Complex Finsler spaces with $(\gamma, |\beta|)$-metric

Sweta Kumari and P. N. Pandey

Abstract. The present paper deals with the differential geometry of a complex Finsler space endowed with $(\gamma, |\beta|)$-metric, where γ is a cubic-root metric and β is a differential $(1, 0)$-form. Expressions for the fundamental metric tensor, complex angular metric tensor, their inverses, Chern-Finsler connection, holomorphic curvature and Euler-Lagrange equations are obtained.

M.S.C. 2010: 53B40, 53C56.
Key words: Complex Finsler space; $(\gamma, |\beta|)$- metric; Chern-Finsler connection coefficients; curvature; Euler-Lagrange equations.

1 Introduction

In 1979, M. Matsumoto [9], introduced the concept of cubic metric on a differentiable manifold with the local coordinates x^i, defined by $L(x, y) = (a_{ijk}(x)y^i y^j y^k)^{\frac{1}{3}}$, where $a_{ijk}(x)$ are components of a symmetric tensor field of $(0, 3)$-type depending upon the position x alone. The Finsler space with a cubic metric is called a cubic Finsler space. There are some papers related to the cubic Finsler space [6, 12, 13] etc. In 2011, T. N. Pandey and V. K. Chaubey [11] introduced the concept of (γ, β)-metric, where γ is a cubic-root metric and β is a 1-form metric defined by $\gamma = \sqrt[3]{a_{ijk}(x)y^i y^j y^k}$ and $\beta = b_i(x)y^i$ respectively. N. Aldea and G. Munteanu [3] introduced a complex Finsler space with Randers metric following the ideas from real case [4, 5, 8, 14] in 2009. The authors of the current paper [7] studied a complex Randers space with metric $L = \alpha + \epsilon|\beta| + k\frac{|\beta|^2}{\alpha}, \epsilon, k \neq 0$.
The aim of the present paper is to introduce and study a complex Finsler space with the fundamental function, $F(\gamma, |\beta|)$ on the lines of the Finsler space with $(\alpha, |\beta|)$ metric as studied by N. Aldea and G. Munteanu [3], such that

$$F(z, \eta) = \gamma(z, \eta) + |\beta|(z, \eta), \tag{1.1}$$

where

$$\begin{cases} \gamma = \sqrt[3]{a_{i\bar{j}\bar{k}}\eta^i\bar{\eta}^j\bar{\eta}^k}; \\ |\beta(z, \eta)| = \sqrt{\beta(z, \eta)\overline{\beta(z, \eta)}} \ \textit{with} \ \beta(z, \eta) = b_i(z)\eta^i. \end{cases} \tag{1.2}$$

Differential Geometry - Dynamical Systems, Vol.19, 2017, pp. 75-86.

In this paper we determine the fundamental metric tensor (it's inverse and determinant), the complex angular metric tensor (it's inverse and determinant), Chern-Finsler connection coefficients, holomorphic curvature and Euler-Lagrange equations for the complex Finsler space (M, F) with $(\gamma, |\beta|)$-metric.

2 Preliminaries

Let M be a complex manifold of dimension n and $(z^k)_{k=1,\ldots,n}$ be complex coordinates in a local chart. The complexified tangent bundle $T_C M$ splits into holomorphic tangent bundle $T'M$ and anti holomorphic tangent bundle $T''M$, i.e. $T_C M = T'M \oplus T''M$. The holomorphic tangent bundle $T'M$ is itself a complex manifold with local coordinates $u = (z^k, \eta^k)$ in a chart, which changes by the following rules

$$z'^k = z'^k(z), \quad \eta'^k = \frac{\partial z'^k}{\partial z^j}\eta^j. \tag{2.1}$$

Further, $T_C(T'M)$ decomposes as a sum of holomorphic and anti holomorphic tangent bundles $T'_u(T'M)$ and $T''_C(T'M)$ respectively. A natural local frame $\{\partial/\partial z^k, \partial/\partial z'^k\}$ for $T'_u(T'M)$ changes according to the rules obtained from Jacobi matrix of (3). Since the changing rule of $\partial/\partial z^k$ contains the second order partial derivatives, the concept of complex non-linear connection(*c.n.c.*) was introduced.

Let $V(T'M) \subset T'(T'M)$ be the vertical bundle spanned by $\{\partial/\partial\eta^k\}$. The complex non-linear connection(*c.n.c.*) determines a supplementary complex subbundle to $V(T'M)$ in $T'(T'M)$, i.e. $T'(T'M) = H(T'M) \oplus V(T'M)$, called the horizontal bundle. It determines an adapted frame $\left\{\frac{\delta}{\delta z^k} = \frac{\partial}{\partial z^k} - N^j_k \frac{\partial}{\partial \eta^j}\right\}$, where $N^j_k(z, \eta)$ are the coefficients of the (*c.n.c.*) [1, 2, 10].

A complex Finsler metric F on complex manifold M is a continuous function $F\colon T'M \to R^+$ satisfying following conditions [10]

1. $L = F^2$ is smooth on $T'M \setminus \{0\}$;
2. $F(z, \eta) \geq 0$, the equality holds if and only if $\eta = 0$;
3. $F(z, \lambda\eta) = |\lambda| F(z, \eta)$, for $\forall\ \lambda \in C$;
4. the Hermitian matrix $(g_{i\bar{j}}(z, \eta))$ where $g_{i\bar{j}} = \frac{\partial^2 L}{\partial\eta^i \partial\bar{\eta}^j}$, is postive definite on $T'M \setminus \{0\}$.

