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SHAPE OPTIMIZATION FOR STOKES PROBLEM WITH THRESHOLD SLIP

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Abstract. We study the Stokes problems in a bounded planar domain Ω with a friction type boundary condition that switches between a slip and no-slip stage. Our main goal is to determine under which conditions concerning the smoothness of Ω solutions to the Stokes system with the slip boundary conditions depend continuously on variations of Ω. Having this result at our disposal, we easily prove the existence of a solution to optimal shape design problems for a large class of cost functionals. In order to release the impermeability condition, whose numerical treatment could be troublesome, we use a penalty approach. We introduce a family of shape optimization problems with the penalized state relations. Finally we establish convergence properties between solutions to the original and modified shape optimization problems when the penalty parameter tends to zero.

Keywords: Stokes problem; friction boundary condition; shape optimization

MSC 2010: 49Q10, 76D07

1. Introduction

An important part of mathematical modeling of fluid flow is the proper choice of boundary conditions. Solid impermeable walls are traditionally described by the no-slip condition, i.e.,

$$\boldsymbol{u} = \mathbf{0},$$

where $\boldsymbol{u}$ denotes the velocity field. In some applications, however, one can observe a tangential velocity along the surface. In this case it is more realistic to use some

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kind of the slip condition. Navier [14] proposed the condition

$$\boldsymbol{u}_\tau = -\lambda \boldsymbol{\sigma}_\tau, \quad \lambda \geqslant 0,$$

saying that the tangential velocity $\boldsymbol{u}_\tau$ should be proportional to the shear stress $\boldsymbol{\sigma}_\tau$. Relations of this type are often used especially in non-Newtonian fluid mechanics, see e.g. [13], [4].

In this paper we introduce a system with a friction-type condition, which switches between a slip and no-slip stage depending on the magnitude of the shear stress. Due to its non-smoothness, the weak formulation of the considered problem leads to a variational inequality. To demonstrate the difficulties arising from this fact and still to keep ideas clear, we consider the Stokes problem in a planar domain Ω.

Problems involving friction-type boundary conditions have been analysed e.g. in [6], [7], [15]. The main goal of this paper is to study under which conditions concerning the smoothness of Ω solutions to the Stokes problem with threshold slip depend continuously on variations of Ω. This is the basic property enabling us to prove the existence of optimal shapes for a large class of optimal shape design problems.

It should be stressed that domain dependence of solutions subject to slip boundary conditions is more delicate than in the case of no-slip. In particular, the control-to-state mapping for problems with slip boundary conditions can be discontinuous for some sequences of equi-Lipschitz domains [1], which cannot happen when no slip is considered. It is also known that uniform $C^{1,1}$ regularity of boundary perturbations is sufficient for continuous dependence of solutions subject to Navier's slip condition [17]. We refer to [3] for more details on this subject.

The slip conditions bring another difficulty also for the numerical treatment. On polygonal computational domains the impermeability condition cannot be applied directly due to insufficient approximation of the normal vector. One possible remedy is to use a penalty approach [12]. We introduce a family of shape optimization problems with the penalized states and establish mutual relations between solutions to the original and modified optimization problems when the penalty parameter tends to zero.

The paper is organized as follows: In the next section we present the fluid flow model and define a class of shape optimization problems. The domain dependence of solutions to the state problem is analysed in Section 3. In Section 4 we define a family of shape optimization problems governed by the Stokes system with threshold slip but with a penalized form of the impermeability condition. Discretizations of these problems together with the convergence analysis are presented in Section 5.

Throughout the paper, the following notation will be used: $H^k(Q)$, $k \geqslant 0$ integer, stands for the classical Sobolev space of functions which are together with their generalized derivatives up to order k square integrable in Q ($H^0(Q) := L^2(Q)$) with the norm denoted by $\|\cdot\|_{k,Q}$. For the norm in $L^\infty(Q)$ we use the notation $\|\cdot\|_{\infty,Q}$. Finally, c denotes a generic, positive constant. To emphasize that c depends on a particular parameter p, we shall write $c := c(p)$.

2. Formulation of the problem

Let us consider the Stokes problem in a bounded domain $\Omega \subset \mathbb{R}^2$ with Lipschitz boundary $\partial\Omega$. The slip boundary conditions are prescribed on a part of the boundary S and the no-slip condition on $\Gamma = \partial\Omega \setminus \overline{S}$:

$$-\Delta \boldsymbol{u} + \nabla p = \boldsymbol{f} \text{ in } \Omega, \tag{2.1a}$$

$$\operatorname{div} \boldsymbol{u} = 0 \text{ in } \Omega, \tag{2.1b}$$

$$\boldsymbol{u} = \boldsymbol{0} \text{ on } \Gamma, \tag{2.1c}$$

$$u_\nu = 0 \text{ on } S, \tag{2.1d}$$

$$\|\boldsymbol{\sigma}_\tau\| \leqslant g \text{ on } S, \tag{2.1e}$$

$$\boldsymbol{u}_\tau \neq \boldsymbol{0} \Rightarrow \|\boldsymbol{\sigma}_\tau\| = g \;\&\; \exists \lambda \geqslant 0\colon\; \boldsymbol{u}_\tau = -\lambda \boldsymbol{\sigma}_\tau \text{ on } S. \tag{2.1f}$$

Here $\boldsymbol{u} = (u_1, u_2)$ is the velocity field, p is the pressure, and $\boldsymbol{f}$ is the external force. Further, $\boldsymbol{\nu}$, $\boldsymbol{\tau}$ denote the unit outward normal and tangential vector to $\partial\Omega$, respectively. If $\boldsymbol{a} \in \mathbb{R}^2$ is a vector, then $a_\nu := \boldsymbol{a} \cdot \boldsymbol{\nu}$, $\boldsymbol{a}_\tau := \boldsymbol{a} - a_\nu \boldsymbol{\nu}$ are its normal component and the tangential part on $\partial\Omega$, respectively. The Euclidean norm of $\boldsymbol{a}$ is denoted by $\|\boldsymbol{a}\|$. Finally, $\boldsymbol{\sigma}_\tau := (\partial \boldsymbol{u}/\partial \boldsymbol{\nu})_\tau$ stands for the shear stress and $g > 0$ a.e. on S is a given slip bound. By the classical solution of (2.1) we mean any couple of sufficiently smooth functions $(\boldsymbol{u}, p)$ satisfying the differential equations and the boundary conditions in (2.1).

To give the weak formulation of (2.1) we shall need the following function spaces:

$$V(\Omega) = \{\boldsymbol{v} \in (H^1(\Omega))^2;\; \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma,\; v_\nu = 0 \text{ on } S\}, \tag{2.2}$$

$$V_{\mathrm{div}}(\Omega) = \{\boldsymbol{v} \in V(\Omega);\; \operatorname{div} \boldsymbol{v} = 0 \text{ a.e. in } \Omega\}, \tag{2.3}$$

$$L^2_0(\Omega) = \Bigl\{q \in L^2(\Omega);\; \int_\Omega q = 0\Bigr\}. \tag{2.4}$$

The weak formulation of (2.1) reads as follows:

($\mathcal{P}$) Find $(\boldsymbol{u}, p) \in V(\Omega) \times L^2_0(\Omega)$ such that

$$\forall \boldsymbol{v} \in V(\Omega)\colon\; a(\boldsymbol{u}, \boldsymbol{v} - \boldsymbol{u}) - b(\boldsymbol{v} - \boldsymbol{u}, p) + j(\boldsymbol{v}_\tau) - j(\boldsymbol{u}_\tau) \geqslant (\boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u})_{0,\Omega},$$

$$\forall q \in L^2_0(\Omega)\colon\; b(\boldsymbol{u}, q) = 0,$$

where

$$a(\boldsymbol{u},\boldsymbol{v}) = \int_\Omega \nabla\boldsymbol{u}\colon \nabla\boldsymbol{v} := \int_\Omega \nabla u_i \cdot \nabla v_i, \tag{2.5a}$$

$$b(\boldsymbol{v},q) = \int_\Omega q \operatorname{div}\boldsymbol{v}, \tag{2.5b}$$

$$j(\boldsymbol{\varphi}) = \int_S g\|\boldsymbol{\varphi}\|. \tag{2.5c}$$

R e m a r k 1. Since we consider a two-dimensional case, we have that $\|\boldsymbol{v}_\tau\| = |\boldsymbol{v}\cdot\boldsymbol{\tau}|$ on S.

The following existence and uniqueness result is known [6].

Theorem 1. *Let $\boldsymbol{f} \in (L^2(\Omega))^2$, $g \in L^\infty(S)$, $g > 0$ a.e. on S. Then $(\mathcal{P})$ has a unique solution $(\boldsymbol{u},p)$ and*

$$\|\nabla\boldsymbol{u}\|_{0,\Omega} + \|p\|_{0,\Omega} \leqslant c(\|\boldsymbol{f}\|_{0,\Omega} + \|g\|_{\infty,S}), \tag{2.6}$$

where c is a positive constant which does not depend on $\boldsymbol{f}$ and g.

Up to now, the domain Ω was given. From now on, we shall consider a specific family of domains, namely

$$\mathcal{O} = \{\Omega(\alpha);\ \alpha \in \mathcal{U}_{\rm ad}\},$$

where (see Figure 1)

$$\Omega(\alpha) = \{(x_1,x_2);\ x_1 \in (0,1),\ x_2 \in (\alpha(x_1),\gamma)\}, \tag{2.7}$$

$$\mathcal{U}_{\rm ad} = \{\alpha \in C^{1,1}([0,1]);\ \alpha_{\min} \leqslant \alpha \leqslant \alpha_{\max} \text{ in } [0,1],\ |\alpha^{(j)}| \leqslant C_j,\ j = 1,2 \text{ a.e. in } (0,1)\}. \tag{2.8}$$

Here γ, $\alpha_{\min}$, $\alpha_{\max}$, C_1, C_2 are given positive constants chosen in such a way that $\mathcal{U}_{\rm ad} \neq \emptyset$.

