JOURNAL OF ADVANCED TRANSPORTATION
J. Adv. Transp. 2013; **47**:105–125
Published online 28 May 2012 in Wiley Online Library (wileyonlinelibrary.com). **DOI**: 10.1002/atr.1192

Discretised route travel time models based on cumulative flows

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SUMMARY

The computation of route travel times is a fundamental work in solving dynamic traffic assignment problems, and hence, it is important to develop accurate and efficient methods. In this paper, both the step function (SF) and linear interpolation (LI) are used to approximate cumulative flows over time and develop formulations of route travel time models. We propose two categories of discretised route travel time models: (i) models based on route cumulative flow curves, and (ii) models based on link cumulative flow curves. We prove that all route travel time models proposed in this paper satisfy some desirable properties such as route first-in-first-out and continuity, and all discretised route travel times converge to the continuous-time route travel times. Finally, numerical methods are developed to evaluate the accuracy and computational efficiency of each type of route travel time models. Copyright © 2012 John Wiley & Sons, Ltd.

KEY WORDS: route travel time; cumulative flow; piecewise linearization; calculation error; computational efficiency

1. INTRODUCTION

Traditionally, analytical dynamic traffic assignment (DTA) models can be formulated either in discrete time space or continuous time space. But most of them are finally solved by numerical methods that involve discretising time. The reason for discretising time is that there have not been satisfying ready-made methods for solving the complex continuous time network models analytically. The DTA problems can be analytically formulated as path-based models [1–5] or link-based models [6–11]. Computation of route travel times is required for most of DTA problems. In general, the path-based models should calculate the travel times of all routes in the path set, and the link-based models should calculate the shortest route travel times. Although some DTA models [9–11] do not need to explicitly calculate route travel times or the shortest route travel times in the solution procedure, the calculation of route travel times is still very useful to evaluate the obtained DTA solutions. Therefore, developing discretised route travel time models is a basic and important work on the path of developing DTA models.

As one of the fundamental DTA issues, dynamic network loading (DNL) model depicts how traffic propagates inside a traffic network along assigned routes and hence governs the network performance in terms of travel times. In general, the DNL model does not directly output travel times but link cumulative flows and/or route cumulative flows. We can retrieve the travel times either from the link cumulative curves or route cumulative curves if they have been obtained. As a result, there are two methods to obtain route travel times: one is to directly formulate the route travel times by route cumulative flow curves [2,12,13], the other is to derive route travel times by link travel times, which can be computed by link cumulative flow curves [3,4,14,15]. It is important to develop accurate and efficient method to derive travel times from cumulative flow curves, because the calculation of route travel times is a fundamental step in the algorithms of many DTA models.

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Many DNL models can be used to simulate the propagation of traffic through the network along the assigned routes and generate the cumulative flows. Peeta and Ziliaskopoulos [16] distinguished three main existing approaches to capture flow propagation on the links and nodes: exit functions [3,17–20], link performance functions [7,8,21–23] and the cell transmission models (CTM) [2,12–14,24–27]. The first two approaches are usually governed by analytical link-model-based loading procedures and output link cumulative flows or link travel times directly, whereas the third approach is governed by simulation-based loading procedures and output link cumulative flows and/or route cumulative flows. The third approach is also referred to advanced exit flow functions, which are developed on the basis of either Daganzo's [28,29] solution scheme (i.e., CTM) or Newell's [30] solution scheme to the Lighthill and Whitham [31] and Richards [32] (LWR) hydrodynamic model of traffic flow. The analytical link-model-based loading procedures are comparatively simple to implement and efficient to compute even for large road networks, but are often believed to be less accurate than simulation-based loading procedures in capturing complex traffic phenomena [33].

In the literature, route travel times based on cumulative flows are calculated according to the proposed methods, and there are no formulations of travel times provided for analysis of the properties of route travel time. Recently, Long *et al.* [15] formulated three types of discretised link travel time models by using the link cumulative flow curves, which are approximated by both the step functions (SF) and linear interpolations (LI). Following Long *et al.* [15], this paper uses the same method to approximate the route cumulative flows over time and develops formulations of the SF-type and the LI-type route travel time models. Because the travel time of a specified route consists of the travel times of links on the route, which is referred to as the additivity property of route travel times, a nested function can be used to formulate route travel times by link travel times [3]. Embedding the disretised link travel times into the nested function, we can formulate alternative route travel time models.

The proposed formulations for route travel times can simplify the calculation and allow us to analyze the properties of the corresponding travel time functions. The properties of each type of route travel time functions concerned in this paper include route first-in-first-out (FIFO), continuity, and monotonicity. FIFO is an actual traffic behavior. A dynamic route travel time model is necessary to satisfy FIFO in order to obtain the solutions of DTA that are consistent with actual traffic behavior. Continuity and monotonicity of travel times are two important properties of DTA. Route travel times must be continuous with respect to route flows for solution existence, and must be strictly monotone with respect to route flows for solution uniqueness. Moreover, we use numerical methods to investigate the accuracy and computational efficiency of all the route travel time models on the basis of cumulative flows.

This paper is an extension of Long *et al.* [15]. The main differences between the current work and the previous work are as follows: (i) We propose two categories of discretised route travel time models, that is, the models based on route cumulative flow curves and the models based on link cumulative flow curves. (ii) The properties of the first category of models directly follow the previous work, but the properties of the second category of models are newly proved in details. (iii) The discretised route travel times are proved to converge to the continuous-time route travel times. (iv) Besides the model accuracy, the computational efficiency of each type of route travel time models is also evaluated.

In the next section, the general concept of route travel times based on cumulative flows is presented in both continuous and discretised time settings. Section 3 formulates route travel time models based on both route cumulative flows and link cumulative flows, and discusses their properties. In Section 4, calculation error estimation methods are proposed to evaluate the accuracy of each travel time model. Numerical experiments are given in Section 4 to illustrate the accuracy and computational efficiency of all route travel time models and how the results are affected by the fineness of the discretisation. Finally, Section 6 concludes the paper.

2. CUMULATIVE FLOWS AND ROUTE TRAVEL TIME

2.1. Continuous route travel time model

Route travel times can be directly calculated by using the route cumulative flow curves. Let $M_p(t)$ ($N_p(t)$) denote the cumulative departure (arrival) flow along route p by time t, and $\tau_p(t)$ the travel time of route p with respect to departure time t. As shown in Figure 1, route travel times can be used to connect the

DOI: 10.1002/atr

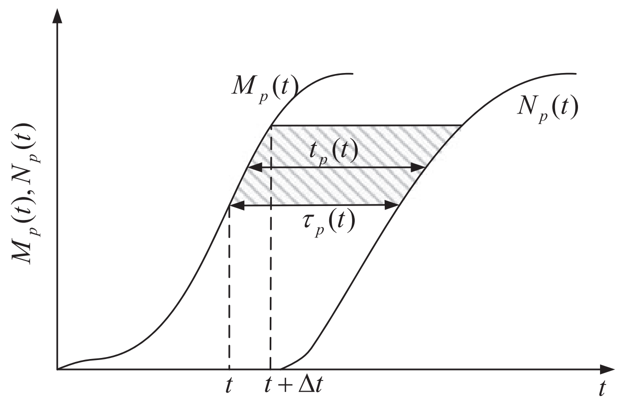

Figure 1. Cumulative vehicle numbers as a function of time.

cumulative departure and arrival flows along each route by the relationship $M_p(t) = N_p(t + \tau_p(t))$. If $M_p(t)$ and $N_p(t)$ are strictly increasing with respect to time t, the route travel times can be mathematically expressed as follows:

$$\tau_p(t) = N_p^{-1}\left(M_p(t)\right) - t \tag{1}$$

where $N_p^{-1}(\cdot)$ is the inversed function of $N_p(\cdot)$.

The additivity property of route travel times can be used to derive the route travel times from link travel times. Let $\tau_a(t)$ denote the travel time of link a with respect to time t. We consider vehicles departing at time t and traveling through path $p = \{a_1, a_2, \cdots, a_m\}$. The travel time for their traveling through link a_1 is $\tau_{a_1}(t)$, and the time instant for their leaving link a_1 and entering link a_2 is $t + \tau_{a_1}(t)$. Then, the travel time for their traveling through link a_2 is $\tau_{a_2}(t + \tau_{a_1}(t))$, and the time instant for their leaving link a_2 and entering link a_3 is $t + \tau_{a_1}(t) + \tau_{a_2}(t + \tau_{a_1}(t))$. Similarly, we can obtain their travel times through other links. The travel time required to traverse path p for vehicles using this path at time t is equal to the sum of travel times of traveling through each link on the path and can be computed by the following nested function [3]:

$$\tau_p(t) = \tau_{a_1}(t) + \tau_{a_2}(t + \tau_{a_1}(t)) + \cdots + \tau_{a_m}(t + \tau_{a_1} + \cdots + \tau_{a_{m-1}}) \tag{2}$$

where $\tau_{a_1} = \tau_{a_1}(t), \tau_{a_2} = \tau_{a_2}(t + \tau_{a_1}(t)), \cdots$, for short. The link travel times can be calculated by using the link cumulative inflows and outflows. Let $U_a(t)$ ($V_a(t)$) be the cumulative number of vehicles that enter (exit) link a by time t. We have the relationship $U_a(t) = V_a(t + \tau_a(t))$. If $U_a(t)$ and $V_a(t)$ are strictly increasing with respect to time t, the link travel times can be mathematically expressed as follows:

$$\tau_a(t) = V_a^{-1}(U_a(t)) - t \tag{3}$$

where $V_a^{-1}(\cdot)$ is the inversed function of $V_a(\cdot)$.