Let us write $L = F^2$. Then, the pair (M, F) is called a complex Finsler space. A Hermitian connection of $(1, 0)$ type named the Chern-Finsler Connection [1] has a special meaning in a complex Finsler space. Notationally, it is $D\Gamma N = (L^i_{jk}, 0, C^i_{jk}, 0)$, where

$$\overset{CF}{N^i_j} = g^{\overline{m}i}\frac{\partial g_{l\overline{m}}}{\partial z^j}\eta^l, L^i_{jk} = g^{\overline{m}i}\frac{\delta g_{j\overline{m}}}{\delta z^k} = \frac{\partial N^i_k}{\partial \eta^j}, C^i_{jk} = g^{\overline{m}i}\frac{\partial g_{j\overline{m}}}{\partial \eta^k}. \tag{2.2}$$

The holomorphic curvature [1, 10] of the complex Finsler space (M, F) in the direction η is

$$\kappa_F(z,\eta) = \frac{2}{L^2(z,\eta)} G(R(\chi,\overline{\chi})\chi,\overline{\chi}), \tag{2.3}$$

where G is the N-lift of the complex Finsler metric tensor $g_{i\overline{j}}$ defined by $G = g_{i\overline{j}}\, dz^i \otimes d\overline{z}^j + g_{i\overline{j}}\, \delta\eta^i \otimes \delta\overline{\eta}^j$ and R is the curvature of Chern-Finsler connection. Locally,

$$\kappa_F(z,\eta) = \frac{2}{L^2} R_{\overline{j}k}\overline{\eta}^j\eta^k, \tag{2.4}$$

where

$$R_{\overline{j}k} = -g_{l\overline{j}}\delta_{\overline{h}}\left(\overset{CF}{N^l_k}\right)\overline{\eta}^h \ \ [4]. \tag{2.5}$$

Consider a C^∞ curve $c(t)$, $t \in R$ on a complex manifold M and $(z^k(t), \eta^k(t) = dz^k/dt)$ be its extension on $T'M$. The Euler-Lagrange equations with respect to a complex Lagrangian $L = F^2$ [2, 10] are given by

$$E_i(L) \equiv \frac{\partial L}{\partial z^i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \eta^i}\right) = 0, \tag{2.6}$$

where L is considered along the curve $c(t)$ on $T'M$. The solutions of the Euler-Lagrange equations are the extremal curves with respect to the arc length.

3 Complex Finsler space with $(\gamma, |\beta|)$-metric

Differentiating (1.2) partially with respect to η^l and $\overline{\eta}^p$ and using the symmetry of $a_{i\overline{j}\,\overline{k}}$ in its indices, we obtain

$$\begin{cases} \frac{\partial\gamma}{\partial\eta^l} = \frac{a_l}{3\gamma^2},\ \frac{\partial\gamma}{\partial\overline{\eta}^p} = \frac{2a_{\overline{p}}}{3\gamma^2},\ \frac{\partial|\beta|}{\partial\eta^i} = \frac{\overline{\beta}b_i}{2|\beta|},\ \frac{\partial|\beta|}{\partial\overline{\eta}^j} = \frac{\beta b_{\overline{j}}}{2|\beta|}, \\ \frac{\partial^2\gamma}{\partial\eta^l\partial\overline{\eta}^p} = \frac{a_{l\overline{p}}}{3\gamma^2} - \frac{4a_l a_{\overline{p}}}{9\gamma^5},\ \frac{\partial^2|\beta|}{\partial\eta^i\partial\overline{\eta}^j} = \frac{b_i b_{\overline{j}}}{4|\beta|}, \end{cases} \tag{3.1}$$

where $a_l = a_{l\overline{j}\overline{k}}\,\overline{\eta}^j\,\overline{\eta}^k$, $a_{\overline{p}} = a_{l\overline{k}\overline{p}}\eta^l\,\overline{\eta}^k$ and $a_{l\overline{p}} = 2a_{l\overline{p}\overline{k}}\overline{\eta}^k$.
The function $L = F^2$ depends on z and η because of $\gamma = \gamma(z,\eta)$ and $|\beta| = |\beta(z,\eta)|$. Also, γ and β are homogeneous with respect to η, i.e. $\gamma(z,\lambda\eta) = |\lambda|\gamma(z,\eta)$ and $\beta(z,\lambda\eta) = \lambda\beta(z,\eta)$, for $\forall\ \lambda \in C$. Therefore, $L(z,\lambda\eta) = \lambda\overline{\lambda}L(z,\eta)$, for $\forall\ \lambda \in C$.
From the homogeneity property, we have the following

$$\frac{\partial\gamma}{\partial\eta^i}\eta^i = \frac{1}{3}\gamma,\ \ \frac{\partial|\beta|}{\partial\eta^i}\eta^i = \frac{1}{2}|\beta|. \tag{3.2}$$

Now,

$$L = F^2 = (\gamma + |\beta|)^2. \tag{3.3}$$

On differentiating (3.3) partially with respect to γ and $|\beta|$ respectively, we have

$$L_\gamma = 2F = L_{|\beta|},\ L_{\gamma\gamma} = 2 = L_{\gamma|\beta|} = L_{|\beta||\beta|}. \tag{3.4}$$

Again differentiating (3.3) partially with respect to η^i and $\overline{\eta}^j$ respectively, we get

$$\eta_i = \frac{\partial L}{\partial \eta^i} = \frac{2F}{3\gamma^2}a_i + \frac{F\overline{\beta}}{|\beta|}b_i;\ \overline{\eta}_j = \frac{\partial L}{\partial \overline{\eta}^j} = \frac{4F}{3\gamma^2}a_{\overline{j}} + \frac{F\beta}{|\beta|}b_{\overline{j}}. \tag{3.5}$$

Using (3.4), we conclude following relations

$$\begin{cases} \gamma L_\gamma + |\beta| L_{|\beta|} = 2L, \gamma L_{\gamma\gamma} + |\beta| L_{|\beta|} = L_\gamma, \gamma L_{\gamma|\beta|} + |\beta| L_{|\beta||\beta|} = L_{|\beta|}, \\ \gamma^2 L_{\gamma\gamma} + 2\gamma|\beta| L_{\gamma|\beta|} + |\beta|^2 L_{|\beta||\beta|} = 2L. \end{cases} \tag{3.6}$$

