C-partner curves and their applications

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Abstract. In this study, we define a new type of partner curves called C-partner curves and give some theorems characterizing C-partner curves. We obtain that the rectifying planes of C-partner curves intersect along a common line at a constant angle. We also introduce some applications between C-partner curves and some special curves such as helices and slant helices. We show that considering C-partner curves, a slant helix can be constructed by another slant helix.

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Key words: alternative frame; C-partner curves; slant helix.

1 Introduction

In 1845, French mathematician Saint-Venant introduced a question about that whether upon the ruled surface generated by the principal normals of a curve in the three-dimensional Euclidean space E^3, another curve can exist for which the principal normals of original curve are also its principal normals [14]. Bertrand answered this question and show that such another curve exists if and only if a linear relationship with constant coefficients shall exists between the curvature and torsion of original curve [2]. Later, such curve pairs have been called Bertrand partner curves or Bertrand curves. Bertrand curves have been studied by many mathematicians and different characterizations and applications of these curves have been introduced [3, 6, 7, 11, 10, 13, 19].

Recently, Liu and Wang have introduced a new definition of curve pairs by originating the notion of Bertrand curve. They have called these new kind of curve pairs as Mannheim partner curves given by the property that the principal normal vectors of original curve coincide with the binormal vectors of second curve [18]. After, Mannheim curves have been studied by some mathematicians and new properties of these curves have been obtained [4, 9, 20].

The goal of this paper is to define a new kind of associated curve pairs and give characterizations for these curves. For this purpose, we use an alternative frame on original curve and define another curve by using this frame. This new curve pair is called C-partner curves. First, we give a brief summary of curve theory and alternative frame. In Section 3, the definition and main characterizations for C-partner curves

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are introduced. In Section 4, some applications of C-partner curves are given. It is shown that original curve is a slant helix (or a helix) if and only if the second curve is also a slant helix (or a helix). In the last section, some numerical examples are given.

2 Preliminaries

Let $\alpha = \alpha(s)$ be a regular space curve in the Euclidean 3-space E^3 and $\{T, N, B\}$ be the Frenet frame of $\alpha(s)$, where T, N, B are unit tangent vector field, principal normal vector field and binormal vector field, respectively. Then the Frenet formulae of the curve is given by

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}, \tag{2.1}$$

where κ, τ are called the curvature and the torsion of the curve, respectively. From (2.1), the unit Darboux vector W of $\alpha(s)$ given by the equation

$$W = \frac{1}{\sqrt{\kappa^2+\tau^2}}\left(\tau T + \kappa B\right) \tag{2.2}$$

is the angular velocity vector of the curve α [8]. It is obvious from (2.2) that the Darboux vector is perpendicular to the principal normal vector field N. Then, defining a unit vector field C by the cross product $C = W \times N$ makes possible to build another orthonormal moving frame along the curve $\alpha(s)$. This frame is represented by $\{N, C, W\}$ and is an alternative frame to curve rather than the Frenet frame $\{T, N, B\}$.

The derivative formulae of the alternative frame is given by

$$\begin{bmatrix} N' \\ C' \\ W' \end{bmatrix} = \begin{bmatrix} 0 & \beta & 0 \\ -\beta & 0 & \gamma \\ 0 & -\gamma & 0 \end{bmatrix} \begin{bmatrix} N \\ C \\ W \end{bmatrix}, \tag{2.3}$$

where $\beta = \sqrt{\kappa^2+\tau^2}$ and $\gamma = \frac{\kappa^2}{\kappa^2+\tau^2}\left(\frac{\tau}{\kappa}\right)'$ [17]]. Since the principal normal vector N is common in both frames, it is possible to form a relationship between the Frenet frame and alternative frame such as

$$\begin{cases} C = -\bar{\kappa}T + \bar{\tau}B, \\ W = \bar{\tau}T + \bar{\kappa}B, \end{cases} \tag{2.4}$$

or

$$\begin{cases} T = -\bar{\kappa}C + \bar{\tau}W, \\ B = \bar{\tau}C + \bar{\kappa}W, \end{cases} \tag{2.5}$$

where $\bar{\kappa} = \kappa/\beta$ and $\bar{\tau} = \tau/\beta$.

