the existence of uncertainties, the transmitter-receiver systems are usually difficult to achieve complete synchronization. So a concept of practical synchronization is introduced in this paper. Under some conditions, Lyapunov stability theory ensures that the transmitter-receiver chaotic systems can achieve practical synchronization, and the information signal hidden in the transmitted chaotic signal can be successfully recovered at the receiver. The designed controllers contain only feedback terms and partial nonlinear terms of the systems, and they are easy to implement in practice. The Lorenz system and Chen system are chosen as the illustrative example to verify the validity of the proposed method.

## 2. SECURE COMMUNICATION SCHEME

Consider a class of chaotic systems with unknown parameters and disturbance as the transmitter

$$\begin{cases} \dot{x} = f(x) + F(x)\alpha + d(x, t) + s \\ x_t = Cx + s \end{cases} \quad (1)$$

where  $x \in R^n$  is the state vector,  $\alpha \in R^m$  is the unknown parameter vector,  $f(x) \in R^{n \times 1}$  and  $F(x) \in R^{n \times m}$  are known function matrices,  $d(x, t) \in R^{n \times 1}$  is the uncertainties including parameter perturbation and external disturbance,  $s \in R^n$  is the information signal vector which will be recovered at the receiver,  $C \in R^{n \times n}$  is a known constant matrix and  $x_t \in R^n$  is the output vector.

Let  $\Omega_x \subset R^n$  be a bounded region containing the whole attractor of transmitter system (1) such that no trajectory of system (1) ever leaves it. This assumption is simply based on the bounded property of chaotic attractor. Also, let  $M \subset R^m$  be the set of parameter under which system (1) is in a chaotic state.

The receiver system with a controller is constructed as follows

$$\begin{cases} \dot{y} = g(y) + G(y)\beta + u(t) \\ y_r = Cy \end{cases} \quad (2)$$

where  $y \in R^n$ ,  $\beta \in R^{m_1}$  is the unknown parameter vector,  $g(y) \in R^{n \times 1}$  and  $G(y) \in R^{n \times m_1}$  are known function matrices,  $u(t) \in R^n$  is the control input vector,  $y_r \in R^n$  is the output vector. Let  $\Omega_y \subset R^n$  be a bounded region containing the whole attractor of receiver system (2) with  $u(t) = 0$ .

Due to the uncertainties in the transmitter system (1), the receiver system (2) is usually difficult to achieve complete synchronization with the transmitter system (1). Therefore a concept of practical synchronization, also called synchronization with uniform ultimate boundedness in other literatures, is introduced in this paper.

**Definition 1** The transmitter-receiver systems (1) and (2) achieve practical synchronization if for any initial state  $x(0) \in \Omega_x$  and  $y(0) \in \Omega_y$ , there exist constants

$h > 0$  and  $T_0 > 0$  such that the trajectory  $x(t, x(0))$  of system (1) and trajectory  $y(t, y(0))$  of system (2) satisfy

$$\|x(t, x(0)) - y(t, y(0))\| < h, \text{ for any } t > T_0, \quad (3)$$

where  $\|\cdot\|$  denotes the Euclidean norm, and  $h$  indicates the error bound.

According to this definition, our control objective is to design a suitable adaptive controller  $u(t)$  such that the receiver system (2) eventually synchronizes with the transmitter system (1), and finally reconstruct the informal signal. To this end, some assumptions are given as follows.

**Assumption 1** The disturbance vector  $d(x, t)$  and information signal  $s$  are norm bounded respectively, namely,  $\|d(x, t)\| \leq L_d$ ,  $\|s(t)\| \leq L_s$ , where  $L_d$  and  $L_s$  are positive constants.