The fundamental metric tensor $g_{i\overline{j}}$ of the complex Randers space (M, F) is given by

$$\begin{aligned} g_{i\overline{j}} &= \frac{\partial^2 L}{\partial \eta^i \partial \overline{\eta}^j} \\ &= L_{\gamma\gamma}\frac{\partial \gamma}{\partial \eta^i}\frac{\partial \gamma}{\partial \overline{\eta}^j} + L_{\gamma|\beta|}\left(\frac{\partial \gamma}{\partial \eta^i}\frac{\partial |\beta|}{\partial \overline{\eta}^j} + \frac{\partial |\beta|}{\partial \eta^i}\frac{\partial \gamma}{\partial \overline{\eta}^j}\right) \\ &+ L_{|\beta||\beta|}\frac{\partial |\beta|}{\partial \eta^i}\frac{\partial |\beta|}{\partial \overline{\eta}^j} + L_\gamma \frac{\partial^2 \gamma}{\partial \eta^i \partial \overline{\eta}^j} + L_{|\beta|}\frac{\partial^2 |\beta|}{\partial \eta^i \partial \overline{\eta}^j}, \end{aligned} \tag{3.7}$$

where $L_\gamma = \frac{\partial L}{\partial \gamma}$, $L_{|\beta|} = \frac{\partial L}{\partial |\beta|}$, $L_{\gamma\gamma} = \frac{\partial^2 L}{\partial \gamma^2}$, $L_{|\beta||\beta|} = \frac{\partial^2 L}{\partial |\beta|^2}$ and $L_{\gamma|\beta|} = \frac{\partial^2 L}{\partial \gamma \partial |\beta|} = L_{|\beta|\gamma}$. Using (3.1) and (3.4) in (3.7), we have

$$\begin{aligned} g_{i\overline{j}} &= \frac{2F}{3\gamma^2}a_{i\overline{j}} - \frac{4}{9\gamma^5}(F + |\beta|)a_i a_{\overline{j}} + \frac{1}{2|\beta|}(F + |\beta|)b_i b_{\overline{j}} \\ &+ \frac{1}{3\gamma^2|\beta|}(\beta a_i b_{\overline{j}} + 2\overline{\beta} b_i a_{\overline{j}}). \end{aligned} \tag{3.8}$$

If we assume $\rho_0 = \frac{L_\gamma}{3\gamma^2} = \frac{2F}{3\gamma^2}$ and $\mu_0 = \frac{L_{|\beta|}}{2|\beta|} = \frac{F}{|\beta|}$, (3.5)gives

$$(\beta a_i b_{\overline{j}} + 2\overline{\beta} b_i a_{\overline{j}}) = \frac{1}{\rho_0 \mu_0}\eta_i \overline{\eta}_j - \frac{\mu_0}{\rho_0}|\beta|^2 b_i b_{\overline{j}} - \frac{2\rho_0}{\mu_0}a_i a_{\overline{j}}. \tag{3.9}$$

Substituting (3.9) in (3.8), we obtain

$$g_{i\overline{j}} = \frac{2F}{3\gamma^2}a_{i\overline{j}} - \frac{8F}{9\gamma^5}a_i a_{\overline{j}} + \frac{F}{2|\beta|}b_i b_{\overline{j}} + \frac{1}{2L}\eta_i \overline{\eta}_j. \tag{3.10}$$

This leads to

Theorem 3.1. *The fundamental metric tensor of a complex Finsler space (M, F) with $(\gamma, |\beta|)$-metric is given by* (3.10).

Next, we have the following proposition [4] given by D. Bao, S. S. Chern and Z. Shen:

Proposition 3.2. *Suppose*
1. $(Q_{i\bar{j}})$ *is a nonsingular* $n \times n$ *complex matrix with inverse* $(Q^{\bar{j}i})$,
2. C_i *and* $C_{\bar{i}} = \overline{C}_i, i = 1, 2, 3, .., n$ *are complex numbers,*
3. $C^i = Q^{\bar{j}i} C_{\bar{j}}$ *and its conjugates,* $C^2 = C^i C_i = \overline{C}^i C_{\bar{i}}; H_{i\bar{j}} = Q_{i\bar{j}} \pm C_i C_{\bar{j}}$.
Then,
(*i*) $det(H_{i\bar{j}}) = (1 \pm C^2) det(Q_{i\bar{j}})$,
(*ii*) *whenever* $1 \pm C^2 \neq 0$ *the matrix* $(H_{i\bar{j}})$ *is invertible and in this case its inverse is* $H^{\bar{j}i} = Q^{\bar{j}i} \mp \frac{1}{1 \pm C^2} C^i \overline{C}^j$.

We use the Proposition 3.2 to find the inverse and determinant of the fundamental metric tensor. (3.10) may be written as

$$g_{i\bar{j}} = \frac{2F}{3\gamma^2} \left\{ a_{i\bar{j}} - \frac{4}{3\gamma^3} a_i a_{\bar{j}} + \frac{3\gamma^2}{4|\beta|} b_i b_{\bar{j}} + \frac{3\gamma^2}{4LF} \eta_i \eta_{\bar{j}} \right\}. \tag{3.11}$$