A regular curve α is called a helix if the tangent lines of the curve make a constant angle with a fixed direction and a helix is characterized by the property that $\frac{\tau}{\kappa}$ is constant [16]. If the principal normal lines of the curve make a constant angle with a fixed direction, then the curve is called a slant helix and characterized by the equality

$$\frac{\kappa^2}{(\kappa^2+\tau^2)^{3/2}}\left(\frac{\tau}{\kappa}\right)' = constant.$$

[8]. Then, the characterization of a slant helix according to alternative frame is given as follows:

Remark 2.1. A regular curve α with curvatures β, γ is a slant helix if and only if $\frac{\gamma}{\beta}(s) = \text{constant}$.

3 C-partner curves in E^3

Associated curves defined by some special properties are the most fascinating subject of curve theory. Bertrand partner curves and Mannheim partner curves are the well-known examples of such curves. Two curves which, at any point, have a common principal normal vector are called Bertrand curves. It is a well-known result that a regular curve α in E^3 is a Bertrand curve if and only if its curvature functions κ and τ satisfy the condition $a\kappa(s) + b\tau(s) = 1$, where $a,\ b$ are constant real numbers [11]. Another classical characterization for Bertrand curves is that the distance between the curves and the angle between unit tangent vectors of the curves are constants [16].

Another kind of partner curves is Mannheim partner curves. A curve α in E^3 is called Mannheim curve if there exists another curve $\vartheta(s)$ such that the principal normal vector fields of α coincide with the binormal vector fields of $\vartheta(s)$. Then $\vartheta(s)$ is called Mannheim partner curve of α. Although the angle between unit tangents of Bertrand curves is constant, it is not a constant for Mannheim partner curves [9, 18].

The purpose of this section is to define a new type of partner curves by considering alternative frame and to find some characterizations for these curves related to distance function between the corresponding points of the curves and angle function between the rectifying planes of the curves. First, we give the following definition.

Definition 3.1. Let $\alpha = \alpha(s)$ and $\alpha^* = \alpha^*(s^*)$ be two regular space curves in the Euclidean 3-space E^3 with Frenet frames $\{T, N, B\}$, $\{T^*, N^*, B^*\}$, curvatures $\kappa,\ \kappa^*$, torsions $\tau,\ \tau^*$, respectively, and let the alternative moving frames and curvatures of curves be $\{N, C, W\}$,$\beta,\ \gamma$ and $\{N^*, C^*, W^*\}$, $\beta^*,\ \gamma^*$, respectively. The curves α and α^* are called C-partner curves if the vector fields C and C^* coincide, i.e., $C = C^*$ holds at the corresponding points of the curves.

From Definition 3.1, the parametric representation of α^* can be given by

$$\alpha^*(s) = \alpha(s) + R(s)C(s),$$

where $R = R(s)$ is the distance function between the curves α and α^*. Since the vector fields C and C^* are the same, we are able to give the relationship between the alternative frames of α and α^* such as

$$\left\{\begin{array}{l} N^* = \cos\theta N - \sin\theta W, \\ W^* = \sin\theta N + \cos\theta W, \end{array}\right. \tag{3.1}$$

or

$$\begin{cases} N = \cos\theta N^* + \sin\theta W^*, \\ W = -\sin\theta N^* + \cos\theta W^*. \end{cases} \tag{3.2}$$

where $\theta = \theta(s)$ is the angle function between vector fields N, N^* or W, W^*. Moreover, from (3.1) it is clear that $\theta = \theta(s)$ is the angle between the rectifying planes of curves at the corresponding points.

Now, we give some theorems characterizing C-partner curves. Whenever we talk about the curves α and α^*, we will assume that the curves have frames and curvatures as given in Definition 3.1.