The boundary $\partial\Omega(\alpha)$ is split into $S(\alpha)$ and $\Gamma(\alpha) = \partial\Omega(\alpha) \setminus \overline{S}(\alpha)$, where

$$S(\alpha) = \{(x_1,x_2);\ x_1 \in (0,1),\ x_2 = \alpha(x_1)\},\ \alpha \in \mathcal{U}_{\rm ad},$$

i.e., $S(\alpha)$ is the graph of α. On any $\Omega(\alpha)$ we shall solve the Stokes system with the slip boundary conditions on $S(\alpha)$ and the no-slip condition on $\Gamma(\alpha)$. To emphasize the fact that the state problem is parametrized by $\alpha \in \mathcal{U}_{\rm ad}$ we shall use the following notation: $V(\alpha) := V(\Omega(\alpha))$, $V_{\rm div}(\alpha) := V_{\rm div}(\Omega(\alpha))$, $L^2_0(\alpha) := L^2_0(\Omega(\alpha))$. Similarly,

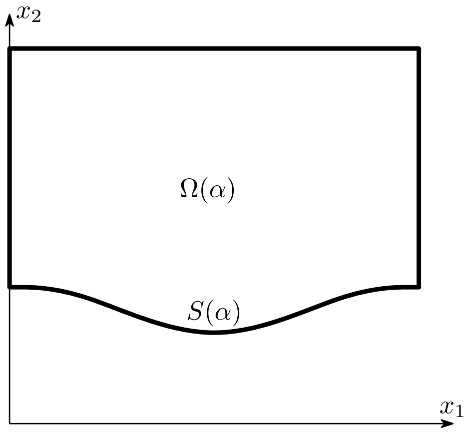

Figure 1. Geometry of the domain $\Omega(\alpha)$.

the bilinear forms a_α, b_α and the non-differentiable term j_α denote the ones from (2.5) with Ω, S replaced by $\Omega(\alpha)$ and $S(\alpha)$, respectively. The weak form of the state problem on $\Omega(\alpha)$, $\alpha \in \mathcal{U}_{\mathrm{ad}}$ reads as follows:

$$(\mathcal{P}(\alpha)) \qquad \begin{aligned} &\text{Find } (\boldsymbol{u}(\alpha), p(\alpha)) \in V(\alpha) \times L_0^2(\alpha) \text{ such that} \\ &\forall\, \boldsymbol{v} \in V(\alpha)\colon\ a_\alpha(\boldsymbol{u}(\alpha), \boldsymbol{v} - \boldsymbol{u}(\alpha)) - b_\alpha(\boldsymbol{v} - \boldsymbol{u}(\alpha), p(\alpha)) \\ &\qquad + j_\alpha(\boldsymbol{v}_\tau) - j_\alpha(\boldsymbol{u}_\tau(\alpha)) \geqslant (\boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u}(\alpha))_{0,\Omega(\alpha)}, \\ &\forall\, q \in L_0^2(\alpha)\colon\ b_\alpha(\boldsymbol{u}(\alpha), q) = 0. \end{aligned}$$

In what follows we shall suppose that $\boldsymbol{f} \in (L^2_{\mathrm{loc}}(\mathbb{R}^2))^2$ and, for simplicity of our analysis, that g is a positive constant.

Finally, let $J\colon \Delta \to \mathbb{R}$ be a cost functional, $\Delta = \{(\alpha, \boldsymbol{y}, q);\ \alpha \in \mathcal{U}_{\mathrm{ad}},\ \boldsymbol{y} \in V(\alpha),\ q \in L_0^2(\alpha)\}$ and $\mathfrak{J}(\alpha) = J(\alpha, \boldsymbol{u}(\alpha), p(\alpha))$, where $(\boldsymbol{u}(\alpha), p(\alpha))$ is the unique solution of $(\mathcal{P}(\alpha))$. Next we shall study the following optimal shape design problem:

$$(\mathbb{P}) \qquad \text{Find } \alpha^* \in \mathcal{U}_{\mathrm{ad}} \text{ such that } \forall\, \alpha \in \mathcal{U}_{\mathrm{ad}}\colon\ \mathfrak{J}(\alpha^*) \leqslant \mathfrak{J}(\alpha).$$

To prove that $(\mathbb{P})$ has a solution we shall need the following lower-semicontinuity property of J:

$$(2.9) \qquad \left.\begin{aligned} &\alpha_n \to \alpha \ \text{ in } C^1([0,1]),\ \alpha_n, \alpha \in \mathcal{U}_{\mathrm{ad}} \\ &\boldsymbol{y}_n \rightharpoonup \boldsymbol{y} \ \text{ in } (H^1(\widehat{\Omega}))^2,\ \boldsymbol{y}_n, \boldsymbol{y} \in (H_0^1(\widehat{\Omega}))^2 \\ &q_n \rightharpoonup q \ \text{ in } L^2(\widehat{\Omega}),\ q_n, q \in L_0^2(\widehat{\Omega}) \end{aligned}\right\} \\ \Rightarrow \liminf_{n\to\infty} J(\alpha_n, \boldsymbol{y}_n|_{\Omega(\alpha_n)}, q_n|_{\Omega(\alpha_n)}) \geqslant J(\alpha, \boldsymbol{y}|_{\Omega(\alpha)}, q|_{\Omega(\alpha)}),$$

where $\widehat{\Omega}$ is a domain which contains all $\Omega(\alpha)$, $\alpha \in \mathcal{U}_{\rm ad}$. Here and in what follows, $\widehat{\Omega} = (0,1) \times (0,\gamma)$ with γ from the definition of $\Omega(\alpha)$. Our first goal will be to prove the following result.

Theorem 2. *Let* (2.9) *be satisfied. Then* ($\mathbb{P}$) *has a solution.*

3. Stability of solutions with respect to shape variations

In this section we shall prove that the solutions of $(\mathcal{P}(\alpha))$ depend on $\alpha \in \mathcal{U}_{\rm ad}$ in a continuous way, which is the basic property used to prove the existence of a solution to ($\mathbb{P}$). To this end we have to introduce convergence of domains belonging to $\mathcal{O}$ and convergence of functions with variable domains of their definition.

Definition 1. Let $\Omega(\alpha_n) \in \mathcal{O}$, $n = 1, 2, \ldots$ be given. We say that the sequence $\{\Omega(\alpha_n)\}$ tends to $\Omega(\alpha) \in \mathcal{O}$ (and write $\Omega(\alpha_n) \to \Omega(\alpha)$) if

$$\alpha_n \to \alpha \text{ in } C^1([0,1]).$$

Definition 2. Let $\boldsymbol{y}_n \in V(\alpha_n)$, $\alpha_n \in \mathcal{U}_{\rm ad}$, $n = 1, 2, \ldots$ be given. We say that the sequence $\{\boldsymbol{y}_n\}$ tends weakly to $\boldsymbol{y} \in V(\alpha)$, $\alpha \in \mathcal{U}_{\rm ad}$ (and write $\boldsymbol{y}_n \rightharpoonup \boldsymbol{y}$) if

$$\pi_{\alpha_n} \boldsymbol{y}_n \rightharpoonup \pi_\alpha \boldsymbol{y} \text{ (weakly) in } (H^1(\widehat{\Omega}))^2, \tag{3.1}$$

where for any $\beta \in \mathcal{U}_{\rm ad}$, $\pi_\beta \in \mathcal{L}(V(\beta), H^1_0(\widehat{\Omega}))$ denotes an extension mapping from $\Omega(\beta)$ on $\widehat{\Omega}$, whose norm can be estimated independently of $\beta \in \mathcal{U}_{\rm ad}$. If weak convergence in (3.1) can be replaced by the strong one, we say that $\{\boldsymbol{y}_n\}$ tends strongly to $\boldsymbol{y}$ (and write $\boldsymbol{y}_n \to \boldsymbol{y}$).

For functions belonging to $H^1_0(\alpha_n) := H^1_0(\Omega(\alpha_n))$ or $L^2_0(\alpha_n)$ the situation is much simpler since one can use the zero extension outside of $\Omega(\alpha_n)$.

Definition 3. Let $z_n \in H^1_0(\alpha_n)$, $\alpha_n \in \mathcal{U}_{\rm ad}$, $n = 1, 2, \ldots$ We say that the sequence $\{z_n\}$ tends to $z \in H^1_0(\alpha)$ weakly, strongly (and write $z_n \rightharpoonup z$, $z_n \to z$, respectively) if

$$\begin{aligned} z_n^0 &\rightharpoonup z^0 \quad \text{in } H^1_0(\widehat{\Omega}),\\ z_n^0 &\to z^0 \quad \text{in } H^1_0(\widehat{\Omega}), \end{aligned}$$

respectively. Here the symbol "^{0}" stands for the zero extension of functions from their domain of definition on $\widehat{\Omega}$ (analogously we define convergence of a sequence $\{q_n\}$, $q_n \in L^2_0(\alpha_n)$).

R e m a r k 2. Since all domains belonging to $\mathcal{O}$ satisfy the so-called uniform cone property, such an extension mapping from Definition 2 can be easily constructed. Indeed, first we use the uniform extension mapping from $V(\beta)$ to $H^1(\mathbb{R}^2)$, whose existence is guarenteed, as follows from [5]. Then extended functions are multiplied by a suitable cut-off function in order to get zero traces on the boundary of $\widehat{\Omega}$.

The following auxiliary result is a direct consequence of the Arzelà-Ascoli and Lebesgue theorem (see e.g. [16], [9] for further details on convergence of domains).

Lemma 1. *It holds:*

(i) *the system $\mathcal{O}$ is compact with respect to convergence from Definition* 1*;*
(ii) *if $\Omega(\alpha_n) \to \Omega(\alpha)$, $\alpha_n, \alpha \in \mathcal{U}_{\rm ad}$, then*

$$\chi_n \to \chi \quad \text{in } L^q(\widehat{\Omega})\ \forall\, q \in [1, \infty),$$

where χ_n, χ are the characteristic functions of $\Omega(\alpha_n)$ and $\Omega(\alpha)$, respectively.

First we show that the constant c in (2.6) can be chosen to be independent of $\alpha \in \mathcal{U}_{\rm ad}$.

Lemma 2. *There exists a constant $c > 0$ such that*

$$\|\pi_\alpha \boldsymbol{u}(\alpha)\|_{1,\widehat{\Omega}} + \|p^0(\alpha)\|_{0,\widehat{\Omega}} \leqslant c \tag{3.2}$$

holds for any $\alpha \in \mathcal{U}_{\rm ad}$.

P r o o f. Using test functions $\boldsymbol{v} \in V_{\rm div}(\alpha)$, $\alpha \in \mathcal{U}_{\rm ad}$, problem $(\mathcal{P}(\alpha))$ takes the form:

$$a_\alpha(\boldsymbol{u}(\alpha), \boldsymbol{v} - \boldsymbol{u}(\alpha)) + j_\alpha(\boldsymbol{v}_\tau) - j_\alpha(\boldsymbol{u}_\tau(\alpha)) \geqslant (\boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u}(\alpha))_{0,\Omega(\alpha)}, \quad \boldsymbol{v} \in V_{\rm div}(\alpha). \tag{3.3}$$

Inserting $\boldsymbol{v} = \boldsymbol{0}$ and $\boldsymbol{v} = 2\boldsymbol{u}(\alpha)$ into (3.3) we obtain:

$$\begin{aligned} |\boldsymbol{u}(\alpha)|^2_{1,\Omega(\alpha)} := \|\nabla \boldsymbol{u}(\alpha)\|^2_{0,\Omega(\alpha)} &\leqslant a_\alpha(\boldsymbol{u}(\alpha), \boldsymbol{u}(\alpha)) + j_\alpha(\boldsymbol{u}_\tau(\alpha)) = (\boldsymbol{f}, \boldsymbol{u}(\alpha))_{0,\Omega(\alpha)} \\ &\leqslant \|\boldsymbol{f}\|_{0,\widehat{\Omega}} \|\pi_\alpha \boldsymbol{u}\|_{1,\widehat{\Omega}}, \end{aligned} \tag{3.4}$$

where for simplicity of notation $\pi_\alpha \boldsymbol{u} := \pi_\alpha \boldsymbol{u}(\alpha)$. The seminorm on the left of (3.4) can be estimated from below by the Friedrichs inequality with a constant $c > 0$ which does not depend on $\alpha \in \mathcal{U}_{\rm ad}$ [9]. Thus

$$c\|\boldsymbol{u}(\alpha)\|^2_{1,\Omega(\alpha)} \leqslant |\boldsymbol{u}(\alpha)|^2_{1,\Omega(\alpha)} \leqslant \|\boldsymbol{f}\|_{0,\widehat{\Omega}} \|\pi_\alpha \boldsymbol{u}\|_{1,\widehat{\Omega}}.$$