Assuming $Q_{i\bar{j}} = a_{i\bar{j}}$ and $C_i = \frac{2}{\sqrt{3\gamma}} \frac{a_i}{\gamma}$, and applying Proposition 3.2, we find $Q^{\bar{j}i} = a^{\bar{j}i}, C^i = \frac{1}{\sqrt{3\gamma}} \frac{\eta^i}{\gamma}, C^2 = \frac{2}{3}$, where $(a^{\bar{j}i})$ is the Hermitian inverse of $(a_{i\bar{j}})$. Since $1 - C^2 = \frac{1}{3} \neq 0$, the matrix $H_{i\bar{j}} = a_{i\bar{j}} - \frac{4}{3\gamma^3} a_i a_{\bar{j}}$ is invertible with the inverse $H^{\bar{j}i} = a^{\bar{j}i} + \frac{\eta^i \overline{\eta}^j}{\gamma^3}$ and $det(H_{i\bar{j}}) = \frac{1}{3} det(a_{i\bar{j}})$.
Now, assuming $Q_{i\bar{j}} = a_{i\bar{j}} - \frac{4}{3\gamma^3} a_i a_{\bar{j}}$ and $C_i = \sqrt{\frac{3}{|\beta|} \frac{\gamma}{2}} b_i$ and applying Proposition 3.2, we obtain $Q^{\bar{j}i} = a^{\bar{j}i} + \frac{\eta^i \overline{\eta}^j}{\gamma^3}$ and $C^i = \sqrt{\frac{3}{|\beta|} \frac{\gamma}{2}} \left(b^i + \frac{\overline{\beta} \eta^i}{\gamma^3} \right)$, where $b^i = a^{\bar{j}i} b_{\bar{j}}$. Therefore $C^2 = \frac{3\gamma^2}{4|\beta|} \left(\|b\|^2 + \frac{|\beta|^2}{\gamma^3} \right)$, where $\|b\|^2 = a^{\bar{j}i} b_i b_{\bar{j}}$. Since $1 + C^2 = \frac{\sigma}{4|\beta|\gamma} \neq 0$, where $\sigma = 4|\beta|\gamma + 3\gamma^3 \|b\|^2 + 3|\beta|^2$, the inverse of $H_{i\bar{j}} = a_{i\bar{j}} - \frac{4}{3\gamma^3} a_i a_{\bar{j}} + \frac{3\gamma^2}{4|\beta|} b_i b_{\bar{j}}$ exists and is given by $H^{\bar{j}i} = a^{\bar{j}i} + \frac{\eta^i \overline{\eta}^j}{\gamma^3} - \frac{3\gamma^2}{\sigma} \left(b^i + \frac{\overline{\beta} \eta^i}{\gamma^3} \right) \left(b^{\bar{j}} + \frac{\beta \overline{\eta}^j}{\gamma^3} \right)$ and $det(H_{i\bar{j}}) = \frac{\sigma}{12|\beta|\gamma} det(a_{i\bar{j}})$.
Finally we set $Q_{i\bar{j}} = a_{i\bar{j}} - \frac{4}{3\gamma^3} a_i a_{\bar{j}} + \frac{3\gamma^2}{4|\beta|} b_i b_{\bar{j}}$ and $C_i = \sqrt{\frac{3}{LF} \frac{\gamma}{2}} \eta_i$. In view of the Proposition 3.2, $Q^{\bar{j}i} = a^{\bar{j}i} + \frac{\eta^i \overline{\eta}^j}{\gamma^3} - \frac{3\gamma^2}{\sigma} \left(b^i + \frac{\overline{\beta} \eta^i}{\gamma^3} \right) \left(b^{\bar{j}} + \frac{\beta \overline{\eta}^j}{\gamma^3} \right)$ and

$$C^i = \sqrt{\frac{3}{LF} \frac{\gamma}{2}} \left\{ a^{\bar{j}i} + \frac{\eta^i \overline{\eta}^j}{\gamma^3} - \frac{3\gamma^2}{\sigma} \left(b^i + \frac{\overline{\beta} \eta^i}{\gamma^3} \right) \left(b^{\bar{j}} + \frac{\beta \overline{\eta}^j}{\gamma^3} \right) \right\} \overline{\eta}_j. \tag{3.12}$$

From (3.5), we get

$$\begin{cases} a^{\bar{j}i} \overline{\eta}_j = \frac{2F}{3\gamma^2} \eta^i + \frac{F\beta}{|\beta|} b^i, b^{\bar{j}} \overline{\eta}_j = \frac{2F}{3\gamma^2} \beta + \frac{F\beta}{|\beta|} \|b\|^2, b^i \eta_i = \frac{F}{3\gamma^2} \overline{\beta} + \frac{F\overline{\beta}}{|\beta|} \|b\|^2, \\ \eta^i \eta_i = \frac{2F}{3} \gamma + F|\beta|, \overline{\eta}^j \overline{\eta}_j = \frac{4F}{3} \gamma + F|\beta|. \end{cases} \tag{3.13}$$

Using (3.13) in (3.12), we have

$$C^i = \sqrt{\frac{3}{LF}} \frac{F}{\gamma} \left\{ \eta^i \left(1 - \frac{|\beta|^2}{\sigma} \right) - \frac{\beta \gamma^3 b^i}{\sigma} \right\}. \tag{3.14}$$

Therefore $C^2 = 1 + \frac{\gamma |\beta|^2}{2F\sigma}$. Since $1 + C^2 \neq 0$, $H_{i\bar{j}} = a_{i\bar{j}} - \frac{4}{3\gamma^3} a_i a_{\bar{j}} + \frac{3\gamma^2}{4|\beta|} b_i b_{\bar{j}} + \frac{3\gamma^2}{4LF} \eta_i \overline{\eta}_j$ is invertible with the inverse

$$H^{\bar{j}i} = a^{\bar{j}i} + A \eta^i \overline{\eta}^j + B b^i b^{\bar{j}} + C(\beta b^i \overline{\eta}^j + \overline{\beta} b^{\bar{j}} \eta^i), \tag{3.15}$$

where

$$\begin{cases} A = \frac{4|\beta|\sigma^2 - 2\gamma\sigma^2 - 12|\beta|^3\sigma - 9\gamma|\beta|^4 + \gamma|\beta|^2\sigma}{\gamma^3\sigma(4F\sigma+\gamma|\beta|^2)}, B = -\frac{3\gamma^3}{\sigma}\frac{(4F\sigma+3\gamma|\beta|^2)}{(4F\sigma+\gamma|\beta|^2)}, \\ C = -\frac{3(4F\sigma+3\gamma|\beta|^2)+6\gamma\sigma}{\sigma(4F\sigma+\gamma|\beta|^2)}. \end{cases} \tag{3.16}$$

Also, the determinant of $H_{i\bar{j}}$ is

$$det(H_{i\bar{j}}) = \frac{(4F\sigma + \gamma|\beta|^2)}{24F|\beta|\gamma} det(a_{i\bar{j}}). \tag{3.17}$$

From (3.11), $g_{i\bar{j}} = \frac{2F}{3\gamma^2} H_{i\bar{j}}$, the inverse of the fundamental metric tensor is given by

$$g^{\bar{j}i} = \frac{3\gamma^2}{2F} H^{\bar{j}i}, \tag{3.18}$$

where $H^{\bar{j}i}$ is given by (3.15).
Also, the determinant of the fundamental metric tensor is given by

$$det(g_{i\bar{j}}) = \left(\frac{2F}{3\gamma^2}\right)^n det(H_{i\bar{j}}), \tag{3.19}$$

where $det(H_{i\bar{j}})$ is given by (3.17). Thus, we have

Theorem 3.3. *The inverse and determinant of the fundamental metric tensor of a complex Finsler space* (M, F) *with* $(\gamma, |\beta|)$*-metric are given by* (3.18) *and* (3.19) *respectively.*

