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INTEGRATION OF SYSTEMS OF ODEs VIA NONLOCAL SYMMETRY-LIKE OPERATORS

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Abstract- We apply nonlocal symmetry-like operators to systems of two first and two second-order ordinary differential equations to seek reduction to quadratures. The reduction of order of such systems is carried out with the help of analytic continuation of scalar equations in the complex plane. Examples are taken from the literature. Precisely it is shown how the reduction to quadratures of a system of two second-order ordinary differential equations that admits four Lie-like operators with certain structure is obtainable from a restricted complex ordinary differential equation possessing a connected two-dimensional complex Lie algebra. A direct method of integration for a system of two first and second-order equations which possess nonlocal symmetry-like operators are given. Moreover, we present the use of nonlocal Noether-like operators to effect double reduction of order of systems of two second-order equations that arise from the corresponding scalar complex Euler-Lagrange equations which admit nonlocal Noether symmetry.

Keywords- Nonlocal operators, systems, reductions, Noether integrals.

1. INTRODUCTION

The reduction of order of an ordinary differential equation (ODE) is at times a first step towards the solution of the equation. Lie point symmetries [1] can be used to reduce a given ODE to quadratures provided there is a sufficient number of symmetries which constitute a solvable algebra. Once a Lie point symmetry (also called local symmetry) is known, there are two Lie group theoretic methods [2, 3, 4, 5] to reduce a first-order ODE to quadrature: one is to make use of canonical variables which converts the given equation to variables separable form while the other is the integrating factor method that makes the given equation exact. When a given symmetry is of a particular form one can use invariants of the known symmetry to reduce the first-order equation to quadrature [6]. However, Lie point symmetries are not the only symmetries which are useful in providing solutions to differential equations (DEs) as we point out below.

Over the last few years, nonlocal symmetries of the DEs have also been promoted as a means to reduce the order of the equation under consideration. A Lie point symmetry that is lost and appears as a nonlocal symmetry can be used to reduce the order of the given equation, see e.g. [6, 7, 8, 9, 10, 11, 12]. In the reduction of the order of an ODE, more than one symmetry may be lost and these can be nonlocal symmetries [7, 8, 9, 10, 11, 12]. In the case when the original equation is of second-order, the reduced first-order equation can be expressed in terms of quadrature [6, 13] via a nonlocal symmetry. It means that Lie point symmetries as well as nonlocal symmetries can be treated on equal footing in the reduction of order of a given differential equation.

Recently, the idea of complex Lie point symmetries was introduced in [14, 15] for complex ordinary differential equations (CODEs) and restricted CODEs (r-CODEs). The latter can be obtained by allowing a complex function to depend only on a single real variable. These authors investigated the symmetry analysis for systems of partial differential equations (PDEs) and ODEs by introducing the analytic continuation of ODEs in the complex plane including those of variational problems. They also extended the Lie table for second-order ODEs having two symmetries. The symmetry analysis of an r-CODE of order two gives nontrivial results for systems of two second-order ODEs. Here, we present a complex nonlocal symmetry method for the integration of first and second-order r-CODEs and the corresponding system of two first and second-order ODEs. A complex nonlocal symmetry splits into two operators. We call such operators nonlocal symmetry-like operators. A discussion about these operators for the local case has been studied in [16]. Herein examples are discussed in detail which show the significance of these operators. It is further demonstrated that this approach can be used for the double reduction of a system of two second-order ODEs which corresponds to a given r-CODE.

This paper is divided into two main sections with the aid of several examples that illustrate our approach. The next Section 2 is devoted to the complex nonlocal symmetry method for systems of two second-order ODEs admitting connected operators. In Section 3, we discuss how a system of two ODEs admitting nonlocal operators can be converted to quadratures. We present examples of first and second-order r-CODEs in order to deal with systems of two first and two second-order ODEs respectively. In Section 4 we consider scalar Euler-Lagrange equations which possess complex nonlocal Noether symmetries and first integrals. We adapt these to associated systems of two real ODEs via complex splitting. The examples used are mainly taken from the references [6, 17].