However, the link/route cumulative flow curves cannot always be strictly increasing. Nie [34] developed a generic approach to retrieve link travel times from the DNL results, in which the curves are only required to be non-decreasing, and the method of linear interpolation is used to deal with the special case that the curves have flat portions for some time. Long *et al.* [15] conducted pretreatments of link inflow rates and outflow capacities for the DNL to ensure that the link cumulative flow curves are strictly increasing. This method can be extended to the calculation of route travel times: setting the route flow rate and link outflow rate as $\max\{f_p(t), \varepsilon\}$ and $\max\{q_a(t), \zeta\}$ during the DNL implementation, respectively, where ε and ζ are two very small positive numbers that satisfy $\varepsilon \to 0^+$ and $\zeta \ll \varepsilon$. These pretreatments can ensure $f_p(t) > 0$ and $q_a(t) > 0$. Therefore, both cumulative route flows and cumulative link flows will be strictly increasing with respect to time t.

The route-based DTA models output route flows, which can be used as input of DNL models and generate both the cumulative route flows and cumulative link flows whereas the link-based DTA models output link flows, which can also be used as input of link-based DNL models and generate

DOI: 10.1002/atr

the cumulative link flows. Therefore, throughout this paper, we assume that the cumulative route departure flows and arrival flows are given for the route travel time models on the basis of route cumulative flow curves, and the cumulative inflows and outflows are given for the route travel time models on the basis of link cumulative flow curves. In the DTA models, travelers are assumed to make their dynamic route choice according to either the instantaneous travel time or experienced travel time. The route travel times calculated by both Equations (1) and (2) are the experienced travel times. However, only Equation (2) can be modified to formulate the instantaneous travel times, by replacing the experienced link travel times as the instantaneous link travel times in Equation (2). In this paper, we only consider to model the experienced route travel time.

2.2. *Discretised route travel time model*

In the discrete time DTA models, time is usually discretised into small time intervals. We use Δt to denote the length of each interval. The average travel time of vehicles that depart the origin of route p during interval $(t, t+\Delta t]$ can be calculated by

$$t_p(t) = \frac{\int_t^{t+\Delta t} \tau_p(v) f_p(v)\mathrm{d}v}{\int_t^{t+\Delta t} f_p(v)\mathrm{d}v} \tag{4}$$

where $f_p(v)$ is the flow rate of route p at time v, the denominator and numerator on the right hand side of Equation (4) are the number of departure vehicles along route p during interval $(t, t+\Delta t]$ and total travel time of those vehicles (i.e., the shade area in Figure 1), respectively. It follows immediately that $t_p(t) = \tau_p(t)$ if $\Delta t = 0$.

Both Equations (1) and (2) can be substituted into Equation (4) to calculate the discretised route travel time exactly. However, the continuous route travel times cannot be generally expressed as explicit functions of time and the computational burden is too high to compute $\tau_p(t)$ for all time instant t.

With the use the route cumulative flow curves, Equation (4) can be equivalently reformulated as follows:

$$t_p(t) = \frac{\int_{M_p(t)}^{M_p(t+\Delta t)} \left(N_p^{-1}(v) - M_p^{-1}(v)\right)\mathrm{d}v}{M_p(t+\Delta t) - M_p(t)} \tag{5}$$

where, $M_p^{-1}(\cdot)$ is the inversed function of $M_p(\cdot)$, the numerator on the right hand side of Equation (5) equals the shade area in Figure 1.

Similar with the calculation of the continuous route travel times, the additivity property of route travel times enables the discretised link travel times to be used to estimate the discretised route travel times. For a given link a, the discretised travel time $t_a(t)$ is also defined as the average travel time of vehicles that enter link a during interval $(t, t+\Delta t]$. The discretised link travel times based on link cumulative flows can be formulated as follows:

$$t_a(t) = \frac{\int_{U_a(t)}^{U_a(t+\Delta t)} \left(V_a^{-1}(v) - U_a^{-1}(v)\right)\mathrm{d}v}{U_a(t+\Delta t) - U_a(t)} \tag{6}$$

where $U_a^{-1}(\cdot)$ is the inversed function of $U_a(\cdot)$. The discretised route travel times can be estimated as follows:

$$t_p(t) = t_{a_1}(t) + t_{a_2}(t + t_{a_1}(t)) + \cdots + t_{a_m}(t + t_{a_1} + \cdots + t_{a_{m-1}}) \tag{7}$$

Following Long *et al.* [15], we use piecewise functions, such as the SFs and LIs, to approximate the profiles of cumulative flows and develop solution schemes to solve Equations (5) and (7) in the next section.

DOI: 10.1002/atr

3. ROUTE TRAVEL TIME FORMULATIONS BASED ON CUMULATIVE FLOWS

3.1. Notations

In a traffic network $G(N, A)$, N denotes the set of nodes, whereas A denotes the set of arcs (links). The time period T of interest is discretised into a finite set of time intervals, $K = \{k = 1, 2, \cdots, \underline{K}\}$, where the length of each interval is δ and satisfies $\delta \underline{K} = T$. We assume that the network is empty at the beginning of the period and the set of intervals with traffic demand is denoted by $K_d = \{k = 1, 2, \cdots, \underline{K_d}\}$, where $\underline{K_d} < \underline{K}$. The following notations will be adopted throughout this paper:

$U_a(k)$	the cumulative arrivals at link a by the end of interval k.
$V_a(k)$	the cumulative departures from link a by the end of interval k.
$y_a(k)$	the number of vehicles entering link a during interval k.
$y_{ak}(l)$	the number of vehicles entering link a during interval k and exiting the link during interval l.
$Y_{ak}(l)$	the cumulative number of vehicles entering link a during interval k and exiting the link by the end of interval l.
τ_a^0	the free-flow travel time of link a.
$t_a(k)$	the average travel time for vehicles entering link a during interval k.
n_k	the critical link outflow interval with respect to interval k.
$M_p(k)$	the cumulative departures of route p by the end of interval k.
$N_p(k)$	the cumulative arrivals of route p by the end of interval k.
$x_p(k)$	the number of vehicles entering the network and selecting route p during interval k.
$\mathbf{x}$	the vector of $(x_p(k), \forall p, k)$.
$x_{pk}(l)$	the number of vehicles entering the network and selecting route p during interval k, and arriving the destination during interval l.
$X_{pk}(l)$	the cumulative number of vehicles entering the network and selecting route p during interval k, and arriving the destination by the end of interval l.
τ_p^0	the free-flow travel time of route p.
$t_p(k)$	the average travel time for vehicles entering the network and selecting route p during interval k.
m_k	the critical route arrival interval with respect to interval k.

3.2. Overview of link travel time models based on link cumulative flows

The profiles of link cumulative flows can be approximated by both the SFs and LIs, which can be employed to develop two types of link travel time models, namely, SF-type and LI-type models [15]. Using link outflow capacity, we can further develop a modified LI-type (MLI-type) link travel time model. In this subsection, we briefly introduce the formulations of the three types of link travel time functions, which will also be extended to the route travel time pattern.

3.2.1. The SF-type link travel time

The actual link travel time of vehicles entering link a at time t is the horizontal distance between the two cumulative flow curves (Figure 1). However, if time is discretised, taking a particular interval k for example, there is no guarantee that the entire packet $y_a(k)$ will exit link a during the same discretised time tick. The following definition will be used to determine each sub-packet that exits the link during each time interval.