Next, we define the complex angular metric tensor of the complex Finsler space (M, F) with $(\gamma, |\beta|)$-metric as

$$\begin{aligned} k_{i\bar{j}} &= \frac{\partial^2 F}{\partial\eta^i \partial\overline{\eta}^j} \\ &= F_{\gamma\gamma}\frac{\partial\gamma}{\partial\eta^i}\frac{\partial\gamma}{\partial\overline{\eta}^j} + F_{\gamma|\beta|}\left(\frac{\partial\gamma}{\partial\eta^i}\frac{\partial|\beta|}{\partial\overline{\eta}^j} + \frac{\partial|\beta|}{\partial\eta^i}\frac{\partial\gamma}{\partial\overline{\eta}^j}\right) \\ &\quad + F_{|\beta||\beta|}\frac{\partial|\beta|}{\partial\eta^i}\frac{\partial|\beta|}{\partial\overline{\eta}^j} + F_\gamma\frac{\partial^2\gamma}{\partial\eta^i\partial\overline{\eta}^j} + F_{|\beta|}\frac{\partial^2|\beta|}{\partial\eta^i\partial\overline{\eta}^j}, \end{aligned} \tag{3.20}$$

where $F_\gamma = \frac{\partial F}{\partial\gamma}, F_{|\beta|} = \frac{\partial F}{\partial|\beta|}, F_{\gamma\gamma} = \frac{\partial^2 F}{\partial\gamma^2}, F_{|\beta||\beta|} = \frac{\partial^2 F}{\partial|\beta|^2}$ and $F_{\gamma|\beta|} = \frac{\partial^2 F}{\partial\gamma\partial|\beta|} = F_{|\beta|\gamma}$.
On differentiating F partially with respect to γ and $|\beta|$ respectively, we have

$$F_\gamma = 1 = F_{|\beta|},\ F_{\gamma\gamma} = 0 = F_{\gamma|\beta|} = F_{|\beta||\beta|}. \tag{3.21}$$

On substituting (3.1) and (3.21) in (3.20), we obtain

$$k_{i\bar{j}} = \frac{1}{3\gamma^2} h_{i\bar{j}} + \frac{1}{4|\beta|} b_i b_{\bar{j}}, \tag{3.22}$$

where

$$h_{i\bar{j}} = a_{i\bar{j}} - \frac{4}{3\gamma^3} a_i a_{\bar{j}}. \tag{3.23}$$

To find out the inverse of the complex angular metric tensor, we will apply Proposition 3.2. From (3.22), $k_{i\bar{j}} = \frac{1}{3\gamma^2}\left\{a_{i\bar{j}} - \frac{4}{3\gamma^3}a_i a_{\bar{j}} + \frac{3\gamma^2}{4|\beta|}b_i b_{\bar{j}}\right\}$. Assuming $Q_{i\bar{j}} = a_{i\bar{j}}$ and $C_i = \frac{2}{\sqrt{3\gamma}}\frac{a_i}{\gamma}$, and applying Proposition 3.2, we get $Q^{\bar{j}i} = a^{\bar{j}i}, C^i = \frac{1}{\sqrt{3\gamma}}\frac{\eta^i}{\gamma}$ and $C^2 = \frac{2}{3}$. Since $1 - C^2 = \frac{1}{3} \neq 0$, the matrix $H_{i\bar{j}} = a_{i\bar{j}} - \frac{4}{3\gamma^3}a_i a_{\bar{j}}$ is invertible with the inverse $H^{\bar{j}i} = a^{\bar{j}i+} + \frac{\eta^i\overline{\eta}^j}{\gamma^3}$ and $det(H_{i\bar{j}}) = \frac{1}{3}det(a_{i\bar{j}})$. Taking $Q_{i\bar{j}} = a_{i\bar{j}} - \frac{4}{3\gamma^3}a_i a_{\bar{j}}$ and $C_i = \sqrt{\frac{3}{|\beta|}}\frac{\gamma}{2}b_i$, Proposition 3.2 gives $Q^{\bar{j}i} = a^{\bar{j}i+} + \frac{\eta^i\overline{\eta}^j}{\gamma^3}$ and $C^i = \sqrt{\frac{3}{|\beta|}}\frac{\gamma}{2}\left(b^i + \frac{\overline{\beta}\eta^i}{\gamma^3}\right)$. Therefore $C^2 = \frac{3\gamma^2}{4|\beta|}\left(\|b\|^2 + \frac{|\beta|^2}{\gamma^3}\right)$. Since $1 + C^2 \neq 0$, the inverse of $H_{i\bar{j}} = a_{i\bar{j}} - \frac{4}{3\gamma^3}a_i a_{\bar{j}} + \frac{3\gamma^2}{4|\beta|}b_i b_{\bar{j}}$ exists and is given by

$$H^{\bar{j}i} = a^{\bar{j}i} + \frac{\eta^i\overline{\eta}^j}{\gamma^3} - \frac{3\gamma^3}{\sigma}\left(b^i + \frac{\overline{\beta}\eta^i}{\gamma^3}\right)\left(b^{\bar{j}} + \frac{\beta\overline{\eta}^j}{\gamma^3}\right) \tag{3.24}$$

and

$$det(H_{i\bar{j}}) = \frac{\sigma}{12|\beta|\gamma}det(a_{i\bar{j}}). \tag{3.25}$$

Since $k_{i\bar{j}} = \frac{1}{3\gamma^2}H_{i\bar{j}}$, the inverse of the angular metric tensor $k_{i\bar{j}}$ is given by

$$k^{\bar{j}i} = 3\gamma^2 H^{\bar{j}i}, \tag{3.26}$$

where $H^{\bar{j}i}$ is given by (3.24).
Also, the determinant of the angular metric tensor $k_{i\bar{j}}$ is

$$det(k_{i\bar{j}}) = \frac{\sigma}{4(3\gamma^2)^{n+1}|\beta|}det(a_{i\bar{j}}). \tag{3.27}$$

Thus, we have

Theorem 3.4. *The inverse and determinant of the complex angular metric tensor of a complex Finsler space (M, F) with $(\gamma, |\beta|)$- metric are given by* (3.26) *and* (3.27) *respectively.*

4 Connection coefficients and curvature

The Chern-Finsler connection coefficients (*c.n.c.*) of a complex Finsler space (M, F) with $(\gamma, |\beta|)$- metric is defined by

$$\overset{CF}{N^i_j} = g^{\overline{m}i}\frac{\partial g_{l\overline{m}}}{\partial z^j}\eta^l = g^{\overline{m}i}\frac{\partial\overline{\eta}_m}{\partial z^j}. \tag{4.1}$$