Theorem 3.1. *Let α and α^* be C-partner curves in E^3. At the corresponding points of the curves, the angle θ between the rectifying planes of curves is constant.*

Proof. By differentiating the second equation of (3.1) with respect to the arc length parameter s of α, we obtain

$$\frac{dW^*}{ds^*}\frac{ds^*}{ds} = \theta' \cos\theta N + \sin\theta N' - \theta' \sin\theta W + \cos\theta W'.$$

By using the derivative formulae given in (2.3), from the last equation we get

$$-\gamma^* C^* \frac{ds^*}{ds} = \theta' \cos\theta N + (\sin\theta\beta - \cos\theta\gamma)\, C - \theta' \sin\theta W. \tag{3.3}$$

Since the curves α and α^* are C-partner curves, from Definition 3.1, we have that $C = C^*$. Thus, from (3.3), it follows

$$\begin{cases} \theta' \cos\theta = 0, \\ \theta' \sin\theta = 0, \end{cases}$$

which means that $\theta = \text{constant}$. □

Theorem 3.1 gives us another geometric definition of C-partner curves. We can define C-partner curves as curves for which at the corresponding points of curves joined with a line, the rectifying planes intersect along this line at a constant angle. Now, using this fact, we can give characterizations according to distance function $R = R(s)$.

Theorem 3.2. *Let α and α^* be C-partner curves. Then, the distance R between the curves is constant if and only if $\cos\theta\,(\bar{\tau}\beta' - \bar{\tau}'\beta) + \sin\theta\,(\bar{\tau}\gamma' - \bar{\tau}'\gamma) = 0$ or equivalently,*

$$\frac{\left(\frac{\bar{\tau}}{\beta}\right)' \beta^2}{\left(\frac{\bar{\tau}}{\gamma}\right)' \gamma^2} = \text{constant}$$

holds.

Proof. From Definition 3.1, the parametric representation of α^* can be given by

$$\alpha^*(s) = \alpha(s) + R(s)C(s),$$

where $R = R(s)$ is the distance function between the curves α and α^*. Then, by differentiating the last equation with respect to s and by using formulae (2.3), we get

$$T^* \frac{ds^*}{ds} = T - R\beta N + R'C + R\gamma W.$$

Since $C = C^*$ and we have $\theta =$ constant, by using equations (2.4), (2.5), (3.1) and (3.2), from last equation we obtain the following system

$$\left\{ \begin{array}{l} \bar{\kappa}^* \frac{ds^*}{ds} = \bar{\kappa} - R', \\ \bar{\tau} \sin\theta + R\beta \cos\theta + R\gamma \sin\theta = 0, \\ \bar{\tau}^* \frac{ds^*}{ds} = \bar{\tau} \cos\theta - R\beta \sin\theta + R\gamma \cos\theta. \end{array} \right. \tag{3.4}$$

From the second equation of system (3.4), we have

$$R = \frac{-\bar{\tau} \sin\theta}{\beta \cos\theta + \gamma \sin\theta}.$$

By differentiating the last equation with respect to s, we get

$$R' = \frac{\sin\theta \cos\theta\, (\bar{\tau}\beta' - \bar{\tau}'\beta) + \sin^2\theta\, (\bar{\tau}\gamma' - \bar{\tau}'\gamma)}{(\beta \cos\theta + \gamma \sin\theta)^2}.$$

Therefore, $R =$ constant if and only if

$$\cos\theta\, (\bar{\tau}\beta' - \bar{\tau}'\beta) + \sin\theta\, (\bar{\tau}\gamma' - \bar{\tau}'\gamma) = 0,$$

or equivalently,

$$\frac{\left(\frac{\bar{\tau}}{\beta}\right)' \beta^2}{\left(\frac{\bar{\tau}}{\gamma}\right)' \gamma^2} = -\tan\theta = \text{constant}.$$