Definition 1. *(Critical link outflow interval)*. A critical outflow interval of a link with respect to interval k is defined as follows:

$$n_k = \min\left\{l \middle| U_a(k) \leqslant V_a(l), l > k + \tau_a^0/\delta, l \in \{1, 2, \cdots\}\right\} \tag{8}$$

Using the critical link outflow interval, we can determine the cumulative number of vehicles entering link a during a particular interval k and leaving the link by the end of each interval:

DOI: 10.1002/atr

$$Y_{ak}(l) = \begin{cases} 0 & \text{if } l < n_{k-1} \\ V_a(l) - U_a(k-1) & \text{if } n_{k-1} \leqslant l < n_k \\ U_a(k) - U_a(k-1) & \text{otherwise} \end{cases} \tag{9}$$

By definition, we have

$$y_{ak}(l) = Y_{ak}(l) - Y_{ak}(l-1) \text{and} \tag{10}$$

$$y_a(k) = \sum_l y_{ak}(l) = U_a(k) - U_a(k-1) \tag{11}$$

If the cumulative flow curves are approximated by SFs, the actual travel time for sub-packet $y_{ak}(l)$ traveling through the link is $(l-k)\delta$ and with a sum of travel time $y_{ak}(l)(l-k)\delta$. According to the definition of discretised link travel time, we can mathematically formulate the SF-type link travel time as follows (see Long *et al*. [15] for details):

$$t_a(k) = \sum_{l=1}^{K} y_{ak}(l)(l-k)\delta / y_a(k) = (n_k - k)\delta - \sum_{l=n_{k-1}}^{n_k - 1} \delta(V_a(l) - U_a(k-1))/y_a(k) \tag{12}$$

3.2.2. The LI-type link travel time

If the cumulative flow curves are linear interpolated, there exists a time instant $(n_k - 1 + \mu_k)\delta$, where μ_k [0, 1] is a cumulative outflow parameter associated with interval k, such that the flows entering link a before the beginning of interval $k+1$ have completely exited the link, and inflows belonging to interval $k+1$ start leaving away from the link. The cumulative outflow parameter can be calculated by

$$\mu_k = \frac{U_a(k) - V_a(n_k - 1)}{V_a(n_k) - V_a(n_k - 1)} \tag{13}$$

The LI-type link travel time can be stated as follows (see Long *et al*.[15] for details):

$$\hat{t}_a(k) = t_a(k) + \frac{1}{2}\delta(y_{ak}(n_{k-1})\mu_{k-1} + y_{ak}(n_k)(\mu_k - 1))/y_a(k) \tag{14}$$

where $t_a(k)$ is the SF-type link travel time with respect to interval k.

3.2.3. The modified LI-type link travel time

Because the LI-type link travel time does not satisfy causality property [15], an improved cumulative outflow parameter can be defined as follows:

$$\bar{\mu}_k = \frac{U_a(k) - V_a(n_k - 1)}{C_a(n_k)} \tag{15}$$

where $C_a(n_k)$ is positive and denotes the outflow capacity of link a within interval n_k.

With the redefined cumulative outflow parameter in Equation (15), we can formulate the MLI-type link travel time function as follows:

$$\bar{t}_a(k) = t_a(k) + \frac{1}{2}\delta(y_{ak}(n_{k-1})\bar{\mu}_{k-1} + y_{ak}(n_k)(\bar{\mu}_k - 1))/y_a(k) \tag{16}$$

3.3. Route travel time models based on route cumulative flows

Similar with the link pattern, we also can use the SFs and the linear interpolations to approximate the profiles of route cumulative flows and obtain the SF-type and the LI-type route travel times, respectively. Because the calculation of the MLI-type link travel times relies on link outflow capacity and we cannot easily obtain route outflow capacity, therefore the MLI-type link travel time model cannot be directly extended for route travel times. This subsection only considers the extensions of the SF-type and LI-type link travel time models.

DOI: 10.1002/atr

3.3.1. *The SF-type route travel time*

When time is discretised, there is also no guarantee that the entire packet $x_p(k)$ will arrive at the destination during the same discretised time tick. The following definition will be used to determine each sub-packet that arrives at the destination during each time interval.

Definition 2. *(Critical route arrival interval).* A critical arrival interval of a route with respect to interval k is defined as follows:

$$m_k = \min\left\{ l \middle| M_p(k) \leqslant N_p(l), l > k + \tau_p^0/\delta, l \in \{1, 2, \cdots\} \right\} \tag{17}$$

Using the critical route arrival interval, we can determine the cumulative number of vehicles entering the network and selecting route p during the same interval, but arriving at the destination by the end of each interval:

$$X_{pk}(l) = \begin{cases} 0 & \text{if } l < m_{k-1} \\ N_p(l) - M_p(k-1) & \text{if } m_{k-1} \leqslant l < m_k \\ M_p(k) - M_p(k-1) & \text{otherwise} \end{cases} \tag{18}$$

By definition, we have

$$x_{pk}(l) = X_{pk}(l) - X_{pk}(l-1) \text{and} \tag{19}$$

$$x_p(k) = \sum_l x_{pk}(l) = M_p(k) - M_p(k-1) \tag{20}$$

If the cumulative flow curves are approximated by SFs, the actual travel time for sub-packet $x_{ak}(l)$ traveling through the route is $(l-k)\delta$ and with a sum of travel time $x_{ak}(l)(l-k)\delta$. Similar to the SF-type link travel time model, we can mathematically formulate the SF-type route travel time as follows:

$$t_p(k) = \sum_l^K x_{pk}(l)(l-k)\delta / x_p(k) = (m_k - k)\delta - \sum_{l=m_{k-1}}^{m_k - 1} \delta\left(N_p(l) - M_p(k-1)\right) / x_p(k) \tag{21}$$

3.3.2. *The LI-type route travel time*

If the cumulative flow curves are linear interpolated, there exists a time instant $(m_k - 1 + \omega_k)\delta$, where $\omega_k \in [0, 1]$ is a cumulative arrival parameter associated with interval k, such that the flows departing the origin before the beginning of interval $k + 1$ have completely arrived at the destination. The cumulative arrival parameter can be calculated by

$$\omega_k = \frac{M_p(k) - N_p(m_k - 1)}{N_p(m_k) - N_p(m_k - 1)} \tag{22}$$

Similar to the LI-type link travel time model, the LI-type route travel time can be formulated as follows:

$$\hat{t}_p(k) = t_p(k) + \frac{1}{2}\delta\left(x_{pk}(m_{k-1})\omega_{k-1} + x_{pk}(m_k)(\omega_k - 1)\right) / x_p(k) \tag{23}$$

3.4. *Route travel time models based on link cumulative flows*

The discretised link travel time models can be used to calculate link travel times by using the link cumulative flows. The additivity property of route travel times enables route travel times to be formulated as a nested function of link travel times. To simplify the formulation of the nested function, we use LIs to approximate the link travel times. For a given time instant $(k + \omega)\delta$, where $0 \leqslant \omega \leqslant 1$, the link travel time of this time instant can be calculated as follows:

$$t_a(k + \omega) = (1 - \omega)t_a(k) + \omega t_a(k + 1) \tag{24}$$

DOI: 10.1002/atr

Proposition 1. If the discretised link travel times satisfy link FIFO, then the linear interpolated link travel time $t_a(k+\omega)$ also satisfies link FIFO. Equivalently, we have

$$k'+\omega' > k''+\omega'' \Rightarrow (k'+\omega')\delta + t_a(k'+\omega') \geqslant (k''+\omega'')\delta + t_a(k''+\omega''), \forall k', k'', \omega' \in [0,1], \omega'' \in [0,1]$$

Proof. Because the discretised link travel times satisfy link FIFO, we have $\delta + t_a(k+1) - t_a(k) \geqslant 0$ for all $k = 1, 2, 3, \cdots$. Because $k' + \omega' > k'' + \omega''$ and $\omega', \omega'' \in [0, 1]$, we have $k' \geqslant k''$. If $k' = k''$, we have $\omega' - \omega'' > 0$, and Equation (24) implies

$$\left(k'+\omega'\right)\delta + t_a\left(k'+\omega'\right) - \left[\left(k''+\omega''\right)\delta + t_a\left(k''+\omega''\right)\right] = \left(\omega'-\omega''\right)\left[\delta + t_a\left(k'+1\right) - t_a\left(k'\right)\right] \geqslant 0$$

Therefore, we have $(k'+\omega')\delta + t_a(k'+\omega') \geqslant (k''+\omega'')\delta + t_a(k''+\omega'')$. If $k' > k''$, we have $(k'+\omega')\delta + t_a(k'+\omega') \geqslant k'\delta + t_a(k') \geqslant (k''+1)\delta + t_a(k''+1) \geqslant (k''+\omega'')\delta + t_a(k''+\omega'')$. This completes the proof. □

Proposition 2. If the discretised link travel times satisfy link strong FIFO (SFIFO), then the linear interpolated link travel time $t_a(k+\omega)$ also satisfies link SFIFO. Equivalently, we have

$$k'+\omega' > k''+\omega'' \Rightarrow (k'+\omega')\delta + t_a(k'+\omega') > (k''+\omega'')\delta + t_a(k''+\omega''), \forall k', k'', \omega' \in [0,1], \omega'' \in [0,1]$$

The proof is similar to that of Proposition 1.