**Assumption 2** The function vectors  $f(\cdot)$  and  $g(\cdot)$  are continuous on a bounded closed region  $\Omega$  containing both  $\Omega_x$  and  $\Omega_y$ . So there exists a positive constant  $L_f$  such that

$$\|f(x) - g(x)\| \leq L_f, \quad x \in \Omega.$$

**Assumption 3** The function vector  $g(\cdot)$  satisfies the Lipschitz condition, that is, there exists a positive constant  $L_g$  such that

$$\|g(x) - g(y)\| \leq L_g \|x - y\|, \text{ for any } x, y \in R^n.$$

**Theorem 1** The transmitter-receiver systems (1) and (2) can achieve practical synchronization and the informal signal  $s$  can be recovered by

$$\tilde{s} = x_r - y_r,$$

if the Assumptions 1-3 hold, and the controller  $u(t)$  is designed as

$$u(t) = F(x)\hat{\alpha} - G(y)\hat{\beta} + ke, \quad (4)$$

where  $e = x - y$  is the error variable, the feedback coefficient  $k$  is a constant to be determined, and the adaptive variables  $\hat{\alpha}$  and  $\hat{\beta}$  satisfy the following adaptation laws

$$\dot{\hat{\alpha}} = F(x)^T e, \quad \dot{\hat{\beta}} = -G(y)^T e. \quad (5)$$

**Proof** The error dynamical system is

$$\dot{e} = \dot{x} - \dot{y} = f(x) - g(y) + F(x)(\alpha - \hat{\alpha}) - G(y)(\beta - \hat{\beta}) - ke + d(x, t) + s. \quad (6)$$

Construct a Lyapunov function

$$V = \frac{1}{2}[e^T e + (\alpha - \hat{\alpha})^T (\alpha - \hat{\alpha}) + (\beta - \hat{\beta})^T (\beta - \hat{\beta})].$$

Using Assumptions 1-3 and Eqs.(4)-(6), the time derivative of  $V$  satisfies,

$$\begin{aligned} \dot{V} &= e^T \dot{e} - (\alpha - \hat{\alpha})^T \dot{\hat{\alpha}} - (\beta - \hat{\beta})^T \dot{\hat{\beta}} \\ &= e^T (f(x) - g(y) + F(x)(\alpha - \hat{\alpha}) - G(y)(\beta - \hat{\beta}) - ke + d(x, t) + s) \\ &\quad - (\alpha - \hat{\alpha})^T F(x)^T e - (\beta - \hat{\beta})^T (-G(y)^T e) \\ &= e^T (f(x) - g(x)) + e^T (g(x) - g(y)) - k \|e\|^2 + e^T (d(x, t) + s) \end{aligned}$$

$$\begin{aligned} &\leq L_f \|e\| + L_g \|e\|^2 - k \|e\|^2 + (L_d + L_s) \|e\| \\ &\leq \frac{\|e\|^2}{2\varepsilon_1} + \frac{\varepsilon_1 L_f^2}{2} + L_g \|e\|^2 - k \|e\|^2 + \frac{\|e\|^2}{2\varepsilon_2} + \frac{\varepsilon_2 (L_d + L_s)^2}{2} \\ &= \left(\frac{1}{2\varepsilon_1} + \frac{1}{2\varepsilon_2} + L_g - k\right) \|e\|^2 + \frac{1}{2} (\varepsilon_1 L_f^2 + \varepsilon_2 (L_d + L_s)^2) \end{aligned}$$

where  $\varepsilon_1, \varepsilon_2$  are small positive constants, and the equality  $ab \leq \frac{a^2}{2c} + \frac{cb^2}{2}$  has been used, in which  $c$  is a positive constant. Let

$$\varepsilon = \frac{1}{2} (\varepsilon_1 L_f^2 + \varepsilon_2 (L_d + L_s)^2). \tag{7}$$

If the feedback coefficient  $k$  satisfies

$$k \geq \frac{1}{2\varepsilon_1} + \frac{1}{2\varepsilon_2} + L_g + 1, \tag{8}$$

then we have

$$\dot{V} \leq -\|e\|^2 + \varepsilon. \tag{9}$$

By the inequality (9), we conclude that the trajectory of state error will approach to a hyper-ball determined by  $\|e\| \leq \sqrt{\varepsilon}$ , so practical synchronization can be achieved. From Eqs.(7)-(9), it is clear that  $\varepsilon$  will be sufficiently small if  $\varepsilon_1$  and  $\varepsilon_2$  are chosen sufficiently small, which implies the synchronization error will also be sufficiently small. This can be realized simply by choosing larger value of  $k$ . Then the informal signal  $s$  can be reconstructed approximately by

$$\tilde{s} = x_t - y_r = Cx + s - Cy = s + Ce.$$

**Remark** The controller (4) contains only the feedback term and partial nonlinear terms of the systems, while the controllers in Refs.[12-17] include all the information appeared in the error dynamical system or the transmitter-receiver systems.