From this and the fact that also the norm of π_α can be estimated uniformly with respect to $\alpha \in \mathcal{U}_{\mathrm{ad}}$, the boundedness of $\|\pi_\alpha \boldsymbol{u}(\alpha)\|_{1,\widehat{\Omega}}$ follows. To prove the boundedness of the pressure we proceed as follows: Using the fact that

$$a_\alpha(\boldsymbol{u}(\alpha), \boldsymbol{u}(\alpha)) - b_\alpha(\boldsymbol{u}(\alpha), p(\alpha)) + j_\alpha(\boldsymbol{u}_\tau(\alpha)) = (\boldsymbol{f}, \boldsymbol{u}(\alpha))_{0,\Omega(\alpha))},$$

we obtain from the inequality in $(\mathcal{P}(\alpha))$:

$$b_\alpha(\boldsymbol{v}, p(\alpha)) \leqslant a_\alpha(\boldsymbol{u}(\alpha), \boldsymbol{v}) + j_\alpha(\boldsymbol{v}_\tau) - (\boldsymbol{f}, \boldsymbol{v})_{0,\Omega(\alpha)} \leqslant c\|\boldsymbol{v}\|_{1,\Omega(\alpha)}, \ \boldsymbol{v} \in V(\alpha), \tag{3.5}$$

where $c > 0$ does not depend on $\alpha \in \mathcal{U}_{\mathrm{ad}}$, making use of the boundedness of $\|\pi_\alpha \boldsymbol{u}\|_{1,\widehat{\Omega}}$ and the uniform boundedness of the trace mapping $\mathrm{Tr}_\alpha \in \mathcal{L}(H^1(\Omega(\alpha)), L^2(\Omega(\alpha)))$ with respect to $\alpha \in \mathcal{U}_{\mathrm{ad}}$ [9]. From (3.5) it follows that

$$\sup_{\boldsymbol{v} \in V(\alpha)} \frac{b_\alpha(\boldsymbol{v}, p(\alpha))}{\|\boldsymbol{v}\|_{1,\Omega(\alpha)}} \leqslant c. \tag{3.6}$$

From [8] we know that there is a mapping $\mathcal{B}_\alpha \in \mathcal{L}(L_0^2(\alpha), (H_0^1(\alpha))^2)$ such that $\operatorname{div} \mathcal{B}_\alpha q = q$ a.e. in $\Omega(\alpha)$, whose norm is bounded independently of $\alpha \in \mathcal{U}_{\mathrm{ad}}$ (see also [3], Section 4)[1]. The choice $\boldsymbol{v} := \mathcal{B}_\alpha p(\alpha)$ in (3.6) yields:

$$\sup_{\boldsymbol{v} \in V(\alpha)} \frac{b_\alpha(\boldsymbol{v}, p(\alpha))}{\|\boldsymbol{v}\|_{1,\Omega(\alpha)}} \geqslant \frac{b_\alpha(\mathcal{B}_\alpha p(\alpha), p(\alpha))}{\|\mathcal{B}_\alpha p(\alpha)\|_{1,\Omega(\alpha)}} = \frac{\|p(\alpha)\|^2_{0,\Omega(\alpha)}}{\|\mathcal{B}_\alpha p(\alpha)\|_{1,\Omega(\alpha)}} \geqslant \bar{c}\|p(\alpha)\|_{0,\Omega(\alpha)},$$

where the constant $\bar{c} > 0$ is independent of $\alpha \in \mathcal{U}_{\mathrm{ad}}$. This concludes the proof. □

We shall also need the following auxiliary result.

Lemma 3. *Let $\alpha_n, \alpha \in \mathcal{U}_{\mathrm{ad}}$ be such that $\alpha_n \to \alpha$ in $C^1([0,1])$ and let $\boldsymbol{v} \in V(\alpha)$ be given. Then there exists a sequence $\{\boldsymbol{v}_k\}$, $\boldsymbol{v}_k \in (H^1(\widehat{\Omega}))^2$ and a function $\overline{\boldsymbol{v}} \in (H^1(\widehat{\Omega}))^2$ such that $\overline{\boldsymbol{v}}|_{\Omega(\alpha)} = \boldsymbol{v}$ and*

$$\boldsymbol{v}_k \to \overline{\boldsymbol{v}} \quad \text{in } (H^1(\widehat{\Omega}))^2, \ k \to \infty. \tag{3.7}$$

In addition, for any $k \in \mathbb{N}$ there exists $n_k \in \mathbb{N}$ such that

$$\boldsymbol{v}_k|_{\Omega(\alpha_{n_k})} \in V(\alpha_{n_k}). \tag{3.8}$$

P r o o f. Let $\boldsymbol{\nu}^\alpha := \boldsymbol{\nu}^\alpha(x_1)$, $\boldsymbol{\nu}^{\alpha_n} := \boldsymbol{\nu}^{\alpha_n}(x_1)$ denote the unit outward normal vector to $S(\alpha)$ and $S(\alpha_n)$, respectively. By the same symbols we shall denote their

[1] In fact, the norm of $\mathcal{B}_\alpha$ depends only on $\|\alpha\|_{1,\infty,[0,1]}$, i.e., it is uniformly bounded for $\alpha \in \{\beta \in C^{0,1}([0,1]);\ 0 \leqslant \beta \leqslant \alpha_{\max},\ |\beta'| \leqslant C_1 \text{ in } [0,1]\}$.

natural extensions defined in $\widehat{\Omega}$, i.e., $\boldsymbol{\nu}^\alpha(x) := \boldsymbol{\nu}^\alpha(x_1)$ and $\boldsymbol{\nu}^{\alpha_n}(x) := \boldsymbol{\nu}^{\alpha_n}(x_1)$, $x = (x_1, x_2) \in \widehat{\Omega}$. We set

$$\varphi(x) := \boldsymbol{v}(x) \cdot \boldsymbol{\nu}^\alpha(x), \quad \boldsymbol{\psi}(x) := \boldsymbol{v}_{\tau^\alpha}(x), \quad x \in \Omega(\alpha).$$

Then $\varphi \in H_0^1(\Omega(\alpha))$, $\boldsymbol{\psi} \in (H^1(\Omega(\alpha)))^2$ and $\boldsymbol{\psi} = \mathbf{0}$ on $\Gamma(\alpha)$. Using the density arguments, one can find sequences $\{\varphi_k\}$, $\varphi_k \in C_0^\infty(\Omega(\alpha))$ and $\{\boldsymbol{\psi}_k\}$, $\boldsymbol{\psi}_k \in (C^\infty(\overline{\Omega}(\alpha)))^2$, $\operatorname{dist}(\operatorname{supp} \boldsymbol{\psi}_k, \Gamma(\alpha)) > 0$ for all $k \in \mathbb{N}$ such that

$$\begin{gathered} \varphi_k \to \varphi \quad \text{in } H_0^1(\Omega(\alpha)), \\ \boldsymbol{\psi}_k \to \boldsymbol{\psi},\ k \to \infty, \quad \text{in } (H^1(\Omega(\alpha))^2 \end{gathered}$$

and also

$$\begin{gathered} \varphi_k^0 \to \varphi^0 \quad \text{in } H_0^1(\widehat{\Omega}), \\ \pi_\alpha \boldsymbol{\psi}_k \to \pi_\alpha \boldsymbol{\psi}, \quad \text{in } (H^1(\widehat{\Omega}))^2. \end{gathered}$$

Moreover, we may assume that $\operatorname{dist}(\operatorname{supp} \pi_\alpha \boldsymbol{\psi}_k, \widehat{\Gamma}) > 0$ for all $k \in \mathbb{N}$ where $\widehat{\Gamma} := \partial\widehat{\Omega} \setminus [0,1] \times \{0\}$. The sequence $\{\boldsymbol{v}_k\}$ satisfying (3.7)–(3.8) will be constructed as follows. Suppose for the moment that there exists a filter of indices $\{n_k\}$, $k \to \infty$, such that for any $k \in \mathbb{N}$ it holds that $S(\alpha_{n_k}) \cap \operatorname{supp} \varphi_k^0 = \emptyset$ and in addition there exist functions $\boldsymbol{N}_{n_k} \in (C^{0,1}(\widehat{\Omega}))^2$ such that $\boldsymbol{N}_{n_k}|_{\partial\Omega(\alpha_{n_k})} = \boldsymbol{\nu}^{\alpha_{n_k}}$ and

$$\boldsymbol{N}_{n_k} \to \boldsymbol{\nu}^\alpha \quad \text{in } (H^1(\widehat{\Omega}))^2,\ k \to \infty. \tag{3.9}$$

Define $\boldsymbol{v}_k$ by:

$$\boldsymbol{v}_k = \varphi_k^0 \boldsymbol{N}_{n_k} + (\pi_\alpha \boldsymbol{\psi}_k)_{\tau_{n_k}} = \varphi_k^0 \boldsymbol{N}_{n_k} + \pi_\alpha \boldsymbol{\psi}_k - (\pi_\alpha \boldsymbol{\psi}_k \cdot \boldsymbol{N}_{n_k}) \boldsymbol{N}_{n_k}. \tag{3.10}$$

From this and the definition of n_k it immediately follows that $\boldsymbol{v}_k \in (H^1(\widehat{\Omega}))^2$, $\boldsymbol{v}_k = \mathbf{0}$ on $\Gamma(\alpha_{n_k})$ and $\boldsymbol{v}_k \cdot \boldsymbol{\nu}^{\alpha_{n_k}}|_{S(\alpha_{n_k})} = \varphi_k^0|_{S(\alpha_{n_k})} = 0$. Hence, $\boldsymbol{v}_k|_{\Omega(\alpha_{n_k})} \in V(\alpha_{n_k})$. Passing to the limit with $k \to \infty$ in (3.10), we obtain:

$$\boldsymbol{v}_k \to \varphi^0 \boldsymbol{\nu}^\alpha + \pi_\alpha \boldsymbol{\psi} - (\pi_\alpha \boldsymbol{\psi} \cdot \boldsymbol{\nu}^\alpha) \boldsymbol{\nu}^\alpha =: \overline{\boldsymbol{v}} \quad \text{in } (H^1(\widehat{\Omega}))^2.$$

It is easy to see that $\overline{\boldsymbol{v}}$ satisfies $\overline{\boldsymbol{v}}|_{\Omega(\alpha)} = \boldsymbol{v}$.

It remains to prove (3.9). Since $\alpha_n \to \alpha$ in $C^1([0,1])$, we have

$$\boldsymbol{\nu}^{\alpha_n} \to \boldsymbol{\nu}^\alpha \quad \text{in } C(\overline{\widehat{\Omega}}) \tag{3.11}$$

and from the definition of $\mathcal{O}$ it follows that

$$\|\nabla \boldsymbol{\nu}^{\beta}\|_{\infty,\widehat{\Omega}} \leqslant C_2 \quad \text{for every } \beta \in \mathcal{U}_{\mathrm{ad}}. \tag{3.12}$$

Let $\xi_k \in C^{\infty}([0,\infty))$ be functions satisfying $0 \leqslant \xi_k \leqslant 1$ in $[0,\infty)$, $\xi_k|_{[0,1/(2k)]} = 1$, and $\xi_k|_{[1/k,\infty)} = 0$ for every $k \in \mathbb{N}$. For $k, n \in \mathbb{N}$ we set

$$\boldsymbol{N}_{n,k}(x) := \xi_k(|x_2 - \alpha(x_1)|)(\boldsymbol{\nu}^{\alpha_n} - \boldsymbol{\nu}^{\alpha}) + \boldsymbol{\nu}^{\alpha}.$$

It is readily seen that $\boldsymbol{N}_{n,k} \in (C^{0,1}(\overline{\widehat{\Omega}}))^2$ for all $k, n \in \mathbb{N}$ and

$$\|\boldsymbol{N}_{n,k} - \boldsymbol{\nu}^{\alpha}\|_{0,\widehat{\Omega}} \leqslant \|\boldsymbol{\nu}^{\alpha_n} - \boldsymbol{\nu}^{\alpha}\|_{0,\widehat{\Omega}} \quad \text{as } n \to \infty \tag{3.13}$$

uniformly with respect to $k \in \mathbb{N}$.