The travel time required to traverse a path $p = \{a_1, a_2, \cdots, a_m\}$ for vehicles using this path during time interval k can be estimated by the following nested function:

$$t_p(k) = t_{a_1}(k) + t_{a_2}(k + t_{a_1}(k)/\delta) + \cdots + t_{a_m}(k + t_{a_1}/\delta + \cdots + t_{a_{m-1}}/\delta) \tag{25}$$

If the SF-type link travel times are substituted in Equation (25) to calculate the route travel times, then the corresponding route travel time model is named by NSF-type route travel time model for short. Similarly, the LI-type (MLI-type) link travel times can be used to develop the NLI-type (NMLI-type) route travel time model by using Equation (25).

3.5. *Properties of route travel time models*

Definition 3. *(Route FIFO).* The route FIFO condition is satisfied if and only if

$$k' > k'' \Rightarrow k'\delta + t_p\left(k'\right) \geqslant k''\delta + t_p\left(k''\right), \forall k', k'' \in K_d$$

Definition 4. *(Route SFIFO).* The route SFIFO condition is satisfied if and only if

$$k' > k'' \Rightarrow k'\delta + t_p\left(k'\right) > k''\delta + t_p\left(k''\right), \forall k', k'' \in K_d$$

Proposition 3. The SF-type route travel times calculated by Equation (21) satisfy route FIFO.The proof is the same as that of Proposition 5 in Long *et al*. [15].

Proposition 4. The LI-type route travel times calculated by Equation (23) satisfy route SFIFO.The proof is the same as that of Proposition 11 in Long *et al*. [15].

Proposition 5. The NSF-type route travel times calculated by Equation (25) satisfy route FIFO.

Proof. Considering two time intervals k' and k'', we assume $k' > k''$. Because the LI-type link travel times satisfy link FIFO, we have $k'\delta + t_{a_1}\left(k'\right) \geqslant k''\delta + t_{a_1}\left(k''\right)$. Equivalently, we have $k' + t_{a_1}\left(k'\right)\delta^{-1} \geqslant k'' + t_{a_1}\left(k''\right)\delta^{-1}$. According to Proposition 1, we have

DOI: 10.1002/atr

$$k^{'} + t_{a_1}\left(k^{'}\right)\delta^{-1} + t_{a_2}\left(k^{'} + t_{a_1}\left(k^{'}\right)\delta^{-1}\right)/\delta \geqslant k^{''} + t_{a_1}\left(k^{''}\right)\delta^{-1} + t_{a_2}\left(k^{''} + t_{a_1}\left(k^{''}\right)\delta^{-1}\right)/\delta$$

Similarly, we have $k'\delta + t_p(k') \geqslant k''\delta + t_p(k'')$. This completes the proof. □

Proposition 6. The NLI-type and NMLI-type route travel times calculated by Equation (25) satisfy route SFIFO.

The proof is similar to that of Proposition 5.

Propositions 3–6 show that all of the five route travel time models satisfy route FIFO or route SFIFO. It is also important to investigate continuity and monotonicity of the route travel time functions, because the existence and uniqueness of the solutions of DTA models depend on these properties. The following assumption is used to discuss the continuity of each route travel time models.

Assumption 1. The route cumulative outflows and link cumulative outflows are continuous with respect to route flows.

Proposition 7. Under Assumption 1, the traffic flow $x_{pk}(l)$ is continuous with respect to the route flow $\mathbf{x}$.

The proof is the same as that of Proposition 7 in Long *et al.* [15].

Proposition 8. Under Assumption 1, the SF-type and the LI-type route travel times calculated by Equations (23) and (25) are continuous functions of route flow $\mathbf{x}$.

Proof. Substituting Equation (20) into Equation (21), we have $t_p(k) = \sum_l x_{pk}(l)(l-k)\delta / \sum_l x_{pk}(l)$. According to Proposition 7, $x_{pk}(l)$ is continuous with respect to the route flow $\mathbf{x}$, and the SF-type route travel times calculated by Equation (23) are also continuous functions of route flow $\mathbf{x}$. According to Equations (18) and (19), we have $M_p(k) - N_p(m_k - 1) = x_{pk}(m_k)$ and $N_p(m_k) - N_p(m_k - 1) = x_{pk+1}(m_k) + x_{pk}(m_k)$. Hence, $\omega_k = x_{pk}(m_k)/(x_{pk+1}(m_k) + x_{pk}(m_k))$. According to Proposition 7, the second term on the right hand side of Equation (23) is continuous with respect to $\mathbf{x}$. We have proved that $t_p(k)$ is continuous with respect to $\mathbf{x}$. Therefore, the LI-type route travel times calculated by Equation (25) are also continuous functions of route flow $\mathbf{x}$. This completes the proof. □

Proposition 9. Under Assumption 1, the NSF-type, the NLI-type and the NMLI-type route travel times calculated by Equation (25) are continuous functions of route flow $\mathbf{x}$.

Proof. Under Assumption 1, the SF-type, the LI-type, and the MLI-type link travel times are continuous of route flow $\mathbf{x}$[15]. The nested function $t_p(k)$ in Equation (25) is continuous with respect to link travel times [3,13]. Therefore the NSF-type, the NLI-type, and the NMLI-type route travel times are continuous functions of route flow $\mathbf{x}$. □

Continuity and strict monotonicity are certainly desirable properties of route travel times, which determine the solution existence and uniqueness of DTA models. Propositions 8 and 9 have proved that the presented route travel time functions are continuous with respect to rout flows under Assumption 1. However, continuity of route travel times may not hold when a physical-queue traffic flow model is adopted [15]. In this case, weak solutions are usually defined, which do not require continuity to hold at each time instant [12,13]. Nevertheless, our experience and that of many authors worked with many DTA models and algorithms are that strict monotonicity of route travel times does not hold even for static user equilibrium problems. This implies that DTA models using discretised route travel time functions may not guarantee the uniqueness of the solutions.

Reference [11] stated that the continuity of route travel time with respect to time cannot always be guaranteed in practice. We here prove that the discretised route travel time will converge to the continuous-time route travel time if the latter is continuous with respect to time.

Assumption 2. The continuous-time route travel time is continuous with respect to time instant t.

Proposition 10. Under Assumption 2, the SF-type and LI-type route travel times converge to the continuous-time route travel times calculated by Equation (1).

The proof is presented in Appendix.

Assumption 3. The continuous-time link travel time is continuous with respect to time instant *t*.

Proposition 11. Under Assumption 3, the NSF-type, the NLI-type, and the NMLI-type route travel times calculated by Equation (25) converge to the continuous-time route travel times calculated by Equation (2).

Proof. Similar to the proof of Proposition 10, we can prove that the SF-type, the LI-type, and the MLI-type link travel time functions converge to the same function if $\delta \to 0$, and the SF-type link travel times converge to the continuous-time link travel times under Assumption 3. Hence, the LI-type and the MLI-type link travel times also converge to the continuous-time link travel times under Assumption 3. The NSF-type, the NLI-type, and the NMLI-type route travel times calculated by Equation (25) are nested function of the SF-type, the LI-type, and the MLI-type link travel times, respectively and, therefore converge to the continuous-time route travel times calculated by Equation (2).

4. MODEL ACCURACY ESTIMATION

4.1. Route travel time calculation error

To evaluate the accuracy of each type of route travel time model, the theoretical value of route travel time should be computed firstly. However, it is difficult to directly calculate the theoretical value of route travel time because of the irregularity of cumulative flow curves. Long *et al.* [15] developed an efficient method to estimate the theoretical value of link travel time. The method can be extended to estimate the theoretical value of route travel time. The DNL model is implemented with a very short interval length to generate the cumulative flow curves and obtain the route travel time (denoted by $\tilde{t}_p(\imath)$) and route flow (denoted by $\tilde{w}_p(\imath)$), where $\imath$ is the index of a particular short interval. The theoretical value of route travel time during a particular interval k can be estimated as follows:

$$tt_p(k) = \frac{\sum_{\imath \in K^{\delta}(k)} \tilde{w}_p(\imath)\tilde{t}_p(\imath)}{w_p(k)} \tag{26}$$

where $tt_p(k)$ is the theoretical value of route travel time at interval k, $K^{\delta}(k)$ is the set of short intervals that belong to time interval $((k-1)\delta, k\delta]$.