### 3. ILLUSTRATIVE EXAMPLE

Secure communication between Lorenz system and Chen system is presented to simulate the proposed method. The Lorenz system with unknown parameters and perturbations is chosen as the transmitter system

$$\begin{cases} \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} 0 \\ -x_2 - x_1 x_3 \\ x_1 x_2 \end{pmatrix} + \begin{pmatrix} x_2 - x_1 & 0 & 0 \\ 0 & x_1 & 0 \\ 0 & 0 & -x_3 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} + \begin{pmatrix} d_1(x,t) \\ d_2(x,t) \\ d_3(x,t) \end{pmatrix} + \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} \\ \begin{pmatrix} x_{1t} \\ x_{2t} \\ x_{3t} \end{pmatrix} = C \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} \end{cases} \tag{10}$$

Compared with Eq.(1), the relative notations are

$$x = (x_1, x_2, x_3)^T, \quad f(x) = (0, -x_2 - x_1 x_3, x_1 x_2)^T, \quad F(x) = \text{diag}(x_2 - x_1, x_1, -x_3),$$

$$\alpha = (\alpha_1, \alpha_2, \alpha_3)^T, \quad d(x, t) = (d_1(x, t), d_2(x, t), d_3(x, t))^T, \quad s = (s_1, s_2, s_3)^T.$$

The Chen system with unknown parameters and controllers is selected as the receiver system

$$\begin{cases} \begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \end{pmatrix} = \begin{pmatrix} 0 \\ -y_1 y_3 \\ y_1 y_2 \end{pmatrix} + \begin{pmatrix} y_2 - y_1 & 0 & 0 \\ -y_1 & y_1 + y_2 & 0 \\ 0 & 0 & -y_3 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} + \begin{pmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{pmatrix} \\ \begin{pmatrix} y_{1r} \\ y_{2r} \\ y_{3r} \end{pmatrix} = C \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \end{cases} \quad (11)$$

Compared with Eq.(2), the relative notations are

$$y = (y_1, y_2, y_3)^T, \quad g(y) = (0, -y_1 y_3, y_1 y_2)^T, \quad G(y) = \begin{pmatrix} y_2 - y_1 & 0 & 0 \\ -y_1 & y_1 + y_2 & 0 \\ 0 & 0 & -y_3 \end{pmatrix},$$

$$\beta = (\beta_1, \beta_2, \beta_3)^T. \quad u(t) = (u_1(t), u_2(t), u_3(t))^T \text{ is determined by Eqs.(4) and (5).}$$

It is easy to verify that the Lorenz system and Chen system satisfy Assumptions 1-3.

**Case 1** No disturbance is applied to the transmitter system, that is,  $d(x, t) = 0$ . Suppose that the transmitted signal is  $(s_1, s_2, s_3)^T = (0.2 \sin 2t, 0.1 \text{sign}(\sin(\pi t / 4)), 0)^T$ , and

$$C = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{pmatrix}.$$

The Lorenz system with signal loading is in a state of chaos, whose attractor is displayed in Fig.1. Under the designed controller, the receiver system (11) can achieve practical synchronization with the transmitter system (10) as shown in Fig.2. It is clearly seen in Figs. 3 and 4 that the reconstructed signal  $\tilde{s}$  coincides with the informal signal  $s$  with good accuracy. The error between the transmitted and recovered signals can be reduced to desired accuracy by simply increasing the feedback coefficient  $k$ . In the simulation, the unknown parameters in systems (10) and (11) are assumed to be "known" as  $\alpha = (10, 28, 8/3)^T$  and  $\beta = (35, 28, 3)^T$ . The feedback coefficient  $k = 50$ , and the initial values are chosen arbitrarily as  $x(0) = (1.4, -1, 2.4)^T$ ,  $y(0) = (-0.5, 1.3, 0.8)^T$ ,  $\hat{\alpha}(0) = (1, 4, -1)^T$ ,  $\hat{\beta}(0) = (-2, 2.1, 1.4)^T$ .