Let $k \in \mathbb{N}$ be fixed. Then from the definition of ξ_k it follows that there exists an index $n_0 := n_0(k) \in \mathbb{N}$ such that $\boldsymbol{N}_{n,k}|_{\partial\Omega_n} = \boldsymbol{\nu}^{\alpha_n}$ for any $n \geqslant n_0$. Furthermore:

$$\begin{aligned} \|\nabla(\boldsymbol{N}_{n,k} - \boldsymbol{\nu}^{\alpha})\|_{0,\widehat{\Omega}} &\leqslant \max_{(x_1,x_2)\in\widehat{\Omega}} |\nabla(\xi_k(|x_2 - \alpha(x_1)|))| \|\boldsymbol{\nu}^{\alpha_n} - \boldsymbol{\nu}^{\alpha}\|_{0,\widehat{\Omega}} \\ &\quad + \|\nabla(\boldsymbol{\nu}^{\alpha_n} - \boldsymbol{\nu}^{\alpha})\|_{0,\{|x_2-\alpha(x_1)|<1/k\}} \\ &\leqslant \sqrt{1 + C_1^2}\, \|\xi_k'\|_{\infty,[0,\infty)} \|\boldsymbol{\nu}^{\alpha_n} - \boldsymbol{\nu}^{\alpha}\|_{0,\widehat{\Omega}} + 2C_2/k. \end{aligned} \tag{3.14}$$

From this we see (still keeping $k \in \mathbb{N}$ fixed) that there exists an index $n_1 := n_1(k) \in \mathbb{N}$ such that $\|\nabla(\boldsymbol{N}_{n,k} - \boldsymbol{\nu}^{\alpha})\|_{0,\widehat{\Omega}} = O(1/k)$ for any $n \geqslant n_1$. Setting $\boldsymbol{N}_{n_k} := \boldsymbol{N}_{n_k,k}$, where $n_k = \max\{n_0, n_1\}$, we obtain (3.9), making use of (3.13). $\square$

The main result of this section is the following stability result.

Theorem 3. *Let α_n, $\alpha \in \mathcal{U}_{\mathrm{ad}}$ be such that $\alpha_n \to \alpha$ in $C^1([0,1])$ and denote by $(\boldsymbol{u}_n, p_n) := (\boldsymbol{u}(\alpha_n), p(\alpha_n)) \in V(\alpha_n) \times L_0^2(\alpha_n)$ the unique solution of $(\mathcal{P}(\alpha_n))$. Suppose that there exists an element $(\bar{\boldsymbol{u}}, \bar{p}) \in (H_0^1(\widehat{\Omega}))^2 \times L_0^2(\widehat{\Omega})$ such that*

$$\pi_{\alpha_n}\boldsymbol{u}_n \rightharpoonup \bar{\boldsymbol{u}} \quad \text{in } (H^1(\widehat{\Omega}))^2, \tag{3.15a}$$

$$p_n^0 \rightharpoonup \bar{p} \quad \text{in } L_0^2(\widehat{\Omega}). \tag{3.15b}$$

Then $(\boldsymbol{u}(\alpha), p(\alpha)) := (\bar{\boldsymbol{u}}|_{\Omega(\alpha)}, \bar{p}|_{\Omega(\alpha)})$ solves $(\mathcal{P}(\alpha))$.

P r o o f. First we show that $(\bar{\boldsymbol{u}}|_{\Omega(\alpha)}, \bar{p}|_{\Omega(\alpha)}) \in V_{\mathrm{div}}(\alpha) \times L_0^2(\alpha)$. The fact that $\boldsymbol{u}(\alpha) := \bar{\boldsymbol{u}}|_{\Omega(\alpha)} = \boldsymbol{0}$ on $\Gamma(\alpha)$ and $p(\alpha) := \bar{p}|_{\Omega(\alpha)} \in L_0^2(\Omega(\alpha))$ is readily seen. It remains to prove that $\operatorname{div} \boldsymbol{u}(\alpha) = 0$ in $\Omega(\alpha)$ and $\boldsymbol{u}(\alpha) \cdot \boldsymbol{\nu}^{\alpha} = 0$ on $S(\alpha)$. This is equivalent to verifying that

$$\int_{\Omega(\alpha)} \boldsymbol{u}(\alpha) \cdot \nabla \varphi = 0 \quad \forall\, \varphi \in H^1(\Omega(\alpha)),\ \varphi = 0 \text{ on } \Gamma(\alpha). \tag{3.16}$$

Let φ from (3.16) be given and denote by $\tilde{\varphi} \in H^1(\widehat{\Omega})$ its extension such that $\tilde{\varphi} = 0$ on $\partial\widehat{\Omega} \setminus [0,1] \times \{0\}$. Since $\boldsymbol{u}_n \in V(\alpha_n)$ for all $n \in \mathbb{N}$, we get

$$\int_{\Omega(\alpha_n)} \boldsymbol{u}_n \cdot \nabla \tilde{\varphi} = 0 \ \Leftrightarrow\ \int_{\widehat{\Omega}} \chi_n \pi_{\alpha_n} \boldsymbol{u}_n \cdot \nabla \tilde{\varphi} = 0, \tag{3.17}$$

where χ_n is the characteristic function of $\Omega(\alpha_n)$. Letting $n \to \infty$ in (3.17), we obtain

$$\int_{\widehat{\Omega}} \chi_n \pi_{\alpha_n} \boldsymbol{u}_n \cdot \nabla \tilde{\varphi} \to \int_{\widehat{\Omega}} \chi \bar{\boldsymbol{u}} \cdot \nabla \tilde{\varphi} = \int_{\Omega(\alpha)} \boldsymbol{u}(\alpha) \cdot \nabla \varphi = 0,$$

where χ is the characteristic function of $\Omega(\alpha)$, making use of Lemma 1 (ii) and (3.15a). Hence, $\boldsymbol{u}(\alpha) \in V_{\mathrm{div}}(\alpha)$. Now we show that the pair $(\boldsymbol{u}(\alpha), p(\alpha))$ satisfies the inequality in $(\mathcal{P}(\alpha))$.

Let $\boldsymbol{v} \in V(\alpha)$ be given and construct the sequence $\{\boldsymbol{v}_k\}$, $\boldsymbol{v}_k \in (H^1(\widehat{\Omega}))^2$ satisfying (3.7) and (3.8). Since $\boldsymbol{v}_k|_{\Omega(\alpha_{n_k})} \in V(\alpha_{n_k})$ for an appropriate $n_k \in \mathbb{N}$, it can be used as a test function in $(\mathcal{P}(\alpha_{n_k}))$ (to simplify notation we shall write $a_{n_k} := a_{\alpha_{n_k}}$, $b_{n_k} := b_{\alpha_{n_k}}$, $j_{n_k} := j_{\alpha_{n_k}}$):

$$\begin{aligned} a_{n_k}(\boldsymbol{u}_{n_k}, \boldsymbol{v}_k - \boldsymbol{u}_{n_k}) - b_{n_k}(\boldsymbol{v}_k - \boldsymbol{u}_{n_k}, p_{n_k}) + j_{n_k}(\boldsymbol{v}_{k\tau}) - j_{n_k}(\boldsymbol{u}_{n_k\tau}) \\ \geqslant (\boldsymbol{f}, \boldsymbol{v}_k - \boldsymbol{u}_{n_k})_{0,\Omega(\alpha_{n_k})}. \end{aligned} \tag{3.18}$$

Letting $k \to \infty$ in (3.18) and using Lemma 1 (ii), (3.7), (3.15) we obtain (for details we refer to [9]):

$$\limsup_{k\to\infty} a_{n_k}(\boldsymbol{u}_{n_k}, \boldsymbol{v}_k - \boldsymbol{u}_{n_k}) \leqslant a_{\alpha}(\boldsymbol{u}(\alpha), \boldsymbol{v} - \boldsymbol{u}(\alpha)), \tag{3.19a}$$

$$\lim_{k\to\infty} b_{n_k}(\boldsymbol{v}_k - \boldsymbol{u}_{n_k}, p_{n_k}) = b_{\alpha}(\boldsymbol{v} - \boldsymbol{u}(\alpha), p(\alpha)), \tag{3.19b}$$

$$\lim_{k\to\infty} (\boldsymbol{f}, \boldsymbol{v}_k - \boldsymbol{u}_{n_k})_{0,\Omega(\alpha_{n_k})} = (\boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u}(\alpha))_{0,\Omega(\alpha)}. \tag{3.19c}$$

The frictional term can be written as

$$\begin{aligned} j_{n_k}(\boldsymbol{v}_{k\tau}) &= g \int_0^1 |\boldsymbol{v}_{k\tau} \circ \alpha_{n_k}| \sqrt{1 + |\alpha'_{n_k}|^2}\, \mathrm{d}x_1 \\ &= g \int_0^1 |\boldsymbol{v}_k \circ \alpha_{n_k} - (\boldsymbol{v}_k \circ \alpha_{n_k} \cdot \boldsymbol{\nu}^{\alpha_{n_k}}) \boldsymbol{\nu}^{\alpha_{n_k}}|^2 \sqrt{1 + |\alpha'_{n_k}|^2}\, \mathrm{d}x_1. \end{aligned}$$

From [9] we know that

$$\boldsymbol{v}_k \circ \alpha_{n_k} \to \boldsymbol{v} \circ \alpha \quad \text{in } (L^2((0,1)))^2,\ k \to \infty.$$

Therefore,

$$j_{n_k}(\boldsymbol{v}_{k\tau}) \to j_\alpha(\boldsymbol{v}_\tau), \quad k \to \infty,$$

using the fact that $\boldsymbol{\nu}^{\alpha_{n_k}} \rightrightarrows \boldsymbol{\nu}^\alpha$, $\alpha'_{n_k} \rightrightarrows \alpha'$ (uniformly) in $[0,1]$ (similarly for $j_{n_k}(\boldsymbol{u}_{n_k\tau})$). From this and (3.19) we see that $(\boldsymbol{u}(\alpha), p(\alpha))$ satisfies the inequality in $(\mathcal{P}(\alpha))$, i.e., $(\boldsymbol{u}(\alpha), p(\alpha))$ solves $(\mathcal{P}(\alpha))$. □

Remark 3. It is easy to show that (3.15a) implies that

$$\chi_n \nabla \pi_{\alpha_n} \boldsymbol{u}_n \to \chi \nabla \bar{\boldsymbol{u}} \quad \text{in } (L^2(\widehat{\Omega}))^2, \tag{3.20}$$

where χ_n, χ are the characteristic functions of $\Omega(\alpha_n)$ and $\Omega(\alpha)$, respectively. To prove (3.20) it is sufficient to show that

$$\|\chi_n \nabla \pi_{\alpha_n} \boldsymbol{u}_n\|_{0,\widehat{\Omega}} \to \|\chi \nabla \bar{\boldsymbol{u}}\|_{0,\widehat{\Omega}}, \quad n \to \infty.$$

Indeed,

$$\begin{aligned} \|\chi_n \nabla \pi_{\alpha_n} \boldsymbol{u}_n\|^2_{0,\widehat{\Omega}} = a_{\alpha_n}(\boldsymbol{u}_n, \boldsymbol{u}_n) &= b_{\alpha_n}(\boldsymbol{u}_n, p_n^0) - j_{\alpha_n}(\boldsymbol{u}_{n\tau}) + (\boldsymbol{f}, \boldsymbol{u}_n)_{0,\Omega(\alpha_n)} \\ &\to b_\alpha(\boldsymbol{u}(\alpha), p(\alpha)) - j_\alpha(\boldsymbol{u}_\tau(\alpha)) + (\boldsymbol{f}, \boldsymbol{u}(\alpha))_{0,\Omega(\alpha)} \\ &= a_\alpha(\boldsymbol{u}(\alpha), \boldsymbol{u}(\alpha)) = \|\chi \nabla \bar{\boldsymbol{u}}\|^2_{0,\widehat{\Omega}}. \end{aligned}$$

From (3.20) it easily follows that

$$\boldsymbol{u}_n \to \boldsymbol{u}(\alpha) \quad \text{in } (H^1_{\text{loc}}(\Omega(\alpha)))^2$$

(see [9]).