The calculation error of a route travel time model for a particular interval is defined as the difference between the estimated travel time obtained from the model and the theoretical value of route travel time at that interval. This definition of route travel time calculation error is used to evaluate the model accuracy. With the use of the SF-type route travel time as an example, the calculation error of the route travel time at a given interval k can be mathematically expressed as follows:

$$e_p(k) = t_p(k) - tt_p(k) \tag{27}$$

4.2. Model accuracy measures

There are two measures to evaluate the maximum error of the whole study period for a particular model: maximum absolute error (MaxAE) and maximum percentage error (MaxPE). The MaxAE and MaxPE can be mathematically expressed as follows:

$$\text{MaxAE} = \max_{k \in K_d}\{|e_p(k)|\}\text{and} \tag{28}$$

$$\text{MaxPE} = \max_{k \in K_d}\{|e_p(k)|/tt_p(k)\} \times 100 \tag{29}$$

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$|e_p(k)|$ is the absolute calculation error (in seconds) of a particular interval k, and therefore Equation (28) gives MaxAE (in seconds). $|e_p(k)|/tt_p(k)$ is the relative calculation error of a particular interval k, and therefore Equation (29) gives MaxPE, which is expressed in terms of per 100.

There are also two measures to evaluate the average error of the whole study period for a particular model: mean absolute error (MAE) and mean percentage error (MPE). The MAE and MPE can be formulated as follows:

$$\mathrm{MAE} = \frac{\sum_{k \in K_d} |e_p(k)|}{\underline{K_d}} \quad \text{and} \tag{30}$$

$$\mathrm{MPE} = \frac{\sum_{k \in K_d} |e_p(k)|/tt_p(k)}{\underline{K_d}} \times 100 \tag{31}$$

The numerator and the denominator of the right hand side of Equation (30) are the total absolute calculation error (in seconds) and the number of intervals with traffic demand. Therefore, Equation (30) gives MAE (in seconds). In Equation (31), MPE is defined as the average of the relative absolute error of each discretised interval and expressed in terms of per 100.

5. NUMERICAL EXAMPLES

The presented route travel time models use the cumulative flows to compute route travel times. The DNL model used in this paper is the point-queue (PQ) model [3,33]. The underlying reasons for choosing the PQ model are as follows [33]: (i) The PQ model is easy to calibrate because its parameters, including free-flow travel time and bottleneck capacity, are all well-defined physical quantities that are relatively easy to measure; and (ii) The PQ model takes advantage of the computational efficiency and behaves identically as the CTM model if queue spillback does not occur. Three examples are developed to demonstrate the model accuracy and the computational efficiency of the proposed travel time models. All the experiments were coded in Visual C# and run on a personal computer with a Pentium IV processor having 2.83 GHz CPU and 3.50 GB of RAM memory.

5.1. Example 1: a linear network

In this example, we use a linear network presented in Figure 2 to illustrate the accuracy of each type of route travel time model. The linear network consists of n nodes, $n-1$ links and n OD pairs, where n is greater than 2. All links have uniform free-flow travel time and outflow capacity, 2 minute and 36 vehicles/minute, respectively. Each two adjacent nodes $(i, i+1)$ form the ith OD pair and nodes $(1, n)$ form the nth OD pair. The interval length used to estimate the theoretical value of route travel time is 0.1 second. The network is empty initially, and only the first 30 minutes has traffic demand. The traffic demands of each OD pairs are set as follows:

$$q_i(t) = q^i_{\max} \sin^2(\pi t/10 + \pi i/n) \tag{32}$$

where $q^i_{\max}$ is the maximum traffic demand of the ith OD pair.

Firstly, we set $n = 10$ and $q^i_{\max} = 120$ vehicles/minute for all $i \leqslant n$. Using the PQ model to implement the DNL, we obtain both link and route cumulative flows and use the presented route travel time models to compute the route travel times. The maximum errors and the mean errors corresponding to each type of route travel time models are presented in Table I. We can observe that both the

Figure 2. A linear network.

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maximum errors and the mean errors have an increase tendency when we increase the value of δ. This result indicates that a small interval length will improve the model accuracy. Table I also shows that the route travel time models based on route cumulative flows (i.e., the SF-type and the LI-type route

Table I. Route travel time calculation errors of the last long route in Example 1.

Measure	Model type	$\delta = 1$ second	$\delta = 5$ seconds	$\delta = 10$ seconds	$\delta = 20$ seconds	$\delta = 30$ seconds	$\delta = 40$ seconds	$\delta = 60$ seconds
MaxAE (seconds)	SF-type	0.748	4.490	11.097	41.196	67.133	60.085	105.773
	LI-type	0.513	4.120	11.374	40.674	66.587	67.546	108.820
	NSF-type	5.030	10.376	27.416	32.043	31.207	54.637	43.054
	NLI-type	4.552	10.097	23.477	25.047	37.074	35.656	33.533
	NMLI-type	4.693	10.097	23.743	25.047	37.074	46.451	33.577
MaxPE (%)	SF-type	0.027	0.217	0.523	1.961	3.118	2.869	4.941
	LI-type	0.025	0.199	0.536	1.936	3.093	3.225	5.084
	NSF-type	0.236	0.436	1.056	1.219	1.406	2.608	2.011
	NLI-type	0.213	0.385	0.904	0.953	1.671	1.702	1.567
	NMLI-type	0.220	0.453	0.914	0.953	1.671	2.218	1.569
MAE (seconds)	SF-type	0.086	0.524	1.186	3.038	5.092	7.514	12.985
	LI-type	0.010	0.176	0.578	1.894	3.633	4.843	8.567
	NSF-type	0.143	0.828	2.084	4.046	6.624	10.213	10.998
	NLI-type	0.019	0.257	0.921	2.286	3.888	5.414	8.263
	NMLI-type	0.016	0.252	0.916	2.380	4.310	6.171	8.316
MPE (%)	SF-type	0.003	0.020	0.045	0.118	0.198	0.292	0.502
	LI-type	0.000	0.007	0.023	0.078	0.148	0.202	0.344
	NSF-type	0.005	0.032	0.081	0.156	0.254	0.391	0.412
	NLI-type	0.001	0.010	0.037	0.089	0.152	0.214	0.316
	NMLI-type	0.001	0.010	0.037	0.092	0.169	0.244	0.312

MaxAE, maximum absolute error; MaxPE, maximum percentage error; MAE, mean absolute error; MPE, mean percentage error.

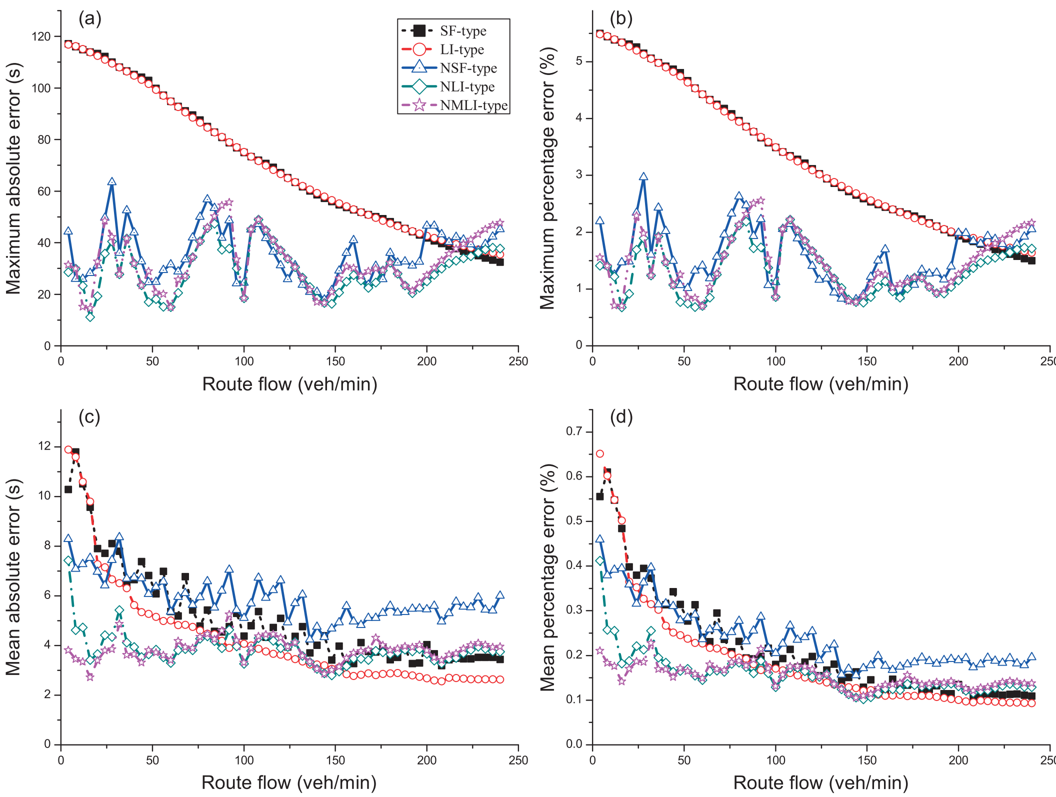

Figure 3. The effect of route flow on the route travel time calculation errors of the last route: (a) the MaxAE, (b) the maximum percentage error, (c) the mean absolute error, and (d) the mean percentage error.