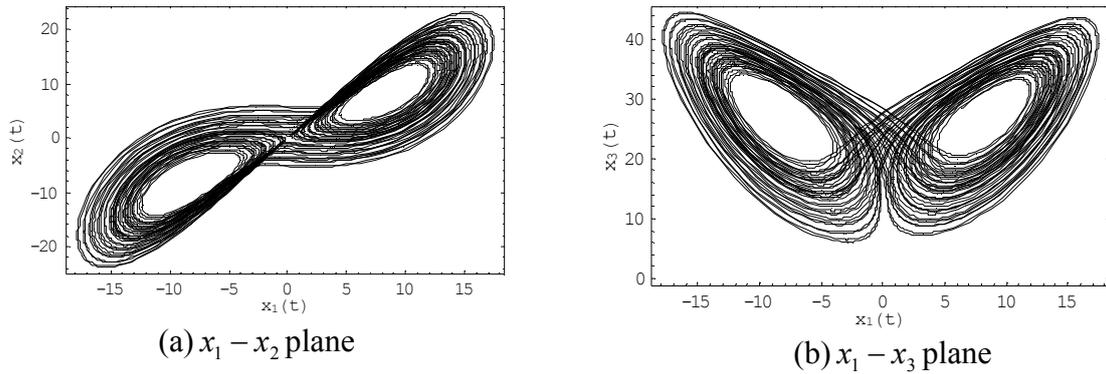


Fig.1 Chaotic attractor of the Lorenz system with information signal loading

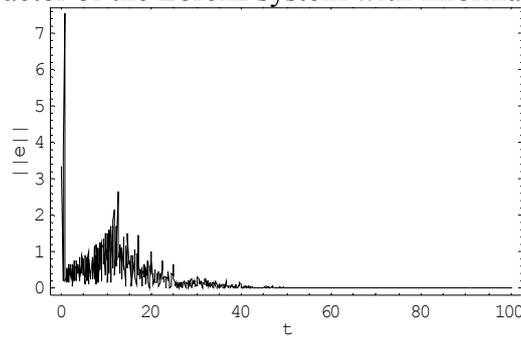


Fig.2 Synchronization error between systems (10) and (11)