□

Theorem 3.3. *Let α and α^* be C-partner curves. Then, the distance function R between the curves is given by*

$$R = \frac{\gamma^* \bar{\tau} - \gamma \bar{\tau}^*}{\beta\beta^* - \gamma\gamma^*}.$$

Proof. From the second equation of system (3.4), we have

$$R\beta = -\frac{(\bar{\tau} + R\gamma) \sin\theta}{\cos\theta}.$$

Substituting the last equation into the third equation of the system (3.4), we get

$$\frac{\bar{\tau} + R\gamma}{\bar{\tau}^*} = \frac{ds^*}{ds} \cos\theta. \tag{3.5}$$

Similarly, from the second equation of system (3.4), we have

$$R\gamma = -\frac{\bar{\tau} \sin\theta + R\beta \cos\theta}{\sin\theta}.$$

If we substitute the last equation into the last equation of system (3.4), we obtain

$$\frac{R\beta}{\bar{\tau}^*} = -\frac{ds^*}{ds}\sin\theta. \tag{3.6}$$

Analogue to the calculations given above, if we take $\alpha = \alpha^* - RC^*$ instead of $\alpha^* = \alpha + RC$ and apply the same process, we have the following equations,

$$\frac{\bar{\tau}^* - R\gamma^*}{\bar{\tau}} = \frac{ds}{ds^*}\cos\theta, \tag{3.7}$$

$$\frac{R\beta^*}{\bar{\tau}^*} = -\frac{ds}{ds^*}\sin\theta. \tag{3.8}$$

By multiplying (3.5) with (3.7) and multiplying (3.6) with (3.8) and adding the obtained results, we have desired equality,

$$R = \frac{\gamma^*\bar{\tau} - \gamma\bar{\tau}^*}{\beta\beta^* - \gamma\gamma^*}.$$

□

Now, from the theorems given above, we can give the following corollary.

Corollary 3.4. *Let α and α^* be C-partner curves. The followings are equivalent:*

1. *R is constant,*
2. $\cos\theta\left(\bar{\tau}\beta' - \bar{\tau}'\beta\right) + \sin\theta\left(\bar{\tau}\gamma' - \bar{\tau}'\gamma\right) = 0,$
3. $\frac{\left(\frac{\bar{\tau}}{\beta}\right)'\beta^2}{\left(\frac{\bar{\tau}}{\gamma}\right)'\gamma^2} = \text{constant},$
4. $\frac{\gamma^*\bar{\tau} - \gamma\bar{\tau}^*}{\beta\beta^* - \gamma\gamma^*} = \text{constant}.$

4 Applications of C-partner curves

There exists a powerful contact between helices and Bertrand curves. So, many applications related to Bertrand curves can be given. For instance, a spherical curve α is a circle if and only if the corresponding Bertrand curves are circular helices [7]. Moreover, a Bertrand curve can be constructed from a general helix [1]. Analogue to Bertrand curves, there are some relations between slant helices and C-partner curves. In this section, we introduce these relations.

Theorem 4.1. *Let α and α^* be C-partner curves. Then, α is a slant helix if and only if α^* is a slant helix.*

Proof. By using the equations (2.3) and (3.1), we have

$$\begin{aligned} \beta^* &= \left\langle \tfrac{dN^*}{ds^*}, C^* \right\rangle = \tfrac{ds}{ds^*}\left(\cos\theta\beta + \sin\theta\gamma\right), \\ \gamma^* &= -\left\langle \tfrac{dW^*}{ds^*}, C^* \right\rangle = \tfrac{ds}{ds^*}\left(\cos\theta\gamma - \sin\theta\beta\right), \end{aligned}$$

respectively. If we divide the last two equations to each other, we obtain

$$\frac{\gamma^*}{\beta^*} = \frac{\cos\theta\frac{\gamma}{\beta} - \sin\theta}{\cos\theta + \sin\theta\frac{\gamma}{\beta}}. \tag{4.1}$$

Let now α be a slant helix. From Remark 2.1, we have $\frac{\gamma}{\beta} = c =$ constant. Since θ is constant, from (4.1) it follows

$$\frac{\gamma^*}{\beta^*} = \frac{c\cos\theta - \sin\theta}{\cos\theta + c\sin\theta} = constant,$$

which means that α^* is a slant helix.