Proof of Theorem 2. Let $\{(\boldsymbol{u}_n, p_n)\}$, where $(\boldsymbol{u}_n, p_n)$ solves $(\mathcal{P}(\alpha_n))$, be a minimizing sequence in $(\mathbb{P})$. Since $\{(\pi_{\alpha_n}\boldsymbol{u}_n, p_n^0)\}$ is bounded in $(H^1(\widehat{\Omega}))^2 \times L^2_0(\widehat{\Omega})$ as follows from Lemma 2, one can find its subsequence (denoted by the same symbol) such that (3.15) holds true. The existence of a solution to $(\mathbb{P})$ is then an easy consequence of (2.9) and Theorem 3. □

4. Shape optimization with the penalized state problem

The aim of this section is to analyse a new shape optimization problem for the Stokes system with threshold slip but with a penalization of the impermeability condition (2.1d). In addition to the notation introduced in the previous sections we denote

$$\begin{aligned}\widetilde{V}(\alpha) &= \{\boldsymbol{v} \in (H^1(\Omega(\alpha)))^2;\ \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma(\alpha)\},\\ \widetilde{V}_{\mathrm{div}}(\alpha) &= \{\boldsymbol{v} \in \widetilde{V}(\alpha);\ b_\alpha(\boldsymbol{v}, q) = 0\ \forall\, q \in L^2_0(\alpha)\}, \quad \alpha \in \mathcal{U}_{\mathrm{ad}},\end{aligned}$$

and define the penalty term

$$c_\alpha(\boldsymbol{u}, \boldsymbol{v}) = \int_0^1 (\boldsymbol{u} \circ \alpha \cdot \boldsymbol{\nu}^\alpha)(\boldsymbol{v} \circ \alpha \cdot \boldsymbol{\nu}^\alpha)\,\mathrm{d}x_1,$$

where $\boldsymbol{u} \circ \alpha \cdot \boldsymbol{\nu}^\alpha := \boldsymbol{u}(x_1, \alpha(x_1)) \cdot \boldsymbol{\nu}^\alpha(x_1)$, $x_1 \in (0,1)$. This bilinear form will be used to approximate the boundary condition $\boldsymbol{u} \cdot \boldsymbol{\nu}^\alpha = 0$ on $S(\alpha)$.

Let $\alpha \in \mathcal{U}_{\mathrm{ad}}$ be fixed and $\varepsilon > 0$ be a penalty parameter. The penalized form of $(\mathcal{P}(\alpha))$ reads as follows

$(\mathcal{P}(\alpha)_\varepsilon)$ Find $(\boldsymbol{u}_\varepsilon, p_\varepsilon) \in \widetilde{V}(\alpha) \times L^2_0(\alpha)$ such that

$$\begin{aligned}&\forall\, \boldsymbol{v} \in \widetilde{V}(\alpha)\colon\ a_\alpha(\boldsymbol{u}_\varepsilon, \boldsymbol{v} - \boldsymbol{u}_\varepsilon) - b_\alpha(\boldsymbol{v} - \boldsymbol{u}_\varepsilon, p_\varepsilon)\\ &\qquad + j_\alpha(\boldsymbol{v}_\tau) - j_\alpha(\boldsymbol{u}_{\varepsilon\tau}) + \frac{1}{\varepsilon} c_\alpha(\boldsymbol{u}_\varepsilon, \boldsymbol{v} - \boldsymbol{u}_\varepsilon) \geqslant (\boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u}_\varepsilon)_{0,\Omega(\alpha)},\\ &\forall\, q \in L^2_0(\alpha)\colon\ b_\alpha(\boldsymbol{u}_\varepsilon, q) = 0.\end{aligned}$$

Using the same technique as in [6] one can show that $(\mathcal{P}(\alpha)_\varepsilon)$ has a unique solution $(\boldsymbol{u}_\varepsilon, p_\varepsilon)$ for any $\varepsilon > 0$. Moreover,

$$\boldsymbol{u}_\varepsilon \to \boldsymbol{u} \quad \text{in } (H^1(\Omega(\alpha)))^2, \tag{4.1a}$$

$$p_\varepsilon \rightharpoonup p \quad \text{in } L^2_0(\alpha),\ \varepsilon \to 0+ \tag{4.1b}$$

and $(\boldsymbol{u}, p)$ is the unique solution of $(\mathcal{P}(\alpha))$.

Now we introduce the following family of shape optimization problems with the state problem $(\mathcal{P}(\alpha)_\varepsilon)$. For any $\varepsilon > 0$ fixed, we define

$$(\mathbb{P}_\varepsilon) \qquad \text{Find } \alpha^*_\varepsilon \in \mathcal{U}_{\mathrm{ad}} \text{ such that } \forall\, \alpha \in \mathcal{U}_{\mathrm{ad}}\colon\ \mathfrak{J}_\varepsilon(\alpha^*_\varepsilon) \leqslant \mathfrak{J}_\varepsilon(\alpha),$$

where $\mathfrak{J}_\varepsilon(\alpha) := J(\alpha, \boldsymbol{u}_\varepsilon(\alpha), p_\varepsilon(\alpha))$ with $(\boldsymbol{u}_\varepsilon(\alpha), p_\varepsilon(\alpha))$ being the solution of $(\mathcal{P}(\alpha)_\varepsilon)$. Using a similar approach as in Section 3 (see also [9]) one can prove the following result.

Theorem 4. *Let (2.9) be satisfied. Then* $(\mathbb{P}_\varepsilon)$ *has a solution for any* $\varepsilon > 0$.

In the subsequent part of this section we shall analyse the mutual relation between solutions of $(\mathbb{P})$ and $(\mathbb{P}_\varepsilon)$ for $\varepsilon \to 0+$. We start with the following result.

Lemma 4. *There exists a constant* $c := c(\|\boldsymbol{f}\|_{0,\widehat{\Omega}}) > 0$ *independent of* $\alpha \in \mathcal{U}_{\rm ad}$ *and* $\varepsilon > 0$ *such that the solution* $(\boldsymbol{u}_\varepsilon(\alpha), p_\varepsilon(\alpha))$ *of* $(\mathcal{P}(\alpha)_\varepsilon)$ *is bounded:*

$$\|\pi_\alpha \boldsymbol{u}_\varepsilon(\alpha)\|_{1,\widehat{\Omega}} + \frac{1}{\varepsilon} c_\alpha(\boldsymbol{u}_\varepsilon(\alpha), \boldsymbol{u}_\varepsilon(\alpha)) + \|p^0_\varepsilon(\alpha)\|_{0,\widehat{\Omega}} \leqslant c. \tag{4.2}$$

P r o o f. The boundedness of the first two terms in (4.2) follows easily from the fact that $\boldsymbol{u}_\varepsilon(\alpha) \in \widetilde{V}_{\rm div}(\alpha)$ and satisfies

$$\begin{aligned} a_\alpha(\boldsymbol{u}_\varepsilon, \boldsymbol{u}_\varepsilon) &+ j_\alpha(\boldsymbol{u}_{\varepsilon\tau}) + \frac{1}{\varepsilon} c_\alpha(\boldsymbol{u}_\varepsilon, \boldsymbol{u}_\varepsilon) \\ &\leqslant a_\varepsilon(\boldsymbol{u}_\varepsilon, \boldsymbol{v}) + j_\alpha(\boldsymbol{v}_\tau) + \frac{1}{\varepsilon} c_\alpha(\boldsymbol{u}_\varepsilon, \boldsymbol{v}) - (\boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u}_\varepsilon)_{0,\Omega(\alpha)} \quad \forall\, \boldsymbol{v} \in \widetilde{V}_{\rm div}(\alpha), \end{aligned} \tag{4.3}$$

making use of the definitions of $(\mathcal{P}(\alpha)_\varepsilon)$ and $\widetilde{V}_{\rm div}(\alpha)$. Inserting $\boldsymbol{v} \equiv \boldsymbol{0}$ into the right-hand side of (4.3) we obtain the claim. To show the boundedness of $\{p_\varepsilon(\alpha)\}$ we proceed as follows: From the inequality in $(\mathcal{P}(\alpha)_\varepsilon)$ we see that

$$b_\alpha(\boldsymbol{v}, p_\varepsilon(\alpha)) \leqslant a_\alpha(\boldsymbol{u}_\varepsilon(\alpha), \boldsymbol{v}) - (\boldsymbol{f}, \boldsymbol{v}) \quad \forall\, \boldsymbol{v} \in (H^1_0(\Omega(\alpha)))^2.$$

Thus (see also Lemma 2)

$$\bar{c}\|p_\varepsilon\|_{0,\Omega(\alpha)} \leqslant \sup_{\substack{\boldsymbol{v} \in (H^1_0(\Omega(\alpha)))^2 \\ \boldsymbol{v} \neq \boldsymbol{0}}} \frac{b_\alpha(\boldsymbol{v}, p_\varepsilon)}{\|\boldsymbol{v}\|_{1,\Omega(\alpha)}} \leqslant c,$$

making use of the boundedness of $\{\|\boldsymbol{u}_\varepsilon(\alpha)\|_{1,\Omega(\alpha)}\}$. Since also $\bar{c}$ does not depend on $\alpha \in \mathcal{U}_{\rm ad}$ and $\varepsilon > 0$, we arrive at (4.2). $\square$

The key role in our analysis plays the following stability type result.

Lemma 5. *Let* $\alpha_n \to \alpha$ *in* $C^1([0,1])$, $\alpha_n, \alpha \in \mathcal{U}_{\rm ad}$ *and* $\{(\boldsymbol{u}_n, p_n)\}$ *be the sequence of solutions to* $(\mathcal{P}(\alpha_n)_{\varepsilon_n})$, $\varepsilon_n \to 0+$. *Then there exist a subsequence of* $\{(\boldsymbol{u}_n, p_n)\}$ *(denoted by the same symbol) and a pair* $(\bar{u}, \bar{p}) \in (H^1_0(\widehat{\Omega}))^2 \times L^2_0(\widehat{\Omega})$ *such that*

$$\pi_{\alpha_n} \boldsymbol{u}_n \rightharpoonup \bar{\boldsymbol{u}} \quad \text{in } (H^1(\widehat{\Omega}))^2, \tag{4.4a}$$

$$p^0_n \rightharpoonup \bar{p} \quad \text{in } L^2_0(\widehat{\Omega}),\ n \to \infty. \tag{4.4b}$$

In addition, the pair $(\bar{\boldsymbol{u}}|_{\Omega(\alpha)}, \bar{p}|_{\Omega(\alpha)})$ *is a solution of* $(\mathcal{P}(\alpha))$.