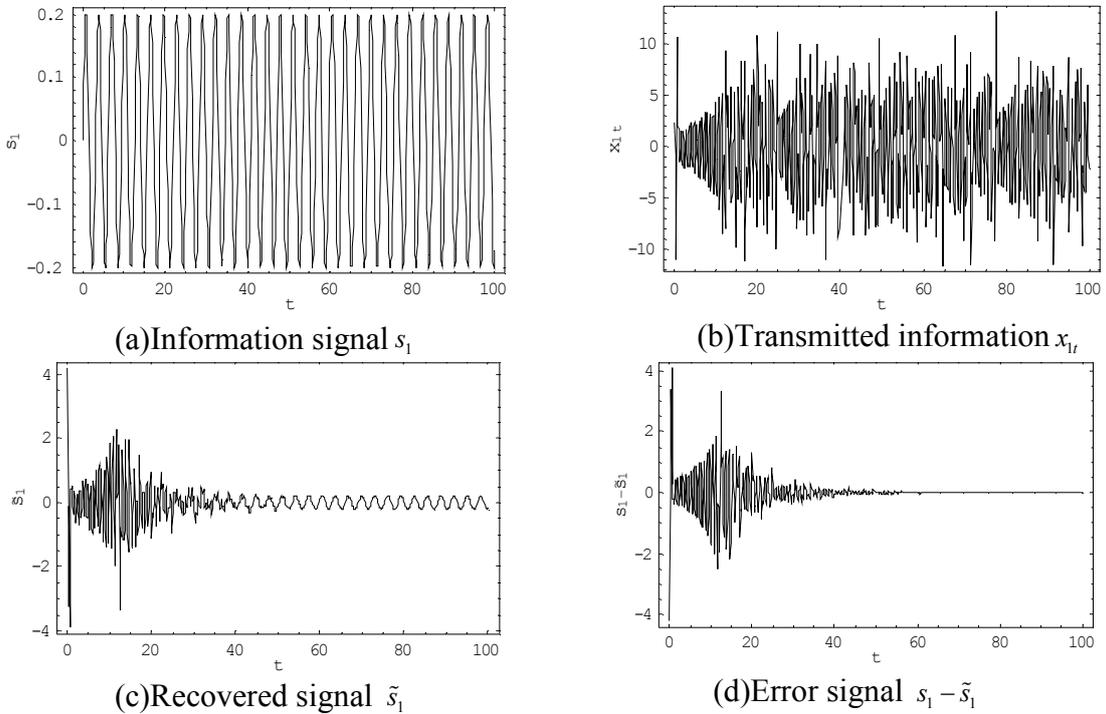


Fig.3 Process of signal  $s_1$  transmission and recovery without disturbance

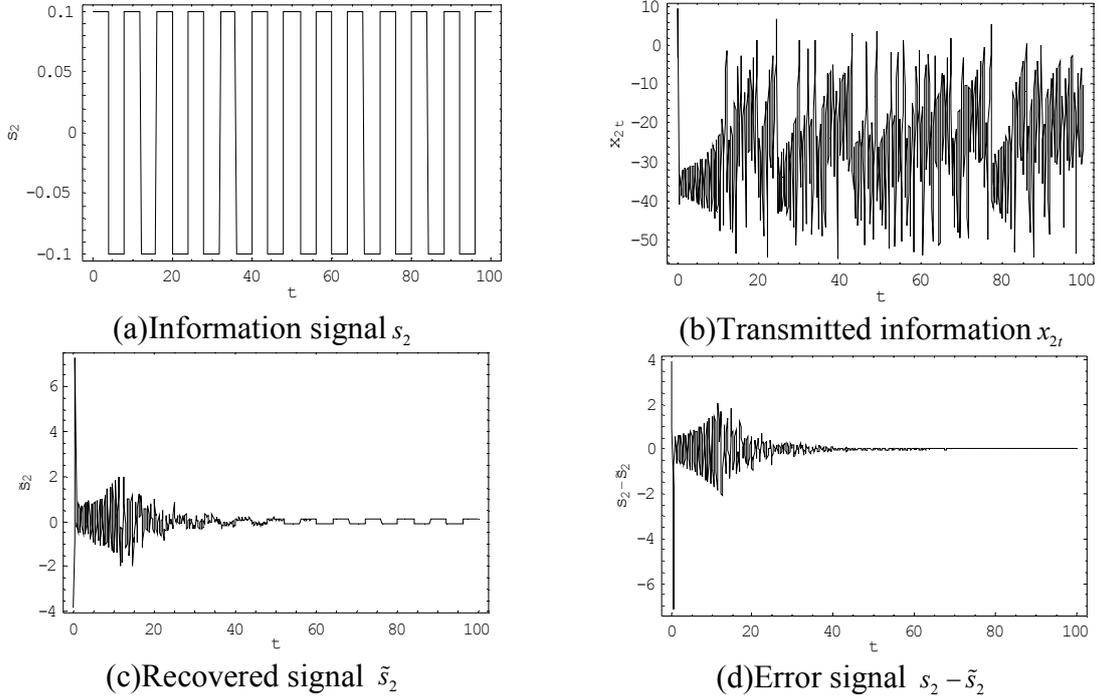


Fig.4 Process of signal  $s_2$  transmission and recovery without disturbance

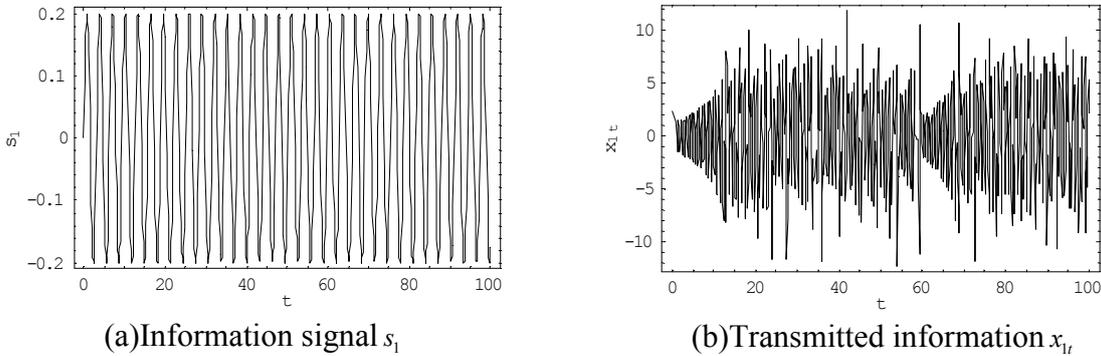
**Case 2** Parameter perturbation is applied to the transmitter system (10), for example,