Differentiating (3.5) with respect to z^j, we have

$$\begin{aligned}\frac{\partial\overline{\eta}_m}{\partial z^j} &= \frac{1}{2|\beta|}\left(\frac{2|\beta|}{3\gamma^2}\frac{\partial a_{i\bar{l}\bar{k}}}{\partial z^j}\eta^i\overline{\eta}^l\overline{\eta}^k + \beta\frac{\partial b_{\bar{l}}}{\partial z^j}\overline{\eta}^l + \overline{\beta}\frac{\partial b_i}{\partial z^j}\eta^i\right)\\ &\times\left(\frac{4a_{\overline{m}}}{3\gamma^2} + \frac{\beta b_{\overline{m}}}{|\beta|}\right) + F\left\{\frac{4}{3\gamma^2}\frac{\partial a_{i\bar{l}\overline{m}}}{\partial z^j}\eta^i\overline{\eta}^l - \frac{8}{9\gamma^5}a_{\overline{m}}\frac{\partial a_{i\bar{l}\bar{k}}}{\partial z^j}\eta^i\overline{\eta}^l\overline{\eta}^k\right.\\ &\left.+ \frac{b_{\overline{m}}}{2|\beta|}\frac{\partial b_i}{\partial z^j}\eta^i + \frac{\beta}{|\beta|}\frac{\partial b_{\overline{m}}}{\partial z^j} - \frac{\beta}{2\overline{\beta}|\beta|}b_{\overline{m}}\frac{\partial b_{\bar{l}}}{\partial z^j}\overline{\eta}^l\right\}.\end{aligned} \tag{4.2}$$

Substituting (3.18) and (4.2) in (4.1), we have

$$\overset{CF}{N}{}^i_j = \frac{3\gamma^2}{4F|\beta|}\Big\{a^{\overline{m}i} + A\eta^i\overline{\eta}^m + Bb^ib^{\overline{m}} + (\beta b^i\overline{\eta}^m + \overline{\beta}b^{\overline{m}}\eta^i)\Big\} \times \Big\{\Big(\frac{2|\beta|}{3\gamma^2}\frac{\partial a_{p\overline{l}k}}{\partial z^j}\eta^p\overline{\eta}^l\overline{\eta}^k + \beta\frac{\partial b_{\overline{l}}}{\partial z^j}\eta^l + \overline{\beta}\frac{\partial b_p}{\partial z^j}\eta^p\Big)\Big(\frac{4a_{\overline{m}}}{3\gamma^2} + \frac{\beta b_{\overline{m}}}{|\beta|}\Big) + F\Big(\frac{4}{3\gamma^2}\frac{\partial a_{p\overline{l}\overline{m}}}{\partial z^j}\eta^p\overline{\eta}^l - \frac{8}{9\gamma^5}a_{\overline{m}}\frac{\partial a_{p\overline{l}\overline{k}}}{\partial z^j}\eta^p\overline{\eta}^l\overline{\eta}^k + \frac{b_{\overline{m}}}{2|\beta|}\frac{\partial b_p}{\partial z^j}\eta^p + \frac{\beta}{|\beta|}\frac{\partial b_{\overline{m}}}{\partial z^j} - \frac{\beta}{2\overline{\beta}|\beta|}b_{\overline{m}}\frac{\partial b_{\overline{l}}}{\partial z^j}\overline{\eta}^l\Big)\Big\}. \tag{4.3}$$

Next, we consider the following complex Cartan tensors [2]

$$C_{j\overline{h}k} = \frac{\partial g_{j\overline{h}}}{\partial\eta^k} = \frac{\partial g_{j\overline{h}}}{\partial\gamma}\frac{\partial\gamma}{\partial\eta^k} + \frac{\partial g_{j\overline{h}}}{\partial|\beta|}\frac{\partial|\beta|}{\partial\eta^k}. \tag{4.4}$$

Differentiating (3.10) with respect to η^k and using (3.1) in (4.4), we have

$$\begin{aligned} C_{j\overline{h}k} = {} & \frac{1}{3\gamma^2}\left\{\frac{\overline{\beta}}{|\beta|}b_k - \frac{2(F+|\beta|)}{3\gamma^3}a_k\right\}a_{j\overline{h}} \\ & + \frac{4}{27\gamma^8F}\left(5\gamma^2 + 10|\beta|^2 - 16|\beta|\gamma\right)a_ka_ja_{\overline{h}} \\ & + \frac{1}{4F\beta|\beta|}\left(\beta|\beta| - F^2\right)b_kb_jb_{\overline{h}} \\ & - \frac{1}{9F|\beta|\gamma^4}\left\{\frac{2\overline{\beta}}{F}\left(F+|\beta|\right)b_ka_ja_{\overline{h}} + \frac{\beta}{\gamma}(2F+\gamma)a_ka_jb_{\overline{h}} + 2\overline{\beta}a_ka_{\overline{h}}b_j\right\} \\ & + \frac{1}{3F\gamma}\left\{\frac{1}{2|\beta|}a_kb_jb_{\overline{h}} + \frac{(|\beta|^2 - F)}{\gamma\beta|\beta|}b_kb_ja_{\overline{h}} + \frac{1}{2\gamma}b_ka_jb_{\overline{h}}\right\}. \end{aligned} \tag{4.5}$$

Also, the vertical coefficients of Chern- Finsler connections are defined as

$$C^i_{jk} = g^{\overline{m}i}\frac{\partial g_{j\overline{m}}}{\partial\eta^k} = g^{\overline{m}i}C_{j\overline{m}k}. \tag{4.6}$$