P r o o f. The existence of a subsequence satisfying (4.4) follows from Lemma 4. Clearly, $\bar{\boldsymbol{u}}|_{\Omega(\alpha)} \in \widetilde{V}_{\rm div}(\alpha)$. Next we show that $\boldsymbol{u} := \bar{\boldsymbol{u}}|_{\Omega(\alpha)}$ satisfies (2.1d) on $S(\alpha)$. From (4.2) we see that

$$0 \leqslant c_n(\boldsymbol{u}_n, \boldsymbol{u}_n) \leqslant \varepsilon_n c \to 0 \quad \text{as } n \to \infty, \tag{4.5}$$

where for brevity $c_n := c_{\alpha_n}$. On the other hand,

$$c_n(\boldsymbol{u}_n, \boldsymbol{u}_n) \to c_\alpha(\boldsymbol{u}, \boldsymbol{u}) \quad \text{as } n \to \infty. \tag{4.6}$$

Indeed,

$$\begin{aligned} &\|\boldsymbol{u}_n \circ \alpha_n \cdot \boldsymbol{\nu}^{\alpha_n} - \boldsymbol{u} \circ \alpha \cdot \boldsymbol{\nu}^\alpha\|_{0,(0,1)} \\ &\quad \leqslant \|(\boldsymbol{u}_n \circ \alpha_n - \boldsymbol{u} \circ \alpha) \cdot \boldsymbol{\nu}^{\alpha_n}\|_{0,(0,1)} + \|\boldsymbol{u} \circ \alpha(\boldsymbol{\nu}^{\alpha_n} - \boldsymbol{\nu}^\alpha)\|_{0,(0,1)} \to 0, \quad n \to \infty. \end{aligned} \tag{4.7}$$

Convergence of the first term on the right of (4.7) is shown in [9], Lemma 2.21. From (4.5) and (4.6) it follows that $\boldsymbol{u} \cdot \boldsymbol{\nu}^\alpha = 0$ on $S(\alpha)$, hence $\boldsymbol{u} \in V_{\rm div}(\alpha)$.

It remains to show that $\boldsymbol{u}$ solves $(\mathcal{P}(\alpha))$. Let $\overline{\boldsymbol{v}} \in V(\alpha)$ be given. Then accordingly to Lemma 3 there exists a sequence $\{\boldsymbol{v}_k\}$, $\boldsymbol{v}_k \in (H^1(\widehat{\Omega}))^2$ satisfying (3.7) and (3.8). Since $\boldsymbol{v}_k|_{\Omega(\alpha_{n_k})}$ can be used as a test function in $(\mathcal{P}(\alpha_{n_k})_{\varepsilon_{n_k}})$, we obtain:

$$a_{n_k}(\boldsymbol{u}_{n_k}, \boldsymbol{v}_k - \boldsymbol{u}_{n_k}) - b_{n_k}(\boldsymbol{v}_k - \boldsymbol{u}_{n_k}, p_{n_k}) + j_{n_k}(\boldsymbol{v}_{k\tau}) - j_{n_k}(\boldsymbol{u}_{n_k}) \geqslant (\boldsymbol{f}, \boldsymbol{v}_k)_{0,\Omega(\alpha_{n_k})}.$$

Here we used the fact that

$$\frac{1}{\varepsilon_{n_k}} c_{n_k}(\boldsymbol{u}_{n_k}, \boldsymbol{v}_k - \boldsymbol{u}_{n_k}) = -\frac{1}{\varepsilon_{n_k}} c_{n_k}(\boldsymbol{u}_{n_k}, \boldsymbol{u}_{n_k}) \leqslant 0.$$

The rest of the proof is identical with the one of Theorem 3. $\square$

To establish a relation between solutions of $(\mathbb{P})$ and $(\mathbb{P}_\varepsilon)$ for $\varepsilon \to 0+$ we shall also need the continuity of J in the following sense

$$\left.\begin{aligned} \alpha_n &\to \alpha \quad \text{in } C^1([0,1]),\ \alpha_n, \alpha \in \mathcal{U}_{\rm ad} \\ \boldsymbol{y}_n &\to \boldsymbol{y} \quad \text{in } (H^1(\widehat{\Omega}))^2,\ \boldsymbol{y}_n, \boldsymbol{y} \in (H^1_0(\widehat{\Omega}))^2 \\ q_n &\rightharpoonup q \quad \text{in } L^2(\widehat{\Omega}),\ q_n, q \in L^2_0(\widehat{\Omega}) \end{aligned}\right\} \tag{4.8}$$
$$\Rightarrow \lim_{n\to\infty} J(\alpha_n, \boldsymbol{y}_n|_{\Omega(\alpha_n)}, q_n|_{\Omega(\alpha_n)}) = J(\alpha, \boldsymbol{y}|_{\Omega(\alpha)}, q|_{\Omega(\alpha)}).$$

Theorem 5. *Let* (2.9) *and* (4.8) *be satisfied. Then from any sequence* $\{\alpha_\varepsilon^*\}$ *of solutions to* $(\mathbb{P}_\varepsilon)$, $\varepsilon \to 0+$, *one can choose a subsequence (denoted by the same symbol) and find a triplet* $(\alpha^*, \boldsymbol{u}^*, p^*) \in \mathcal{U}_{\rm ad} \times (H_0^1(\widehat{\Omega}))^2 \times L_0^2(\widehat{\Omega})$ *such that*

$$\alpha_\varepsilon^* \to \alpha^* \quad in\ C^1([0,1]), \tag{4.9a}$$

$$\pi_{\alpha_\varepsilon^*}\boldsymbol{u}_\varepsilon(\alpha_\varepsilon^*) \rightharpoonup \boldsymbol{u}^* \quad in\ (H^1(\widehat{\Omega}))^2, \tag{4.9b}$$

$$p_\varepsilon^0(\alpha_\varepsilon^*) \rightharpoonup p^* \quad in\ L_0^2(\widehat{\Omega}),\ \varepsilon \to 0+. \tag{4.9c}$$

Moreover, α^* *is a solution of* $(\mathbb{P})$ *and* $(\boldsymbol{u}^*|_{\Omega(\alpha^*)}, p^*|_{\Omega(\alpha^*)})$ *solves* $(\mathcal{P}(\alpha^*))$. *Besides that, any accumulation point of* $\{(\alpha_\varepsilon^*, \boldsymbol{u}_\varepsilon(\alpha_\varepsilon^*), p_\varepsilon(\alpha_\varepsilon^*))\}$ *in the sense of* (4.9) *has this property.*

Proof. The existence of a subsequence $\{\alpha_\varepsilon^*\}$ satisfying (4.9a) follows from the Arzelà-Ascoli theorem. Furthermore, (4.9b), (4.9c), and the fact that $(\boldsymbol{u}^*|_{\Omega(\alpha^*)}, p^*|_{\Omega(\alpha^*)})$ solves $(\mathcal{P}(\alpha^*))$ are proven in Lemma 5. Let $\overline{\alpha} \in \mathcal{U}_{\rm ad}$ be given and $(\boldsymbol{u}(\overline{\alpha}), p(\overline{\alpha}))$ be the unique solution of $(\mathcal{P}(\overline{\alpha}))$. From (4.1) we know that

$$\begin{aligned} \boldsymbol{u}_\varepsilon(\overline{\alpha}) &\to \boldsymbol{u}(\overline{\alpha}) \quad \text{in } (H^1(\Omega(\overline{\alpha})))^2,\\ p_\varepsilon(\overline{\alpha}) &\rightharpoonup p(\overline{\alpha}) \quad \text{in } L_0^2(\Omega(\overline{\alpha})),\ \varepsilon \to 0+ \end{aligned}$$

and also

$$\begin{aligned} \pi_{\overline{\alpha}}\boldsymbol{u}_\varepsilon(\overline{\alpha}) &\to \pi_{\overline{\alpha}}\boldsymbol{u}(\overline{\alpha}) \quad \text{in } (H^1(\widehat{\Omega}))^2,\\ p_\varepsilon^0(\overline{\alpha}) &\rightharpoonup p^0(\overline{\alpha}) \quad \text{in } L_0^2(\widehat{\Omega}),\ \varepsilon \to 0+. \end{aligned} \tag{4.10}$$

The definition of $(\mathbb{P}_\varepsilon)$ yields

$$J(\alpha_\varepsilon^*, \boldsymbol{u}_\varepsilon(\alpha_\varepsilon^*), p_\varepsilon(\alpha_\varepsilon^*)) \leqslant J(\overline{\alpha}, \boldsymbol{u}_\varepsilon(\overline{\alpha}), p_\varepsilon(\overline{\alpha})).$$

Letting ε tend to zero on the filter of indices for which (4.9) holds, we obtain

$$J(\alpha^*, \boldsymbol{u}^*|_{\Omega(\alpha^*)}, p^*|_{\Omega(\alpha^*)}) \leqslant J(\overline{\alpha}, \boldsymbol{u}(\overline{\alpha}), p(\overline{\alpha}))\ \forall\, \overline{\alpha} \in \mathcal{U}_{\rm ad},$$

making use of (2.9), (4.8), and (4.10). $\square$

5. Approximation of $(\mathbb{P}_\varepsilon)$

In this section, a finite-dimensional approximation of $(\mathbb{P}_\varepsilon)$ will be proposed and analysed. Next we shall assume that $\varepsilon > 0$ is fixed. We introduce a finite element discretization of $(\mathcal{P}(\alpha)_\varepsilon)$ and a discretization of the set $\mathcal{U}_{\rm ad}$. We will show that the discrete shape optimization problem has a solution. Finally, we will study convergence properties of such solutions if the discretization parameter $h \to 0+$.

5.1. Formulation of the discrete problem. We start with the approximation of the admissible set $\mathcal{U}_{\rm ad}$. Since for finite element methods it is convenient to use polygonal domains, we will consider piecewise linear approximations of $\mathcal{U}_{\rm ad}$. On the other hand, as $\mathcal{U}_{\rm ad}$ contains $C^{1,1}$-functions, this approximation of $\mathcal{U}_{\rm ad}$ becomes external and some technical difficulties arise, especially in the convergence analysis.