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travel time models) outperform those based on link cumulative flows in terms of the maximum calculation errors when we use a small value of δ, but an opposite result occurs when we use a large value of δ. If the mean calculation errors are considered, we find that the LI-type, the NLI-type and the NMLI-type route travel time models outperform the SF-type and the NSF-type models. In this example, the NLI-type and the NMLI-type route travel time model have a very close accuracy.

It is also interesting to discuss the effect of route flows on the route travel time calculation errors. We change the maximum route flow of the last route from 1.0 to 240 and set $\delta = 30$ seconds. Other input parameters are the same as previous experiments. The maximum calculation errors and the mean calculation errors with respect to each route flow pattern are graphically displayed in Figure 3. We can observe that the MaxAE and the MaxPE of the route travel time models based on route cumulative flows are decreasing when we increase the route flow of the last route, but the MaxAE and the MaxPE of the route travel time models based on link cumulative flows do not have a decrease tendency but with a great fluctuation (Figure 3(a and b)). This implies the maximum errors of the route travel time models based on link cumulative flows may be independent with network congestion. We also can observe that the MAE of the route travel time models based on route cumulative flows are decreasing when we increase the route flow of the last route, and the MAE of the route travel time models based on link cumulative flows does not have a clear decrease tendency but with a slight fluctuation (Figure 3 (c)). Because the route travel times are increasing when the route flow grows up, the MPE of all types of route times is decreasing when we increase the route flow of the last route (Figure 3(d)). The results presented in Figure 3(c and d) also show that the LI-type route travel time models outperform the SF-type route travel time models, and the NLI-type and the NMLI-type route travel time models outperform the NSF-type models in general. Another interesting result is that route travel time models based on route cumulative flows outperform those based on link cumulative flows when route flows are great enough. This example also confirms that the NLI-type and the NMLI-type route travel time model have a very close accuracy.

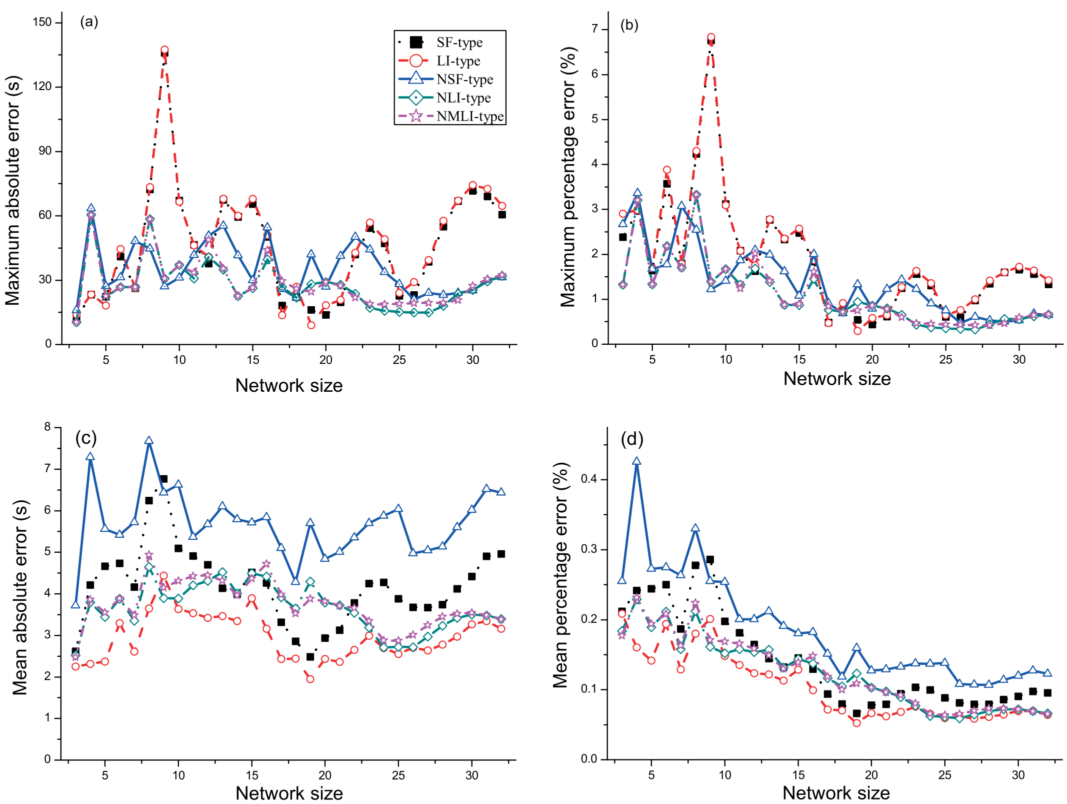

Figure 4. The effect of network size on the route travel time calculation errors of the last route: (a) the maximum absolute error, (b) the maximum percentage error, (c) the mean absolute error, and (d) the mean percentage error.

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Furthermore, we concern the effect of network size on the route travel time calculation errors. In this example, we reset $q^i_{\max} = 120$ vehicles/minute for all $i \leqslant n$, but with the number of nodes varying from 3 to 32. The maximum calculation errors and the mean calculation errors with respect to each network size pattern are graphically displayed in Figure 4. We can observe that the route travel time models based on link cumulative flows outperform those based on route cumulative flows in average in terms of the MaxAE and the MaxPE, but an opposite result occurs when we consider the MAE and the MPE. The results presented in Figure 4(c and d) confirm that the LI-type route travel time models outperform the SF-type route travel time models, and the NLI-type and the NMLI-type route travel time models outperform the NSF-type models in general.

5.2. *Example 2: the Nguyen and Dupius network*

In this example, the Nguyen and Dupius network is adopted (Figure 5), which consists of 13 nodes, 19 links, 4 OD pairs and 25 routes [35]. The free-flow travel time of each link is listed in Table II [2,14]. The number of lanes is as follows: 1 for links 8–2, 12–8 and 13–3, 2 for link 9–13, and 3 for other 15 links. The outflow capacity is: 0.3 vehicles/second/lane for links leading to nodes 2, 3, 5, 6, 8, 9, 10, 11, and 0.5 vehicles/second/lane for other links. The traffic demand lasts for the first 30 minutes.

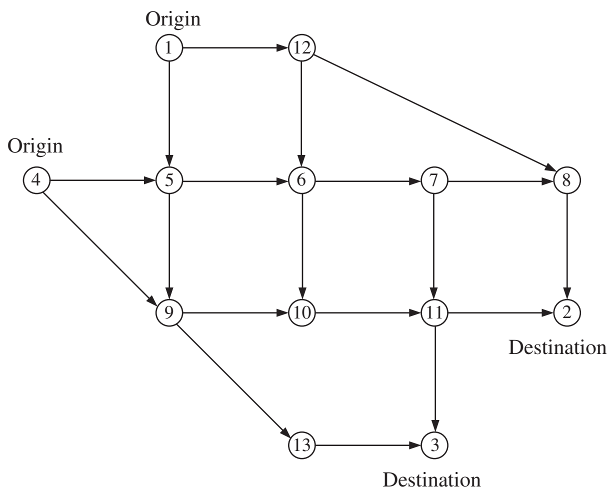

Figure 5. The Nguyen and Dupius network.

Table II. The free-flow travel time of each link in the Nguyen and Dupius network.

Free-flow travel time (seconds)	30	40	40	50
Link	1–12 7–8 7–11 9–13 8–2 10–11	1–5 4–9 5–9 6–10 11–2 12–6	4–5 5–6 6–7 9–10 11–3 13–3	12–8

Table III. Basic OD demand of the Nguyen and Dupius network (10^3 vehicles/hour).

From	To	
	2	3
1	7.2	4.8
4	4.8	7.2

The interval length used to estimate the theoretical value of route travel time is also 0.1 s. The basic OD demand of each OD pair is given in Table III, and we use a function with plateau shaped profile to represent the time-dependent demand level, given by:

$$\psi(t) = \begin{cases} \sin(\pi t/10) & \text{if } 0 \leqslant t \leqslant 5 \\ 1 & \text{if } 5 < t < 20 \\ 0.6 + 0.4\sin^5(\pi(t-10)/20) & \text{if } 20 \leqslant t \leqslant 30 \end{cases} \tag{33}$$

where the unit for the time t is minute. Multiplying the time-dependent demand level with the basic OD demand, we obtain the time-dependent OD demand.

To investigate the performance of the proposed route travel time models, we should predetermine the route flows, which are used to implement the DNL and generate both cumulative route flows and cumulative link flows. In this example, a logit-based dynamic stochastic user equilibrium (DSUE) model [36,37] is used to assign the OD demand to each route. The SF-type route travel time model is adopted to calculate the route travel times, and the method of successive averages is employed to solve the DSUE model. The other parameters for the logit-based DSUE model are as follows: the dispersion parameter equals to 0.1, the interval length is 10 seconds, the total iteration number is 400, and all the 25 routes are used. By solving the DSUE model, we can obtain the time-dependent route choice probability, which is used to determine the time-dependent path flows.