Conversely, if α^* is a slant helix, i.e., $\gamma^*/\beta^* = d =$ constant, from (4.1) we get

$$\frac{\gamma}{\beta} = \frac{d\cos\theta + \sin\theta}{\cos\theta - d\sin\theta\gamma} = constant,$$

which leads us to the result that α is a slant helix. □

As an application of C-partner curves, Theorem 4.1 shows that a slant helix can be constructed by another slant helix. Moreover, as a result of Theorem 4.1, we can give the following corollaries.

Corollary 4.2. *Let α and α^* be C-partner curves. If α is a helix, then α^* is a slant helix. If α^* is a helix, then α is a slant helix.*

Corollary 4.3. *C-partner curves of a helix α form a family of slant helices.*

Theorem 4.4. *Let α and α^* be C-partner curves. If one of the curves α and α^* is a slant helix, then the distance function R and curvatures $\bar{\tau}$, $\bar{\tau}^*$, β, β^* satisfy the condition*

$$R = \mu\frac{\bar{\tau}^*}{\beta^*} + \lambda\frac{\bar{\tau}}{\beta},$$

where μ and λ are real constants.

Proof. Since α and α^* are C-partner curves, from Theorem 3.3, we have

$$R = \frac{\gamma^*\bar{\tau} - \gamma\bar{\tau}^*}{\beta\beta^* - \gamma\gamma^*}. \tag{4.2}$$

Moreover, if one of the curves α and α^* is a slant helix, then from Theorem 4.1 we have that other curve is also a slant helix. So, $\gamma/\beta = d = constant$ and$\gamma^*/\beta^* = d^* =$ constant. Dividing the numerator and denominator of (4.2) with β, we obtain

$$R = \frac{\frac{\gamma}{\beta}^*\bar{\tau} - \frac{\gamma}{\beta}\bar{\tau}^*}{\beta^* - \frac{\gamma}{\beta}\gamma^*}.$$

Dividing again with β^* and substituting $\gamma/\beta = d$ and $\gamma^*/\beta^* = d^*$ in last equality, we get

$$R = \left(\frac{d}{dd^* + 1}\right)\frac{\bar{\tau}^*}{\beta^*} + \left(\frac{d^*}{dd^* + 1}\right)\frac{\bar{\tau}}{\beta}.$$

Finally, by taking $\mu = \frac{d}{dd^*+1}$, $\lambda = \frac{d^*}{dd^*+1}$, it follows

$$R = \mu \frac{\bar{\tau}^*}{\beta^*} + \lambda \frac{\bar{\tau}}{\beta},$$

where $\mu,\ \lambda$ are constants. □

Theorem 4.5. *Let one of the curves α, α^* and α^{**} be a slant helix. If both α^* and α^{**} are C-partner curves of α, then*

$$c_1 \frac{\bar{\tau}^{**}}{\beta^{**}} + c_2 \frac{\bar{\tau}^*}{\beta^*} + c_3 \frac{\bar{\tau}}{\beta} = 0, \tag{4.3}$$

holds, where $\bar{\tau}^{**},\ \beta^{**}$ are the curvatures of α^{**} and $c_1,\ c_2,\ c_3$ are constants.