Let $d \in \mathbb{N}$ be given and set $h := 1/d$. By δ_h we denote the equidistant partition of $[0,1]$:

$$\delta_h\colon\ 0 = a_0 < a_1 < \ldots < a_d = 1,$$

where

$$a_j = jh,\ j = 0,1,\ldots,d.$$

The set of discrete admissible shapes $\mathcal{U}^h_{\rm ad}$ consists of continuous, piecewise linear functions on δ_h which satisfy constraints analogous to those imposed in (2.8):

$$\begin{aligned}\mathcal{U}^h_{\rm ad} := \{\alpha_h \in C([0,1]);\ & \alpha_{h|[a_{i-1},a_i]} \in P_1([a_{i-1},a_i])\ \forall\, i = 1,\ldots,d;\\ & \alpha_{\min} \leqslant \alpha_h(a_i) \leqslant \alpha_{\max}\ \forall\, i = 0,\ldots,d;\\ & |\alpha_h(a_i) - \alpha_h(a_{i-1})| \leqslant C_1 h\ \forall\, i = 1,\ldots,d;\\ & |\alpha_h(a_{i+1}) - 2\alpha_h(a_i) + \alpha_h(a_{i-1})| \leqslant C_2 h^2\ \forall\, i = 1,\ldots,d-1\}.\end{aligned}$$

The positive constants $\alpha_{\min}$, $\alpha_{\max}$, C_1 and C_2 are the same as in (2.8). We denote the set of discrete admissible shapes by

$$\mathcal{O}_h := \{\Omega(\alpha_h);\ \alpha_h \in \mathcal{U}^h_{\rm ad}\}.$$

The symbol $\mathcal{T}_h(\alpha_h)$ will denote a triangulation of $\overline{\Omega}(\alpha_h)$ with the norm h. We will consider the system $\{\mathcal{T}_h(\alpha_h);\ \alpha_h \in \mathcal{U}^h_{\rm ad}\}$ which consists of *topologically equivalent triangulations*, i.e.:

(T1) the number of nodes as well as the neighbours of each triangle in $\mathcal{T}_h(\alpha_h)$ is the same for all $\alpha_h \in \mathcal{U}^h_{\rm ad}$;

(T2) the position of the nodes in $\mathcal{T}_h(\alpha_h)$ depends continuously on α_h;

(T3) the triangulations $\mathcal{T}_h(\alpha_h)$ are compatible with the decomposition of $\partial\Omega(\alpha_h)$ into $S(\alpha_h)$ and $\Gamma(\alpha_h)$ for any $\alpha_h \in \mathcal{U}^h_{\text{ad}}$.

In order to establish convergence results we will also need:

(T4) the system $\{\mathcal{T}_h(\alpha_h);\ \alpha_h \in \mathcal{U}^h_{\text{ad}}\}$ is *uniformly regular* with respect to $h > 0$ and $\alpha_h \in \mathcal{U}^h_{\text{ad}}$, i.e., there exists a constant $\theta_0 > 0$ such that

$$\theta_h(\alpha_h) \geqslant \theta_0 \ \forall\, h > 0 \ \forall\, \alpha_h \in \mathcal{U}^h_{\text{ad}},$$

where $\theta_h(\alpha_h)$ denotes the minimal interior angle of all triangles from $\mathcal{T}_h(\alpha_h)$.

In order to give a finite element discretization of the state problem, we define the spaces of piecewise polynomial functions

$$\widetilde{V}_h(\alpha_h) := \{\boldsymbol{v}_h \in (C(\overline{\Omega}(\alpha_h)))^2;\ \boldsymbol{v}_{h|T} \in (P_2(T))^2\ \forall T \in \mathcal{T}_h(\alpha_h),\ \boldsymbol{v}_h = \mathbf{0} \text{ on } \Gamma(\alpha_h)\},$$
$$L_h(\alpha_h) := \Big\{q_h \in C(\overline{\Omega}(\alpha_h));\ q_{h|T} \in P_1(T)\ \forall T \in \mathcal{T}_h(\alpha_h),\ \int_{\Omega(\alpha_h)} q_h = 0\Big\}.$$

Let $\varepsilon > 0$, $h > 0$ and $\alpha_h \in \mathcal{U}^h_{\text{ad}}$ be given. The discrete penalized state problem reads as follows:

$$(\mathcal{P}_{h\varepsilon}(\alpha_h)) \qquad \begin{aligned} &\text{Find } (\boldsymbol{u}_{h\varepsilon}, p_{h\varepsilon}) := (\boldsymbol{u}_{h\varepsilon}(\alpha_h), p_{h\varepsilon}(\alpha_h)) \in \widetilde{V}_h(\alpha_h) \times L_h(\alpha_h) \text{ s.t.}\\ &\forall\, \boldsymbol{v}_h \in \widetilde{V}_h(\alpha_h)\colon\ a_{\alpha_h}(\boldsymbol{u}_{h\varepsilon}, \boldsymbol{v}_h - \boldsymbol{u}_{h\varepsilon}) - b_{\alpha_h}(\boldsymbol{v}_h - \boldsymbol{u}_{h\varepsilon}, p_{h\varepsilon})\\ &\qquad\qquad + j_{\alpha_h}(\boldsymbol{v}_{h\tau}) - j_{\alpha_h}(\boldsymbol{u}_{h\varepsilon\tau}) + \frac{1}{\varepsilon} c_{\alpha_h}(\boldsymbol{u}_{h\varepsilon}, \boldsymbol{v}_h - \boldsymbol{u}_{h\varepsilon})\\ &\qquad\qquad \geqslant (\boldsymbol{f}, \boldsymbol{v}_h - \boldsymbol{u}_{h\varepsilon})_{0,\Omega(\alpha_h)},\\ &\forall\, q_h \in L_h(\alpha_h)\colon\ b_{\alpha_h}(\boldsymbol{u}_{h\varepsilon}, q_h) = 0. \end{aligned}$$

Since the pair $\widetilde{V}_h(\alpha_h)$ and $L_h(\alpha_h)$ satisfies the Babuška-Brezzi condition (see (5.2) below), problem $\mathcal{P}_{h\varepsilon}(\alpha_h)$ has a unique solution.

Lemma 6. *There exists a constant $c := c(\|\boldsymbol{f}\|_{0,\widehat{\Omega}}) > 0$ independent of $\varepsilon > 0$, $h > 0$ and $\alpha_h \in \mathcal{U}^h_{\text{ad}}$ such that the solution $(\boldsymbol{u}_{h\varepsilon}, p_{h\varepsilon})$ of $(\mathcal{P}_{h\varepsilon}(\alpha_h))$ is bounded:*

$$\|\pi_{\alpha_h} \boldsymbol{u}_{h\varepsilon}\|_{1,\widehat{\Omega}} + \frac{1}{\varepsilon} c_{\alpha_h}(\boldsymbol{u}_{h\varepsilon}, \boldsymbol{u}_{h\varepsilon}) + \|p^0_{h\varepsilon}\|_{0,\widehat{\Omega}} \leqslant c. \tag{5.1}$$

P r o o f. The boundedness of the first two terms in (5.1) can be shown exactly as in the proof of Lemma 4. The pressure estimate will be proven provided that the discrete inf-sup condition

$$\inf_{q \in L_h(\alpha_h)\setminus\{0\}} \ \sup_{\boldsymbol{v} \in \widetilde{V}_h(\alpha_h)\setminus\{\mathbf{0}\}} \frac{b_{\alpha_h}(q, \boldsymbol{v})}{\|q\|_{0,\Omega(\alpha_h)} \|\boldsymbol{v}\|_{1,\Omega(\alpha_h)}} \geqslant c \tag{5.2}$$

holds with a constant $c > 0$ independent of $h > 0$ and $\alpha_h \in \mathcal{U}_{\rm ad}^h$. Indeed, in [2], Chapter VI.6, it is shown that (5.2) holds with a constant $c := c(\bar{c})$, where $\bar{c}$ is the constant in the inf-sup condition for the spaces $L_0^2(\alpha_h)$ and $\widetilde{V}(\alpha_h)$. As we have pointed out before, $\bar{c}$ does not depend on α_h, and so neither does c. □

Analogously to the continuous setting, the discrete shape optimization problem is defined as the minimization of $\mathfrak{J}_{h\varepsilon}$ on $\mathcal{U}_{\rm ad}^h$, where

$$\mathfrak{J}_{h\varepsilon}(\alpha_h) := J(\alpha_h, \boldsymbol{u}_{h\varepsilon}(\alpha_h), p_{h\varepsilon}(\alpha_h)),$$

with $(\boldsymbol{u}_{h\varepsilon}(\alpha_h), p_{h\varepsilon}(\alpha_h))$ being the solution of $(\mathcal{P}_{h\varepsilon}(\alpha_h))$. Thus, for each $\varepsilon > 0$ and $h > 0$, the discrete shape optimization problem reads:

$$(\mathbb{P}_{h\varepsilon}) \qquad \text{Find } \alpha_{h\varepsilon}^* \in \mathcal{U}_{\rm ad}^h \text{ such that } \forall\, \alpha_h \in \mathcal{U}_{\rm ad}^h\colon\ \mathfrak{J}_{h\varepsilon}(\alpha_{h\varepsilon}^*) \leqslant \mathfrak{J}_{h\varepsilon}(\alpha_h).$$

Adapting the approach from the previous section to the discrete case, one can easily show that the graph

$$\begin{aligned}\mathcal{G}_{h\varepsilon} := \{&(\alpha_h, \boldsymbol{u}_{h\varepsilon}(\alpha_h), p_{h\varepsilon}(\alpha_h));\ \alpha_h \in \mathcal{U}_{\rm ad}^h,\\ &(\boldsymbol{u}_{h\varepsilon}(\alpha_h), p_{h\varepsilon}(\alpha_h)) \text{ is the solution of } \mathcal{P}_{h\varepsilon}(\alpha_h)\}\end{aligned}$$

is compact for any $\varepsilon > 0$ and $h > 0$, so the following result is straightforward.

Theorem 6. *Let $h, \varepsilon > 0$ be fixed and $\mathfrak{J}_{h\varepsilon}$ be lower semicontinuous on $\mathcal{U}_{\rm ad}^h$. Then $(\mathbb{P}_{h\varepsilon})$ has a solution.*

5.2. Convergence analysis. In this section we will analyse the mutual relation between solutions to $(\mathbb{P}_{h\varepsilon})$ and $(\mathbb{P}_{\varepsilon})$ as $h \to 0+$ keeping $\varepsilon > 0$ fixed, aiming to show that the discrete optimal shapes converge in some sense to an optimal shape of the continuous setting.

We start by recalling some auxiliary results concerning the relationship between $\mathcal{U}_{\rm ad}^h$, $h \to 0+$, and $\mathcal{U}_{\rm ad}$, which can be proven using the same arguments as in [10], [11].

Lemma 7. *For any $\alpha \in \mathcal{U}_{\rm ad}$ there exists a sequence $\{\alpha_h\}$, $\alpha_h \in \mathcal{U}_{\rm ad}^h$ such that $\alpha_h \to \alpha$ in $C([0,1])$, $h \to 0+$.*

Lemma 8. *Let $\{\alpha_h\}$, $\alpha_h \in \mathcal{U}_{\rm ad}^h$ be such that $\alpha_h \to \alpha$ in $C([0,1])$, $h \to 0+$. Then $\alpha \in \mathcal{U}_{\rm ad}$ and there exists a subsequence $\{\alpha_{h_m}\} \subset \{\alpha_h\}$ satisfying:*

$$\alpha_{h_m}' \to \alpha' \quad \text{in } L^\infty(0,1), \quad h_m \to 0+. \tag{5.3}$$

In order to pass to the limit in the variational inequality we also need the following result.