2. COMPLEX NONLOCAL SYMMETRY APPROACH

Consider the system of second-order ODEs

$$\begin{aligned} f'' &= w_1(x, f, g, f', g'), \\ g'' &= w_2(x, f, g, f', g'), \end{aligned} \tag{1}$$

where w_1 and w_2 are both analytic functions of the arguments f, g, f', g'. Suppose that this system can be obtained from the r-CODE

$$u'' = w(x, u, u'), \tag{2}$$

by (x is real)

$$u = f + ig,\ w = w_1 + iw_2. \tag{3}$$

We have restricted u to depend on x alone. Complex-valued functions of real variables have been widely used in Fourier Analysis. These have also been utilized in Fluids (see, e.g. [18]). They have been invoked in recent papers on complex symmetry analysis. We utilize them in the sense used by Ali et al [14, 15]. In our case although w depends on x which is a single real variable and so the Cauchy-Riemann (CR) equations do not hold, it also depends on u and u' which are complex and thus w satisfies the CR equations not in x and y (through $z = x + iy$) but in f, g, f' and g' in which case (partial) analytic structure is still intact. Therefore, in this sense we do use analytic continuation.

The algebraic analysis of the system (1) under real transformations can be understood from the invariance of (2) under complex transformations. The algebraic analysis of systems of the form (1) was considered in [14] with respect to the r-CODE (2). A complex Lie symmetry of an r-CODE (2) gives rise to two Lie-like operators of the corresponding system (2) [14]. Our aim is to utilize these operators from a nonlocal viewpoint. The authors [14] used the analytic continuation of ODEs in the restricted domain to obtain useful results for systems of two second-order ODEs that correspond to the complex ODEs.

Consider an r-CODE of the form (2) which admits two complex Lie point symmetries $\mathbf{Z}_1$, $\mathbf{Z}_2$, such that $[\mathbf{Z}_1, \mathbf{Z}_2] = \mathbf{Z}_1$, in appropriate basis and $\mathbf{Z}_2 = \alpha(x, u)\mathbf{Z}_1$, where $\alpha(x,u)$ is a complex function, i.e. $\alpha(x,u)=\alpha_1(x,f)+i\alpha_2(x,g)$. The reduction of $\mathbf{Z}_1$, $\mathbf{Z}_2$ to canonical form, i.e. $\mathbf{Z}_1=\partial/\partial U$, $\mathbf{Z}_2=U\partial/\partial U$, transforms (2) to ($X$ real)

$$U''=R(X)U', \quad {}'=d/dX \tag{4}$$

which upon invocation of $U=F+iG$ and $R(X)=R_1(X)+iR_2(X)$ in (4) yields the following system of two ODEs

$$\begin{aligned} F'' &= R_1(X)F'-R_2(X)G', \\ G'' &= R_1(X)G'+R_2(X)F'. \end{aligned} \tag{5}$$

The system of ODEs (5) admits the four Lie-like operators $\mathbf{X}_1$, $\mathbf{Y}_1$, $\mathbf{X}_2$ and $\mathbf{Y}_2$ which can easily be deduced from $\mathbf{Z}_1=\mathbf{X}_1+i\mathbf{Y}_1$ and $\mathbf{Z}_2=\mathbf{X}_2+i\mathbf{Y}_2$. From $[\mathbf{Z}_1,\mathbf{Z}_2]=\mathbf{Z}_1$ one can write the condition on the real symmetries for the system of ODEs (5) or (1), *viz.*

$$\begin{aligned} [\mathbf{X}_1,\mathbf{X}_2]-[\mathbf{Y}_1,\mathbf{Y}_2] &= \mathbf{X}_1, \\ [\mathbf{X}_1,\mathbf{Y}_2]+[\mathbf{Y}_1,\mathbf{X}_2] &= \mathbf{Y}_1, \end{aligned} \tag{6}$$

and $\mathbf{Z}_2=\alpha(x,u)\mathbf{Z}_1$ where $\alpha=\alpha_1(x,f)+i\alpha_2(x,g)$ gives rise to the conditions

$$\begin{aligned} \mathbf{X}_2 &= \alpha_1(x,f)\mathbf{X}_1-\alpha_2(x,g)\mathbf{Y}_1, \\ \mathbf{Y}_2 &= \alpha_1(x,f)\mathbf{Y}_1+\alpha_2(x,g)\mathbf{X}_1. \end{aligned} \tag{7}$$