The route travel time calculation errors under various interval lengths are given in Table IV. We can observe that the LI-type route travel time model performs better than the SF-type route travel time model in terms of the average calculation error, and the NLI-type and NMLI-type route travel time models outperform the NSF-type route travel time model in terms of both the maximum relative calculation error and the average calculation error. This result is consistent with that in Example 1. In this example, the SF-type route travel time model has higher accuracy than the NSF-type route travel time model, whereas the NLI-type and NMLI-type route travel time models have higher accuracy than the LI-type route travel time model. Therefore, we also cannot determine whether the route travel time models based on link cumulative flows or that based on route cumulative flows are better in terms of model accuracy. The results presented in Table IV also confirm that the proposed route travel time models have higher accuracy if a smaller interval length is adopted.

The CPU times required for implementing the DNL and the route travel time models are listed in Table V. We can observe that the CPU time required for implementing the DNL is about ten times of that for calculating route travel times by the proposed methods. This result implies that the proposed formulations of the route travel time models have good performance in terms of computational efficiency. From Table V, we can also observe that the CPU time increases approximately linearly as the interval length decreases. This result implies that selection of interval length will be limited

Table IV. Route travel time calculation errors of Example 2.

Interval length	Model type	MaxAE (seconds)	MaxPE (%)	MAE (seconds)	MPE (%)
$\delta = 1$ second	SF-type	0.4973	0.0728	0.0266	0.0029
	LI-type	0.7357	0.0949	0.0035	0.0003
	NSF-type	0.4500	0.1007	0.0670	0.0077
	NLI-type	0.5474	0.0205	0.0015	0.0001
	NMLI-type	0.2753	0.0137	0.0015	0.0001
$\delta = 5$ seconds	SF-type	3.0244	0.3340	0.1344	0.0146
	LI-type	4.5917	0.2016	0.0435	0.0033
	NSF-type	1.9270	0.5490	0.3311	0.0385
	NLI-type	1.8208	0.0661	0.0205	0.0020
	NMLI-type	1.4292	0.0657	0.0208	0.0020
$\delta = 10$ seconds	SF-type	7.5507	0.7653	0.3144	0.0342
	LI-type	9.3507	0.3984	0.1964	0.0179
	NSF-type	4.4557	0.7767	0.6763	0.0765
	NLI-type	5.0369	0.2882	0.1431	0.0148
	NMLI-type	2.3714	0.2882	0.1409	0.0147

MaxAE, maximum absolute error; MaxPE, maximum percentage error; MAE, mean absolute error; MPE, mean percentage error.

DOI: 10.1002/atr

Table V. The CPU times required to implement the dynamic network loading (DNL) and the route travel time models in Example 2 (seconds).

Interval length	DNL	SF-type	LI-type	NSF-type	NLI-type	NMLI-type
$\delta = 0.1$ second	1.5300	0.0745	0.1130	0.1820	0.2050	0.1977
$\delta = 1$ second	0.1425	0.0078	0.0114	0.0184	0.0206	0.0202
$\delta = 5$ seconds	0.0261	0.0016	0.0023	0.0036	0.0041	0.0041
$\delta = 10$ seconds	0.0127	0.0006	0.0011	0.0019	0.0020	0.0020

by the computational burden. The results presented in Table V also show that The CPU time required for computing the SF-type (NSF-type) route travel times is less than that for computing the LI-type (NLI-type or NMLI-type) route travel times. This is because the formulation of SF-type (NSF-type) route travel time model has a simpler form than that of the LI-type (NLI-type and MLI-type) model. Comparing Equation (21) with Equation (23), we can find that the calculation of SF-type route travel times is a sub-process to calculate the LI-type route travel times, and therefore, the latter one needs more computational time.

5.3. *Example 3: the Sioux Falls network*

To further illustrate the performance of the proposed route travel time models, we tested them in the Sioux Falls network as shown in Figure 6, which is larger than the Nguyen and Dupius network. The Sioux Falls network consists of 24 nodes, 76 links, and 528 OD pairs. The basic OD demand and the capacity of each link are the same as that in Leblanc *et al*. [38], but only the link free-flow travel times are slightly changed (Table VI). The same as Example 2, the traffic demand lasts for the first 30 minutes, and the time-dependent demand level also follows Equation (33). Following Long *et al*. [39], we used a combination of the link elimination method (see Bekhor *et al*. [40] for details) and Dial's [41] STOCH method to generate the path choice set, where the STOCH method can be used to generate the basic path set, and the link elimination method can generate some shorter paths omitted

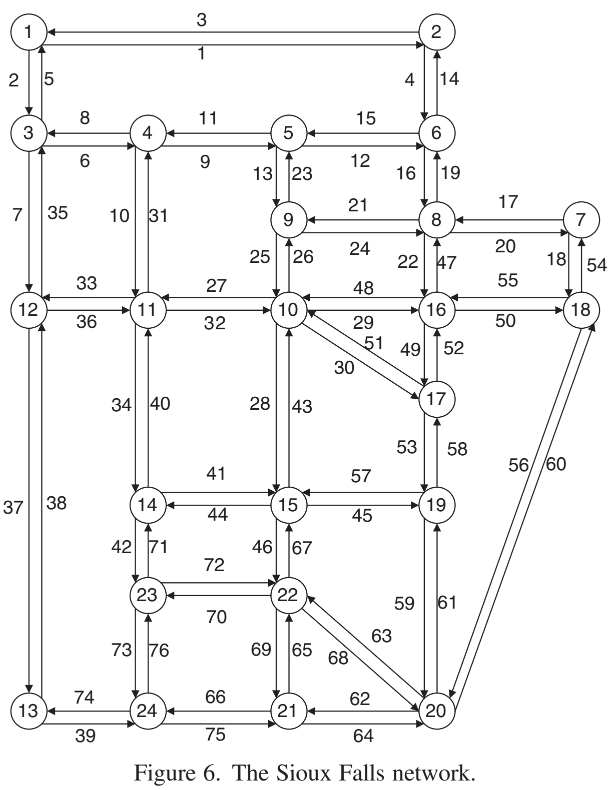

Figure 6. The Sioux Falls network.

Table VI. The free-flow travel time of each link in the Sioux Falls network.

Link order	τ_a^0 (seconds)	Link order	τ_a^0 (seconds)	Link order	τ_a^0 (seconds)	Link order	τ_a^0 (seconds)
1 and 3	140	17 and 20	80	34 and 40	100	53 and 58	60
2 and 5	100	18 and 54	60	37 and 38	80	56 and 60	100
4 and 14	120	21 and 24	240	39 and 74	100	59 and 61	100
6 and 8	100	22 and 47	120	41 and 44	120	62 and 64	140
7 and 35	100	25 and 26	80	42 and 71	100	63 and 68	120
9 and 11	60	27 and 32	120	45 and 57	100	65 and 69	60
10 and 31	140	28 and 43	140	46 and 67	100	66 and 75	80
12 and 15	100	29 and 48	120	49 and 52	60	70 and 72	100
13 and 23	120	30 and 51	200	50 and 55	80	73 and 76	60
16 and 19	60	33 and 36	140				

by the STOCH method. There are totally 1875 paths generated, the average number of paths per OD pair is 3.6, and the maximum number of routes between an OD pair is 15. The path flow generation method is the same as that in Example 2. Because there is not enough RAM memory to implement the DNL when the length of interval is 0.1 second, the interval length used to estimate the theoretical value of route travel time is 0.5 second in this example.

We repeated the experiments under various interval lengths in the Sioux Falls network and reported the route travel time calculation errors and the CPU times required for implementing the DNL and the route travel time models in Tables VII and VIII, respectively. Very similar results with that in Example 2 can be observed: (i) Using a small interval length can improve the accuracy of all route travel time models, but will also increase the computational burden; (ii) The LI-type (NLI-type or NMLI-type) route travel time model outperforms the SF-type (NSF-type) route travel time model in terms

Table VII. Route travel time calculation errors of Example 3.