Lemma 9. *Let $\{\alpha_h\}$, $\alpha_h \in \mathcal{U}_{\mathrm{ad}}^h$ be such that $\alpha_h \to \alpha$ in $C([0,1])$, $h \to 0+$ and let $\boldsymbol{v} \in \widetilde{V}(\alpha)$ be given. Then there exist a sequence $\{\boldsymbol{v}_h\}$, $\boldsymbol{v}_h \in (H^1(\widehat{\Omega}))^2$, and a function $\overline{\boldsymbol{v}} \in (H^1(\widehat{\Omega}))^2$ such that $\boldsymbol{v}_h|_{\Omega(\alpha_h)} \in \widetilde{V}_h(\alpha_h)$, $\overline{\boldsymbol{v}}|_{\Omega(\alpha)} = \boldsymbol{v}$ and*

$$\boldsymbol{v}_h \to \overline{\boldsymbol{v}} \quad \text{in } (H^1(\widehat{\Omega}))^2,\ h \to 0+. \tag{5.4}$$

P r o o f. Let $\eta > 0$ be arbitrary and set $\overline{\boldsymbol{v}} := \pi_\alpha \boldsymbol{v} \in (H_0^1(\widehat{\Omega}))^2$. By the density argument one can find $\boldsymbol{\varphi} \in (C_0^\infty(\widehat{\Omega}))^2$ such that

$$\|\boldsymbol{\varphi} - \overline{\boldsymbol{v}}\|_{1,\widehat{\Omega}} < \frac{\eta}{2}. \tag{5.5}$$

Let $\Theta(\alpha_h) = \widehat{\Omega} \setminus \overline{\Omega}(\alpha_h)$ and $\widehat{\mathcal{T}}_h(\alpha_h)$ be a triangulation of $\overline{\Theta}(\alpha_h)$ such that the nodes of $\mathcal{T}_h(\alpha_h)$ and $\widehat{\mathcal{T}}_h(\alpha_h)$ on $S(\alpha_h)$ coincide and, moreover, the family $\{\widehat{\mathcal{T}}_h(\alpha_h)\}$, $h \to 0$, satisfies (T1), (T2) and (T4). By r_h we denote the piecewise quadratic Lagrange interpolation operator in $\overline{\widehat{\Omega}}$ with the triangulation $\mathcal{T}_h(\alpha_h) \cup \widehat{\mathcal{T}}_h(\alpha_h)$. From (T4) it follows that there exists a constant $c > 0$ independent of $h > 0$ and $\alpha_h \in \mathcal{U}_{\mathrm{ad}}^h$ such that

$$\|r_h\boldsymbol{\varphi} - \boldsymbol{\varphi}\|_{1,\widehat{\Omega}} \leqslant ch\|\boldsymbol{\varphi}\|_{2,\widehat{\Omega}} \quad \forall\, \boldsymbol{\varphi} \in (H^2(\widehat{\Omega}))^2. \tag{5.6}$$

We set $\boldsymbol{v}_h := r_h\boldsymbol{\varphi}$. Then clearly $\boldsymbol{v}_h|_{\Omega(\alpha_h)} \in \widetilde{V}_h(\alpha_h)$ for every $h > 0$. Moreover, from (5.6) we see that there exists $h_0 := h_0(\eta) > 0$ such that for any $h \leqslant h_0$ it holds that

$$\|\boldsymbol{v}_h - \boldsymbol{\varphi}\|_{1,\widehat{\Omega}} < \frac{\eta}{2},$$

which together with (5.5) completes the proof. □

The following lemma establishes convergence properties of solutions to $(\mathcal{P}_{h\varepsilon}(\alpha_h))$ as $h \to 0+$.

Lemma 10. *Let $\{\alpha_h\}$, $\alpha_h \in \mathcal{U}_{\mathrm{ad}}^h$, $h \to 0+$, be an arbitrary sequence. Then there exist its subsequence (denoted by the same symbol), a function $\alpha \in \mathcal{U}_{\mathrm{ad}}$, and a pair $(\bar{\boldsymbol{u}}, \bar{p}) \in (H_0^1(\widehat{\Omega}))^2 \times L_0^2(\widehat{\Omega})$ such that*

$$\begin{aligned} \alpha_h &\to \alpha \quad \text{in } C([0,1]),\\ \pi_{\alpha_h}\boldsymbol{u}_{h\varepsilon}(\alpha_h) &\rightharpoonup \bar{\boldsymbol{u}} \quad \text{in } (H^1(\widehat{\Omega}))^2,\\ p_{h\varepsilon}(\alpha_h)^0 &\rightharpoonup \bar{p} \quad \text{in } L^2(\widehat{\Omega}),\ h \to 0+. \end{aligned}$$

Moreover, $(\bar{\boldsymbol{u}}|_{\Omega(\alpha)}, \bar{p}|_{\Omega(\alpha)})$ is the solution to $(\mathcal{P}(\alpha)_\varepsilon)$.

P r o o f. The existence of convergent subsequences follows from Lemma 6, the Arzelà-Ascoli theorem and Lemma 8. From Lemma 9 we know that for any $\boldsymbol{v} \in \widetilde{V}(\alpha)$ one can find a sequence $\{\boldsymbol{v}_h\}$, $\boldsymbol{v}_h|_{\Omega(\alpha_h)} \in \widetilde{V}_h(\alpha_h)$ satisfying (5.4). The limit passage for $h \to 0+$ in $(\mathcal{P}_{h\varepsilon}(\alpha_h))$ can be done as in the proof of Theorem 3, making use of (5.3). $\square$

To establish the convergence of solutions to $(\mathbb{P}_{h\varepsilon})$ as $h \to 0+$ we shall need the continuity of J in the following sense:

$$\left.\begin{aligned} \alpha_h \to \alpha \quad & \text{in } C([0,1]),\ \alpha_h \in \mathcal{U}_{\rm ad}^h, \alpha \in \mathcal{U}_{\rm ad} \\ \pi_{\alpha_h}\boldsymbol{y}_h \rightharpoonup \boldsymbol{y} \quad & \text{in } (H^1(\widehat{\Omega}))^2,\ \boldsymbol{y}_h \in \widetilde{V}_h(\alpha_h), \boldsymbol{y} \in (H_0^1(\widehat{\Omega}))^2 \\ q_h^0 \rightharpoonup q \quad & \text{in } L^2(\widehat{\Omega}),\ q_h \in L_h(\alpha_h), q \in L_0^2(\widehat{\Omega}) \end{aligned}\right\} \tag{5.7}$$
$$\Rightarrow \lim_{h\to 0+} J(\alpha_h, \boldsymbol{y}_h, q_h) = J(\alpha, \boldsymbol{y}|_{\Omega(\alpha)}, q|_{\Omega(\alpha)}).$$

We have the following convergence result.

Theorem 7. *Let* $\{\alpha_{h\varepsilon}^*\}$, $h \to 0+$, *be a sequence of solutions to* $(\mathbb{P}_{h\varepsilon})$, $h \to 0+$, *and let* (5.7) *be satisfied. Then there exist: a subsequence of* $\{\alpha_{h\varepsilon}^*\}$ *(denoted by the same symbol) and a triplet* $(\alpha_\varepsilon^*, \boldsymbol{u}_\varepsilon^*, p_\varepsilon^*) \in \mathcal{U}_{\rm ad} \times (H_0^1(\widehat{\Omega}))^2 \times L_0^2(\widehat{\Omega})$ *such that*

$$\begin{aligned} \alpha_{h\varepsilon}^* \to \alpha_\varepsilon^* \quad & \text{in } C([0,1]), \\ \pi_{\alpha_{h\varepsilon}^*}\boldsymbol{u}_{h\varepsilon}(\alpha_{h\varepsilon}^*) \rightharpoonup \boldsymbol{u}_\varepsilon^* \quad & \text{in } (H^1(\widehat{\Omega}))^2, \\ p_{h\varepsilon}^0(\alpha_{h\varepsilon}^*) \rightharpoonup p_\varepsilon^* \quad & \text{in } L^2(\widehat{\Omega}),\ h \to 0+. \end{aligned}$$

Moreover, α_ε^* *is a solution of* $(\mathbb{P}_\varepsilon)$ *and* $(\boldsymbol{u}_\varepsilon^*|_{\Omega(\alpha_\varepsilon^*)}, p_\varepsilon^*|_{\Omega(\alpha_\varepsilon^*)})$ *solves* $(\mathcal{P}_\varepsilon(\alpha_\varepsilon^*))$.

The p r o o f is analogous to the one of Theorem 5.

References

[1] *J. Březina*: Asymptotic properties of solutions to the equations of incompressible fluid mechanics. J. Math. Fluid Mech. *12* (2010), 536–553.

[2] *F. Brezzi, M. Fortin*: Mixed and Hybrid Finite Element Methods. Springer Series in Computational Mathematics 15, Springer, New York, 1991.

[3] *D. Bucur, E. Feireisl, Š. Nečasová*: Influence of wall roughness on the slip behaviour of viscous fluids. Proc. R. Soc. Edinb., Sect. A, Math. *138* (2008), 957–973.

[4] *M. Bulíček, P. Gwiazda, J. Málek, A. Świerczewska-Gwiazda*: On unsteady flows of implicitly constituted incompressible fluids. SIAM J. Math. Anal. *44* (2012), 2756–2801.

[5] *D. Chenais*: On the existence of a solution in a domain identification problem. J. Math. Anal. Appl. *52* (1975), 189–219.

[6] *H. Fujita*: A mathematical analysis of motions of viscous incompressible fluid under leak or slip boundary conditions. RIMS Kokyuroku *888* (1994), 199–216.

[7] *H. Fujita*: A coherent analysis of Stokes flows under boundary conditions of friction type. J. Comput. Appl. Math. *149* (2002), 57–69.
[8] *G. P. Galdi*: An Introduction to the Mathematical Theory of the Navier-Stokes Equations. Vol. 1: Linearized Steady Problems. Springer Tracts in Natural Philosophy 38, Springer, New York, 1994.
[9] *J. Haslinger, R. A. E. Mäkinen*: Introduction to Shape Optimization: Theory, Approximation, and Computation. Advances in Design and Control 7. SIAM Society for Industrial and Applied Mathematics, Philadelphia, 2003.
[10] *J. Haslinger, J. V. Outrata, R. Pathó*: Shape optimization in 2D contact problems with given friction and a solution-dependent coefficient of friction. Set-Valued Var. Anal. *20* (2012), 31–59.
[11] *I. Hlaváček, R. Mäkinen*: On the numerical solution of axisymmetric domain optimization problems. Appl. Math., Praha *36* (1991), 284–304.
[12] *A. Liakos*: Weak imposition of boundary conditions in the Stokes and Navier-Stokes equation. PhD thesis, University of Pittsburgh, 1999.
[13] *J. Málek, K. R. Rajagopal*: Mathematical issues concerning the Navier-Stokes equations and some of its generalizations. Evolutionary Equations. Vol. II (C. M. Dafermos, ed.). Handbook of Differential Equations, Elsevier/North-Holland, Amsterdam, 2005, pp. 371–459.
[14] *C. L. Navier*: Mémoire sur les lois du mouvement des fluides. Mem. Acad. R. Sci. Paris *6* (1823), 389–416.
[15] *N. Saito*: On the Stokes equation with the leak and slip boundary conditions of friction type: regularity of solutions. Publ. Res. Inst. Math. Sci. *40* (2004), 345–383; errata ibid. *48* (2012), 475–476.
[16] *J. Sokołowski, J.-P. Zolesio*: Introduction to Shape Optimization: Shape Sensitivity Analysis. Springer Series in Computational Mathematics 16, Springer, Berlin, 1992.
[17] *J. Stebel*: On shape stability of incompressible fluids subject to Navier's slip condition. J. Math. Fluid Mech. *14* (2012), 575–589.

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