Now by the use of the invariants $T=X$ and $S=U'/U$ of the complex symmetry $\mathbf{Z}_2$, the r-CODE (4) is converted to the complex Bernoulli equation

$$\frac{dS}{dT}=F(T)S-S^2, \tag{8}$$

which can be split into the system of two real ODEs

$$\begin{aligned} \frac{dS_1}{dT} &= F(T)S_1-(S_1^2-S_2^2), \\ \frac{dS_2}{dT} &= F(T)S_2-2S_1S_2, \end{aligned} \tag{9}$$

once we set $S=S_1+iS_2$. The symmetry $\mathbf{Z}_1$ is lost as a point symmetry of equation (8) and it becomes the nonlocal symmetry

$$\mathbf{Z}_1=-\,S\exp(-\int SdT)\frac{\partial}{\partial S}, \tag{10}$$

which is in fact complex and gives two real nonlocal symmetries for the system of first-order ODEs (9). However, if we commence reduction of the second-order ODE

(2) using the complex invariants $t(x,u)$, $s(x,u,u')$ of $\mathbf{Z}_2$, the resulting first-order r-CODE

$$\frac{ds}{dt} = H(t,s),$$

has complex nonlocal symmetry

$$\mathbf{Z}_1 = \exp(\int L(t,s)dt)\ (\zeta(t)\frac{\partial}{\partial t} + \chi(t,s)\frac{\partial}{\partial s}), \tag{11}$$

which has canonical form (10). The transformation that converts the form of the complex nonlocal symmetry (11) to its canonical form (10) is given by $T = r(t)$, $S = (\alpha'/\alpha)_{(t,s)}$, $r'(t) \neq 0$. This also results in the transformation of the first-order equation in t, s variables to its standard form (8). Here $r(t)$ is an invariant of the nonlocal operator of the form (11). If one performs the reduction of order by $\mathbf{Z}_1$, $\mathbf{Z}_2$ becomes the local symmetry of the first-order ODE which allows for quadrature. The foregoing discussions results in the following theorem which is a complex extension of the theorem in [13]. The ideas are similar except that we have complex functions and operators as well as importantly decomposition in the real domain which is applicable to systems of two second-order ODEs. Hence we have the following result for systems of two second-order ODEs.

Theorem. A system of two second-order ODEs of the form (1) which admits four real Lie-like operators $\mathbf{X}_1$, $\mathbf{Y}_1$, $\mathbf{X}_2$ and $\mathbf{Y}_2$ with commutation relations, in suitable basis, (6) subject to the relations (7) is reducible to quadratures.

The proof is evident from the previous discussions. For if the system (1) arises from the second-order r-CODE of the form (2) which admits a two-dimensional complex Lie algebra of point symmetries with $[\mathbf{Z}_1, \mathbf{Z}_2] = \mathbf{Z}_1$ in suitable basis and $\mathbf{Z}_2 = \alpha(x,u)\mathbf{Z}_1$ is integrable, then one sets $\mathbf{Z}_1 = \mathbf{X}_1 + i\mathbf{Y}_1$ and $\mathbf{Z}_2 = \mathbf{X}_2 + i\mathbf{Y}_2$. The associated system is reducible to quadratures.

In the following we present examples which illustrate our nonlocal approach.

3. REDUCTIONS FOR SYSTEMS

We illustrate the nonlocal complex variables approach by investigating the following two examples: the first is of second-order and the second is a first-order Riccati system.

Example 1. Consider the system of second-order ODEs

$$\begin{aligned} f'' + xf' - f &= 0, \\ g'' + xg' - g &= 0, \end{aligned} \tag{12}$$

which admits the four real Lie-like operators

$$\begin{aligned}\mathbf{X}_1 &= x\frac{\partial}{\partial f},\ \mathbf{Y}_1 = x\frac{\partial}{\partial g},\\ \mathbf{X}_2 &= f\frac{\partial}{\partial f} + g\frac{\partial}{\partial g},\ \mathbf{Y}_2 = f\frac{\partial}{\partial g} - g\frac{\partial}{\partial f}.\end{aligned} \tag{13}$$

Also the conditions of the theorem of the previous section are satisfied. Thus one can discuss the symmetry and reduction of the system (12) via that of the single complex equation (set $u = f + ig$)

$$u'' + xu' - u = 0. \tag{14}$$

This r-CODE admits two complex Lie point symmetries (as well as six more) (cf. also [6])

$$\mathbf{Z}_1 = x\frac{\partial}{\partial u},\ \mathbf{Z}_2 = u\frac{\partial}{\partial u}\ \text{ which satisfy }\ [\mathbf{Z}_1, \mathbf{Z}_2] = \mathbf{Z}_1.$$