Interval length	Model type	MaxAE (seconds)	MaxPE (%)	MAE (seconds)	MPE (%)
$\delta = 5$ seconds	SF-type	46.1910	1.4952	0.4073	0.0153
	LI-type	46.2058	1.5069	0.2554	0.0092
	NSF-type	32.1256	1.1106	0.5320	0.0218
	NLI-type	32.3895	1.0469	0.1815	0.0064
	NMLI-type	32.3895	1.0469	0.1116	0.0048
$\delta = 10$ seconds	SF-type	78.9225	3.4859	0.9387	0.0370
	LI-type	78.9348	3.5242	0.6585	0.0263
	NSF-type	49.6618	1.9211	1.1454	0.0469
	NLI-type	58.7199	1.9023	0.4042	0.0165
	NMLI-type	58.7199	1.9023	0.2390	0.0126
$\delta = 20$ seconds	SF-type	290.9873	12.3036	4.5690	0.2158
	LI-type	292.3109	12.3342	4.4461	0.2090
	NSF-type	226.2913	7.5367	4.5814	0.2137
	NLI-type	183.6029	6.9464	3.9910	0.1801
	NMLI-type	183.6029	6.9464	3.4164	0.1667

MaxAE, maximum absolute error; MaxPE, maximum percentage error; MAE, mean absolute error; MPE, mean percentage error.

Table VIII. The CPU times required to implement the dynamic network loading (DNL) and the route travel time models in Example 3 (seconds).

Interval length	DNL	SF-type	LI-type	NSF-type	NLI-type	NMLI-type
$\delta = 0.5$ second	11.0609	1.2219	1.7797	0.9875	1.0477	1.0422
$\delta = 5$ seconds	0.9677	0.1227	0.1789	0.1016	0.1039	0.1031
$\delta = 10$ seconds	0.4805	0.0617	0.0898	0.0506	0.0534	0.0530
$\delta = 20$ seconds	0.2366	0.0313	0.0455	0.0252	0.0267	0.0264

of higher average calculation errors, but the former is inferior to the latter in terms of requiring more CUP time; (iii) The NMLI-type route travel time model slightly outperforms the NLI-type route travel time model in terms of both accuracy and computational efficiency; (iv) The CPU time required to implement the proposed route travel time models are far less than that required to implement the DNL; and (v) The CPU times required to implement both the DNL and the route travel time models have an approximately linearly increase tendency when we shorten the interval length. Comparing these with the results presented in Example 2, we found that the accuracy of the proposed models may decrease, whereas the corresponding CPU time will increase a lot if the network size is enlarged. We also found that the route travel time models based on link cumulative flows outperform the models based on route cumulative flows in terms of the computational efficiency. This implies that the route travel time models based on link cumulative flows should be recommended for applications in medium-scale and large-scale networks.

6. CONCLUSIONS

This paper develops discretised route travel time models based on both route cumulative flow curves and link cumulative flow curves, which can be generated by DNL models. By using SFs and LIs to approximate the profile of route cumulative flows, we obtain the SF-type and LI-type route travel time models, respectively. We also develop route travel time models by using a nested function of link travel times based on link cumulative flows. The profiles of link cumulative flows are approximated by SFs and LIs, and we can obtain the SF-type, the LI-type, and the modified LI-type link travel times. Substituting the three types of link travel times into the nested function, we obtain the NSF-type, the NLI-type, and the NMLI-type route travel time models.

On the basis of the analysis, we found that all the five models satisfy route FIFO and continuity, no matter how large is the time step used for discretisation. Numerical methods are developed to evaluate the accuracy and computational efficiency of the route travel time models presented in this paper. Both maximum errors and mean errors are formulated to measure the calculation error of each type of route travel time models. Three networks are used to demonstrate the performance of each type of route travel time models. The numerical results show that the LI-type route travel time models outperform the SF-type route travel time models, and the NLI-type and the NMLI-type route travel time models outperform the NSF-type models. Furthermore, it is demonstrated that the CPU time required to implement the proposed route travel time models are far less than that required to implement the DNL.

In the future, we will discuss other properties of the presented route travel time models and apply them into DTA models. Most of DTA models assume that only the deterministic component of traffic flow is taken into account and the corresponding DNL results in deterministic route travel times. However, traffic uncertainties, such as stochastic speed–density relationship [27,42], stochastic traffic demand [43], stochastic road capacity [44] and so on, exist extensively in real life, which lead to stochastic route travel times. In the future, we are also interested in developing route travel time models that can address traffic uncertainties.

ACKNOWLEDGEMENTS

The authors would like to thank the anonymous referees for their useful comments on this paper. This work is supported by the National Natural Science Foundation of China (70821061, 71001001, 71171124) and the National Basic Research Program of China (2012CB725401).

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APPENDIX

Below is the proof of Proposition 10.

Proof. Firstly, we prove that the SF-type route travel time function and the LI-type route travel time function converges to the same function if $\delta \to 0$. Equation (23) can be applied to calculate the gap of the two types of travel times, which is denoted by $\Delta\hat{t}_p(k)$ and can be expressed as follows

$$\Delta\hat{t}_p(k) = \frac{1}{2}\delta\big(x_{pk}(m_{k-1})\omega_{k-1} + x_{pk}(m_k)(\omega_k - 1)\big)/x_p(k) \tag{A1}$$

Because $0 \leqslant \omega_k \leqslant 1$ and $0 \leqslant x_{pk}(l)/x_p(k) \leqslant 1$, for all $k \in K_d, l \in K$, we have

$$\Delta\hat{t}_p(k) \leqslant \frac{1}{2}\delta x_{pk}(m_{k-1})\omega_{k-1}/x_p(k) \leqslant \frac{1}{2}\delta \quad \text{and} \tag{A2}$$

$$\Delta\hat{t}_p(k) \geqslant -\frac{1}{2}\delta x_{pk}(m_k)(1-\omega_k)/x_p(k) \geqslant -\frac{1}{2}\delta \tag{A3}$$

Equations (A2) and (A3) indicate that the difference of the LI-type route travel time from the SF-type route travel time will approach to zero if $\delta \to 0$.

According to the definition of m_k in Equation (17), we have

$$N_p(m_k - 1) \leqslant M_p(k) \leqslant N_p(m_k) \tag{A4}$$

Using the relationship between the cumulative departure flows and arrival flows of path p, we have

$$(m_k - k - 1)\delta \leqslant \tau_p(k\delta) \leqslant (m_k - k)\delta \tag{A5}$$

where $\tau_p(k\delta)$ is the route travel time for vehicles entering the network and selecting route p at time instant $k\delta$.

The first inequality and the second inequality of Equation (A5), respectively, imply that

$$(m_k - k)\delta \leqslant \tau_p(k\delta) + \delta \quad \text{and} \tag{A6}$$

$$\tau_p((k-1)\delta) - \delta \leqslant (m_{k-1} - k)\delta \tag{A7}$$

According to Equation (18), we have

$$t_p(k) = \sum_{l=1}^{K} x_{pk}(l)(l-k)\delta/x_p(k) = \sum_{l=m_{k-1}}^{m_k} x_{pk}(l)(l-k)\delta/x_p(k) \tag{A8}$$

DOI: 10.1002/atr

Therefore, we have

$$t_p(k) = \sum_{l=m_{k-1}}^{m_k} x_{pk}(l)(l-k)\delta/x_p(k) \geqslant \sum_{l=m_{k-1}}^{m_k} x_{pk}(l)(m_{k-1}-k)\delta/x_p(k) = (m_{k-1}-k)\delta \quad \text{and} \tag{A9}$$

$$t_p(k) = \sum_{l=m_{k-1}}^{m_k} x_{pk}(l)(l-k)\delta/x_p(k) \leqslant \sum_{l=m_{k-1}}^{m_k} x_{pk}(l)(m_k-k)\delta/x_p(k) = (m_k-k)\delta \tag{A10}$$

Substituting Equations (A6) and (A7) into Equations (A9) and (A10), respectively, we have

$$\tau_p((k-1)\delta) - \delta \leqslant (m_{k-1}-k)\delta \leqslant t_p(k) \leqslant (m_k-k)\delta \leqslant \tau_p(k\delta) + \delta \tag{A11}$$

Let $\Delta t = \delta$, $t = k\delta$, and $t_p(t, \Delta t) = t_p(k)$, where $t_p(t, \Delta t)$ is a function of Δt. Equation (A11) can be equivalently rewritten as follows:

$$\tau_p(t-\Delta t) - \Delta t \leqslant t_p(t, \Delta t) \leqslant \tau_p(t) + \Delta t \tag{A12}$$

According to Assumption 2, the continuous-time route travel time is continuous with respect to time instant t, we have

$$\lim_{\Delta t \to 0}\left[\tau_p(t-\Delta t) - \Delta t\right] = \lim_{\Delta t \to 0} \tau_p(t-\Delta t) = \tau_p(t) = \lim_{\Delta t \to 0}\left[\tau_p(t) + \Delta t\right] \tag{A13}$$

Combining Equations (A12) and (A13), we have

$$\lim_{\Delta t \to 0} t_p(t, \Delta t) = \tau_p(t) \tag{A14}$$

This implies that the SF-type route travel times converge to the continuous-time route travel times. Because the SF-type route travel time function and the LI-type route travel time function converge to the same function if $\delta \to 0$, the LI-type route travel times also converge to the continuous-time route travel times. □

DOI: 10.1002/atr