Substituting (3.18) and (4.5) in (4.6), we obtain

$$(4.7)\qquad \begin{aligned} C^i_{jk} = {} & \frac{3\gamma^2}{2F}\left\{a^{\overline{h}i} + A\eta^i\overline{\eta}^h + Bb^ib^{\overline{h}} + C(\beta b^i\overline{\eta}^h + \overline{\beta}b^{\overline{h}}\eta^i)\right\} \\ & \times \Bigg[\frac{1}{3\gamma^2}\left\{\frac{\overline{\beta}}{|\beta|}b_k - \frac{2(F+|\beta|)}{3\gamma^3}a_k\right\}a_{j\overline{h}} \\ & + \frac{4}{27\gamma^8 F}\left(5\gamma^2 + 10|\beta|^2 - 16|\beta|\gamma\right)a_k a_j a_{\overline{h}} \\ & + \frac{1}{4F\beta|\beta|}\left(\beta|\beta| - F^2\right)b_k b_j b_{\overline{h}} \\ & - \frac{1}{9F|\beta|\gamma^4}\left(\frac{2\overline{\beta}}{F}(F+|\beta|)\,b_k a_j a_{\overline{h}} + \frac{\beta}{\gamma}(2F+\gamma)a_k a_j b_{\overline{h}} + 2\overline{\beta}a_k a_{\overline{h}} b_j\right) \\ & + \frac{1}{3F\gamma}\left\{\frac{1}{2\beta}a_k b_{\overline{h}} b_j + \frac{(|\beta|^2 - F)}{\gamma\beta|\beta|}b_k b_j a_{\overline{h}} + \frac{1}{2\gamma}a_j b_{\overline{h}} b_k\right\}\Bigg]. \end{aligned}$$

This leads to

Theorem 4.1. *The coefficients of Chern - Finsler connection, complex Cartan tensors and the vertical coefficients of Chern-Finsler connections of a complex Finsler space (M, F) with $(\gamma, |\beta|)$-metric are given by* (4.3), (4.5)*and* (4.7) *respectively.*

The holomorphic curvature (in the direction of η) of the complex Finsler space (M, F) with $(\gamma, |\beta|)$- metric is defined by (2.4).
Substituting values of $g_{l\overline{j}}$ and $\overset{CF}{N^l_k}$ from (3.10) and (4.3) in (2.5), we have

$$(4.8)\qquad \begin{aligned} R_{\overline{j}k} = {} & -\frac{3\gamma^2}{4F|\beta|}\left\{\frac{2F}{3\gamma^2}a_{l\overline{j}} - \frac{8F}{9\gamma^5}a_l a_{\overline{j}} + \frac{F}{2|\beta|}b_l b_{\overline{j}} + \frac{1}{2L}\eta_l\overline{\eta}_j\right\}\delta_{\overline{h}} \\ & \times\left\{a^{\overline{p}l} + A\eta^l\overline{\eta}^p + Bb^l b^{\overline{p}} + C(\beta b^l\overline{\eta}^p + \overline{\beta}b^{\overline{p}}\eta^l)\right\} \\ & \times\Bigg\{\left(\frac{2|\beta|}{3\gamma^2}\frac{\partial a_{t\overline{s}\overline{r}}}{\partial z^k}\eta^t\overline{\eta}^s\overline{\eta}^r + \beta\frac{\partial b_{\overline{s}}}{\partial z^k}\overline{\eta}^s + \overline{\beta}\frac{\partial b_t}{\partial z^k}\eta^t\right)\left(\frac{4a_{\overline{p}}}{3\gamma^2} + \frac{\beta b_{\overline{p}}}{|\beta|}\right) \\ & + F\left(\frac{4}{3\gamma^2}\frac{\partial a_{t\overline{s}\overline{p}}}{\partial z^k}\eta^t\overline{\eta}^s - \frac{8}{9\gamma^5}a_{\overline{p}}\frac{\partial a_{t\overline{s}\overline{r}}}{\partial z^k}\eta^t\overline{\eta}^s\overline{\eta}^r\right) \\ & + F\left(\frac{b_{\overline{p}}}{2|\beta|}\frac{\partial b_t}{\partial z^k}\eta^t + \frac{\beta}{|\beta|}\frac{\partial b_{\overline{p}}}{\partial z^k} - \frac{\beta}{2\overline{\beta}|\beta|}b_{\overline{p}}\frac{\partial b_{\overline{s}}}{\partial z^k}\eta^{\overline{s}}\right)\Bigg\}\overline{\eta}^h. \end{aligned}$$

Thus, we have

Theorem 4.2. *The holomorphic curvature of a complex Finsler space (M, F) with $(\gamma, |\beta|)$-metric is given by* (2.4) *together with* (4.8).

5 Euler-Lagrange equations

The Lagrangian L given by (3.3), $L = F^2 = (\gamma + |\beta|)^2$ depends on the parameter $t \in R$ by means of $z^k(t)$ and $\eta^k(t)$ and their conjugates. Differentiating (3.3) with respect to t, we have

$$\frac{dL}{dt} = \frac{\partial L}{\partial z^i}\eta^i + \frac{\partial L}{\partial \eta^i}\frac{d\eta^i}{dt} + \frac{\partial L}{\partial \overline{z}^i}\overline{\eta}^i + \frac{\partial L}{\partial \overline{\eta}^i}\frac{d\overline{\eta}^i}{dt}. \tag{5.1}$$

Since, L is homogeneous of degree one in η^i, $\frac{\partial L}{\partial \eta^i}\eta^i = L$, on differentiating with respect to t, we get

$$\frac{dL}{dt} = \frac{d}{dt}\left(\frac{\partial L}{\partial \eta^i}\right)\eta^i + \frac{\partial L}{\partial \eta^i}\frac{d\eta^i}{dt}. \tag{5.2}$$

But $E_i(L) = 0$, implies $\frac{d}{dt}\left(\frac{\partial L}{\partial \eta^i}\right) = \frac{\partial L}{\partial z^i}$ along the extrmal curve $c(t)$ on $T'M$. Therefore (5.2) gives

$$\frac{dL}{dt} = \frac{\partial L}{\partial z^i}\eta^i + \frac{\partial L}{\partial \eta^i}\frac{d\eta^i}{dt}. \tag{5.3}$$

By conjugation (as t and L are real valued functions), we have

$$\frac{dL}{dt} = \frac{\partial L}{\partial \overline{z}^i}\overline{\eta}^i + \frac{\partial L}{\partial \overline{\eta}^i}\frac{d\overline{\eta}^i}{dt}. \tag{5.4}$$

Adding (5.3) and (5.4) and using in (5.1), we conclude $\frac{dL}{dt} = 0$, which further implies $\frac{dF}{dt}$. This leads to