$$d(x, t) = (\lambda \sin(2t) \cdot (10x_1), \lambda \cos t \cdot x_2, \lambda \sin(3t) \cdot (\frac{8}{3}x_3))^T.$$

Now the transmitter system (10) becomes,

$$\begin{aligned} \dot{x}_1 &= 10x_2 - 10(1 - \lambda \sin 2t)x_1 + 0.2 \sin 2t, \\ \dot{x}_2 &= 28x_1 - (1 - \lambda \cos t)x_2 - x_1x_3 + 0.1 \text{sign}(\sin(\pi t / 4)), \\ \dot{x}_3 &= x_1x_2 - \frac{8}{3}(1 - \lambda \sin 3t)x_3, \end{aligned}$$

where  $\lambda$  represents the perturbed strength compared with the magnitude of state variables  $x_i (i=1,2,3)$ . The informal signals also can be well recovered. The results are shown in Figs. 5 and 6 with the perturbed strength  $\lambda=1\%$ . The other values of parameters are the same as those of Case 1. The above simulations are performed in computer algebraic system *Mathematica*, where instruction *NDSolve* is used.



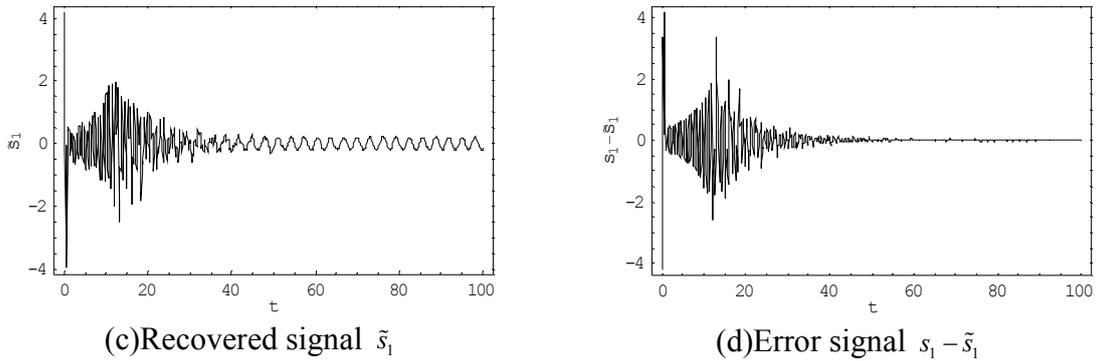


Fig.5 Process of signal  $s_1$  transmission and recovery under disturbances

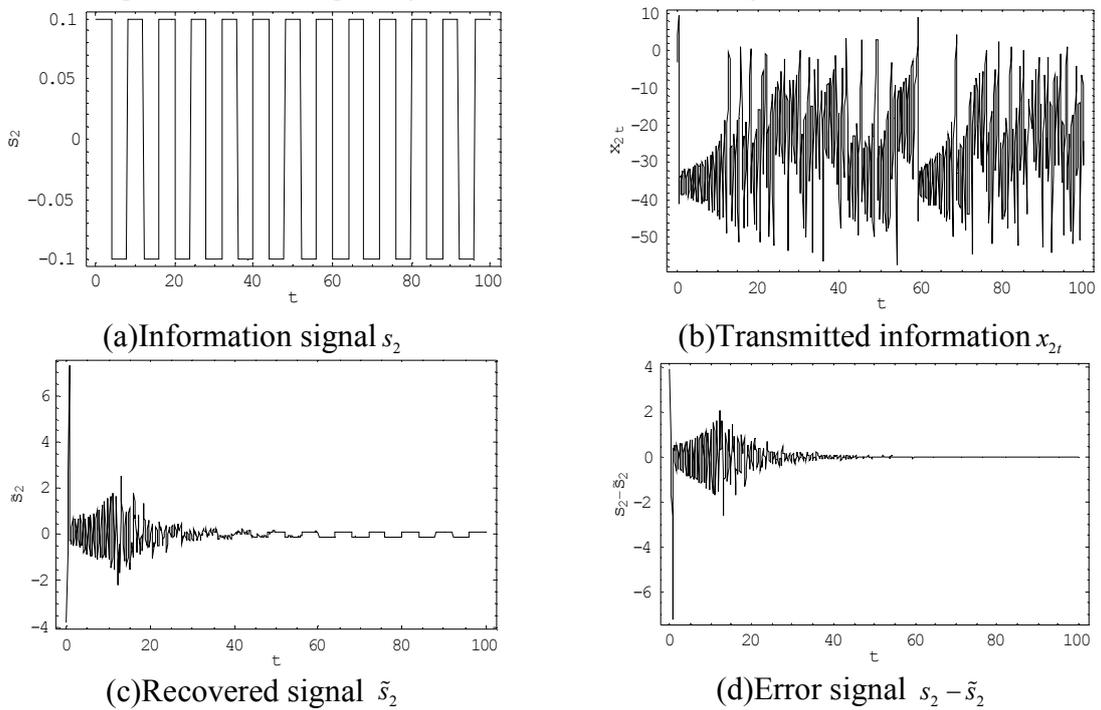
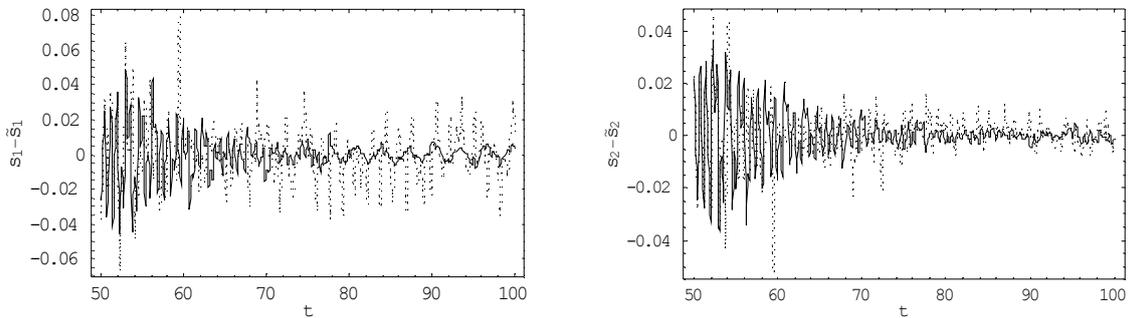


Fig.6 Process of signal  $s_2$  transmission and recovery under disturbances

Now we discuss the effect of disturbance on the error between the transmitted signal and recovered one. The enlarged figure of Fig.3(d) combined with Fig.5(d) at  $t \in [50, 100]$  is shown in Fig.7(a), and that of Fig.4(d) combined with Fig.6(d) is shown in Fig.7(b). We see clearly that the disturbances result in larger error between the transmitted and recovered signals.



(a) Block curve for  $s_1 - \tilde{s}_1$  without disturbance and dashed curve for  $s_1 - \tilde{s}_1$  under disturbances  
 (b) Block curve for  $s_2 - \tilde{s}_2$  without disturbance and dashed curve for  $s_2 - \tilde{s}_2$  under disturbances

Fig.7 Effect of disturbance on the error between the transmitted and recovered signals

#### 4. CONCLUSIONS

In this paper we design adaptive controllers to synchronize two different chaotic systems with uncertainties, containing unknown parameters, internal or external perturbations. Using Lyapunov stability theory, we prove that under some conditions the transmitter-receiver systems can achieve practical synchronization, and the information signal hidden in the transmitted chaotic signal can be successfully recovered at the receiver. Furthermore the error between the transmitted and recovered signals can be reduced to desired accuracy. The designed controllers contain only feedback terms and partial nonlinear terms of the systems, and they are easy to implement in practice. The Lorenz system and Chen system are chosen as the illustrative example to verify the validity of the proposed method.

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