Proof. Since both α^* and α^{**} are C-partner curves of α, from Definition 3.1 it is clear that α^{**} is also a C-partner curve of α^*. Then, from Theorem 4.1 and Theorem 4.4, we have

$$R_1 = \mu_1 \frac{\bar{\tau}^*}{\beta^*} + \lambda_1 \frac{\bar{\tau}}{\beta}, R_2 = \mu_2 \frac{\bar{\tau}^{**}}{\beta^{**}} + \lambda_2 \frac{\bar{\tau}}{\beta}, R_3 = \mu_3 \frac{\bar{\tau}^{**}}{\beta^{**}} + \lambda_3 \frac{\bar{\tau}^*}{\beta^*}, \tag{4.4}$$

where $R_1,\ R_2,\ R_3$ are distance functions between the curves α and α^*, α and α^{**}, α^* and α^{**}, respectively, and $\mu_i,\ \lambda_i\ \ (1 \leq i \leq 3)$ are constants. Since $R_3 = R_2 - R_1$, from (4.4) we have

$$R_3 = \mu_2 \frac{\bar{\tau}^{**}}{\beta^{**}} - \mu_1 \frac{\bar{\tau}^*}{\beta^*} + (\lambda_2 - \lambda_1) \frac{\bar{\tau}}{\beta}. \tag{4.5}$$

From (4.4) and (4.5) it follows,

$$c_1 \frac{\bar{\tau}^{**}}{\beta^{**}} + c_2 \frac{\bar{\tau}^*}{\beta^*} + c_3 \frac{\bar{\tau}}{\beta} = 0,$$

where $c_1 = \mu_3 - \mu_2,\ \ c_2 = \lambda_3 + \mu_1,\ \ c_3 = \lambda_1 - \lambda_2$ are constants. □

5 Some examples

Example 5.1. *(Cylindrical Helix)* Consider a cylindrical helix α given by the parametrization

$$\alpha_1(s) = \left(\cos \frac{s}{\sqrt{2}}, \sin \frac{s}{\sqrt{2}}, \frac{s}{\sqrt{2}}\right).$$

Then, two C-partner curves $\alpha_1^*,\ \alpha_1^{**}$ for $R = 2$ and $R = s + \frac{1}{2}$ are given in Figure 1 and Figure 2, respectively.

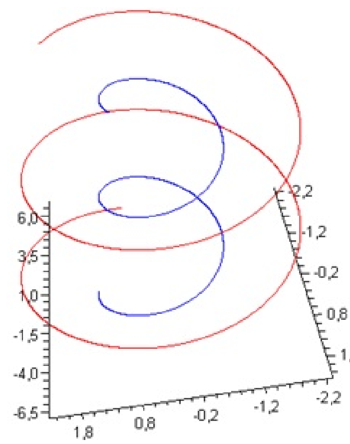

Figure 1. Curves α_1(blue) and α_1^*(red).

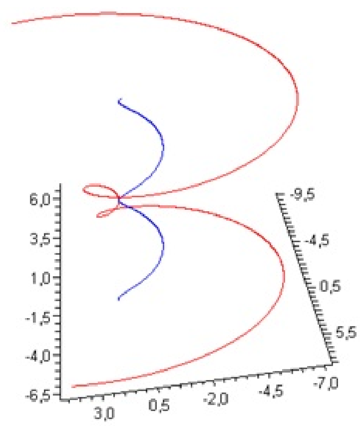

Figure 2. Curves α_1(blue) and α_1^{**}(red).

Example 5.2. *(Salkowski curve)* Salkowski curve is a space curve with a constant curvature κ and non-constant torsion τ [15]. Monterde has characterized them as space curves with constant curvature $\kappa \equiv 1$ and whose normal vector makes a constant angle with a fixed line [12]. This characterization of Monterde also gives that such curves are slant helices. A general parametrization of a slant helix is given by

$$\begin{aligned}\alpha_2(s) = \tfrac{1}{\sqrt{1+m^2}} \Big(& -\tfrac{1-n}{4(1+2n)} \sin((1+2n)s) - \tfrac{1+n}{4(1-2n)} \sin((1-2n)s) - \tfrac{1}{2}\sin s, \\ & \tfrac{1-n}{4(1+2n)} \cos((1+2n)s) + \tfrac{1+n}{4(1-2n)} \cos((1-2n)s) + \tfrac{1}{2}\cos s, \\ & \tfrac{1}{4m}\cos(2ns)\Big)\end{aligned}$$

where $n = \frac{m}{\sqrt{1+m^2}}$ and m are constants [12]. Choosing $m = 1/5$ and taking $R = 3$ and $R(s) = \frac{s}{3}$, we have two C-partner curves α_2^*, α_2^{**} as given in Figure 3 and Figure 4, respectively.