Theorem 5.1. *For the complex Finsler space (M, F) with $(\gamma, |\beta|)$-metric, $\frac{dL}{dt} = 0 = \frac{dF}{dt}$ along an extremal curve $c(t)$ on $T'M$.*

Now, we derive the Euler-Lagrange equations for $L = F^2 = (\gamma + |\beta|)^2$. Differentiating L with respect to z^i and η^i respectively, we have

$$\frac{\partial L}{\partial z^i} = L_\gamma \frac{\partial \gamma}{\partial z^i} + L_{|\beta|}\frac{\partial |\beta|}{\partial z^i} = 2F\left(\frac{\partial \gamma}{\partial z^i} + \frac{\partial |\beta|}{\partial z^i}\right) \tag{5.5}$$

$$\frac{\partial L}{\partial \eta^i} = L_\gamma \frac{\partial \gamma}{\partial \eta^i} + L_{|\beta|}\frac{\partial |\beta|}{\partial \eta^i} = 2F\left(\frac{\partial \gamma}{\partial \eta^i} + \frac{\partial |\beta|}{\partial \eta^i}\right). \tag{5.6}$$

Further differentiation of (5.6) with respect to t implies

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial L}{\partial \eta^i}\right) &= 2\frac{dF}{dt}\left(\frac{\partial \gamma}{\partial \eta^i} + \frac{\partial |\beta|}{\partial \eta^i}\right)\\ &\quad + 2F\left\{\frac{d}{dt}\left(\frac{\partial \gamma}{\partial \eta^i}\right) + \frac{d}{dt}\left(\frac{\partial |\beta|}{\partial \eta^i}\right)\right\}.\end{aligned} \tag{5.7}$$

In view of theorem 5.1, (5.7) gives

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \eta^i}\right) = 2F\left\{\frac{d}{dt}\left(\frac{\partial \gamma}{\partial \eta^i}\right) + \frac{d}{dt}\left(\frac{\partial |\beta|}{\partial \eta^i}\right)\right\}. \tag{5.8}$$

Now,

$$E_i(\gamma) = \frac{1}{3\gamma^2} E_i(\gamma^3) + \frac{1}{\gamma^2} \frac{d\gamma^2}{dt} \frac{\partial \gamma}{\partial \eta^i}, \tag{5.9}$$

$$E_i(|\beta|) = \frac{1}{2|\beta|} E_i(|\beta|^2) + \frac{1}{|\beta|} \frac{d|\beta|}{dt} \frac{\partial |\beta|}{\partial \eta^i} \tag{5.10}$$

respectively. For the Lagrangian $L = F^2 = (\gamma + |\beta|)^2$, (2.6) gives

$$E_i(L) \equiv 2F \{E_i(\gamma) + E_i(|\beta|)\} = 0. \tag{5.11}$$

Substituting values of $E_i(\gamma)$ and $E_i(|\beta|)$ from (5.9) and (5.10) in (5.11), we obtain

$$2|\beta| E_i(\gamma^3) + 3\gamma^2 E_i(|\beta|^2) + 6|\beta| \frac{d\gamma^2}{dt} \frac{\partial \gamma}{\partial \eta^i} + 6\gamma^2 \frac{d|\beta|}{dt} \frac{\partial |\beta|}{\partial \eta^i} = 0. \tag{5.12}$$

Thus, we have

Theorem 5.2. *The Euler-Lagrange equations of the complex Finsler space* (M, F) *with* $(\gamma, |\beta|)$*-metric are given by* (5.12).

References

[1] M. Abate and A. Patrizio, *Finsler Metric-A Global Approach with Applications to Geometric Function Theory, Lecture Notes in Mathematics, 1591*, Springer-Verlag, Berlin, (1994).

[2] T. Aikou, *Projective Flatness of Complex Finsler Metrics*, Publ. Math. Debrecen, 63(3) (2003), 343-362.

[3] N. Aldea and G. Munteanu, *On complex Finsler spaces with Randers metric*, J. Korean Math. Soc. 46(5) (2009), 949-966.

[4] D. Bao, S. S. Chern and Z. Shen, *An Introduction to Riemann Finsler Geometry, Graduate Texts in Mathematics, 200*, Springer-Verlag, New York, 2000.

[5] D. Bao, C. Robles and Z. Shen, *Zermelo Navigation on Riemannian Manifolds*, J. Differential Geometry, 66(3) (2004), 377-435.

[6] V. K. Kropina, *Projective Two-dimentional Finsler Spaces with Special Metric* (Russian), Trudy Sem. Vektor. Tenzor. Anal. 11 (1961), 277-292.

[7] S. Kumari and P. N. Pandey, *On a complex Randers Spaces*, Nat. Acad. Sci. Lett., NASI, (Accepted for Publication) (2017).

[8] M. Matsumoto, *Randers spaces of constant curvature*, Rep. Math. Phy., 28(2) (1989), 249-261.

[9] M. Matsumoto and S. Numata, *On Finsler spaces with cubic metric*, Tensor N.S. 33 (1979), 153-162.

[10] G. Munteanu, *Complex spaces in Finsler, Lagrange and Hamilton geometries*, Kluwer Academic Publishers, Dordrecht, 2004.

[11] T. N. Pandey and V. K. Chaubey, *Theory of Finsler spaces with* (γ, β)*-metrics*, Bulletin of the Transilvania University of Brasov, Series III: Mathematics, Informatics, Physics 4(53)(2) (2011), 43-56.

[12] V. V. Wagner, *Two dimentional space with the metric defined by a cubic differential form* (Russian and English), Abh. Tscherny, Staatuniv. Saratow 1 (1938), 29-40.

[13] J. M. Wegener, *Untersuchung Der Zwei-und Dreidimenionlen Finslerschen raumemit der Grundform* $L = (a_{ikl}x'^i x'^k x'^l)^{1/3}$, Akad. Wetensch. Proc. 38 (1935), 949-955.

[14] H. Yasuda and H. Shimada, *On Randers Spaces of Scalar Curvature*, Rep. Mathematical Phys., 11(3) (1977), 347-360.

Authors' address:

Sweta Kumari and Paras Nath Pandey
Department of Mathematics,
University of Allahabad, Allahabad-211002, India.
E-mail: jhasweta735@gmail.com, pnpiaps@gmail.com