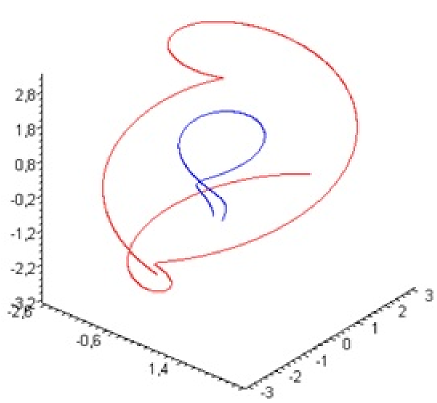

Figure 3. Curves α_2(blue) and α_2^*(red).

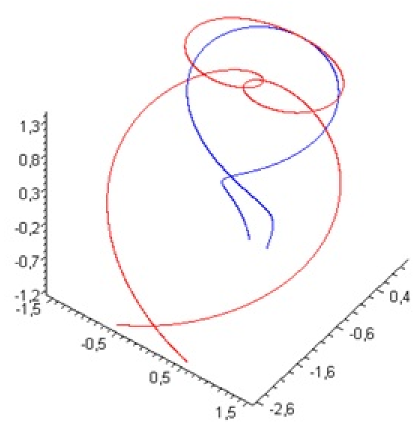

Figure 4. Curves α_2(blue) and α_2^{**}(red).

Example 5.3. *(Slant helix)* In Theorem 4.1, as an application of C-partner curves, we have obtained that a slant helix can be constructed by another slant helix. Furthermore, Choi and Kim have shown that a slant helix can be constructed by a circle

[5]. Then, it is clear that a new slant helix can be constructed by the same circle by using principal-direction curves given in [5] and application of C-partner curves. For this reason, let consider the following slant helix α_3 constructed by the circle $\delta(s) = (\cos s, \sin s)$ in [5],

$$\alpha_3(s) = -\left(\frac{3}{2}\cos\left(\frac{s}{2}\right) + \frac{1}{6}\cos\left(\frac{3s}{2}\right),\ \frac{3}{2}\sin\left(\frac{s}{2}\right) + \frac{1}{6}\sin\left(\frac{3s}{2}\right),\ \sqrt{3}\cos\left(\frac{s}{2}\right)\right).$$

Then, two C-partner curves α_3^*, α_3^{**} for $R = 1$ and $R = s$ are given in Figure 5 and Figure 6, respectively.

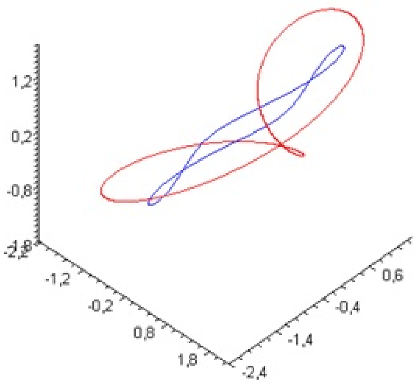

Figure 5. Curves α_3(blue) and α_3^*(red).

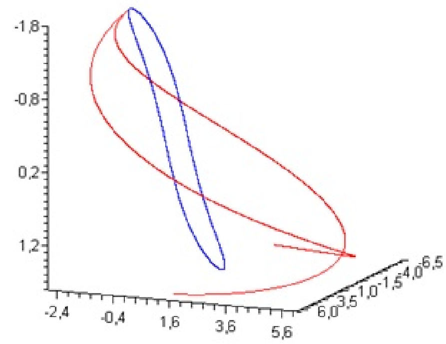

Figure 6. Curves α_3(blue) and α_3^{**}(red).

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