The use of the transformations (3) in the above r-CODE (14) gives rise to system (12). Here we use nonlocal symmetries to integrate our system. The invocation of the real invariants $t = x$ and $s_1 = (ff' + gg')/f^2 + g^2$, $s_2 = (fg' - f'g)/f^2 + g^2$ of $\mathbf{X}_2$ and $\mathbf{Y}_2$, in the above system (12) gives the system of first-order Riccati equations

$$\begin{aligned}\frac{ds_1}{dt} + s_1^2 - s_2^2 + s_1 t - 1 &= 0,\\ \frac{ds_2}{dt} + 2s_1 s_2 + s_2 t &= 0,\end{aligned} \tag{15}$$

and evidently $\mathbf{X}_1$, $\mathbf{Y}_1$ are lost as point symmetries of the system (15) and become nonlocal symmetries, *viz.*

$$\begin{aligned}\mathbf{X}_1 &= \exp(-\int s_1 dt)[\{(1 - ts_1)(\cos(\int s_2 dt) - s_2 t\sin(\int s_2 dt)\}\frac{\partial}{\partial s_1}\\ &\quad -\{(1 - s_1 t)\sin(\int s_2 dt + s_2 t\cos(\int s_2 dt)\}\frac{\partial}{\partial s_2}],\\ \mathbf{Y}_1 &= -\exp(-\int s_1 dt)[\{(1 - ts_1)\cos(\int s_2 dt) - s_2 t\sin(\int s_2 dt)\}\frac{\partial}{\partial s_2}\\ &\quad +\{(1 - ts_1)(\sin(\int s_2 dt)\frac{\partial}{\partial s_2} + \cos(\int s_2 dt)\}\frac{\partial}{\partial s_1}].\end{aligned} \tag{16}$$

This shows that there are nonlocal real symmetries in the integration of the first-order system (15). The transformations $T = t$, $S_1 = s_1 - 1/t$ and $S_2 = s_2$, reduces the above system of Riccati equations (15) to the system of first-order Bernoulli equations

$$\begin{aligned}\frac{dS_1}{dT} + (T + \frac{2}{T})S_1 &= -(S_1^2 - S_2^2),\\ \frac{dS_2}{dT} + (T + \frac{2}{T})S_2 &= -2S_1 S_2.\end{aligned} \tag{17}$$

The system (17) is solvable. This is seen by setting $S = S_1 + iS_2$ which converts the system to the first-order r-CODE

$$\frac{dS}{dT} + (T + \frac{2}{T})S = -S^2.$$

The complex solution of above equation yields two real solutions of system of first-order ODEs (17). Ultimately, we arrive at the solution of the system of two coupled equations (12).

Thus one can discuss the symmetry and reduction of the considered system via that of a single equation in the complex domain. Here this is illustrated via nonlocal symmetries.

Example 2. Consider the system of first-order Riccati ODEs

$$\begin{aligned}\frac{ds_1}{dt} &= f_1(t)(s_1^2 - s_2^2) + g_1(t)s_1 + h(t),\\ \frac{ds_2}{dt} &= 2f_1(t)s_1s_2 + g_1(t)s_2.\end{aligned} \tag{18}$$

The functions f_1, g_1 and h are real functions (they can be considered as complex). Here again we resort to the algebraic properties of the Riccati system (18) by means of the first-order Riccati equation in the restricted complex domain

$$\frac{ds}{dt} = f_1(t)s^2 + g_1(t)s + h(t), \tag{19}$$

where s is a complex function of t. This equation (19) can easily be transformed via $s = s_1 + is_2$ to the Riccati system (18). It admits the complex nonlocal symmetry (cf. [6])

$$\mathbf{Z}_1 = \exp(\int f_1(t)sdt)(sk(t) + \frac{k'(t)}{f_1(t)})\frac{\partial}{\partial s}, \tag{20}$$

where $u = k(x)$ is a solution of the second-order linear equation associated with (19) via the transformation, $t = x$, $s = -u'/uf_1$. The complex transformation $S = s$ and $T = -f_1(t)s - k'(t)/k(t)$ reduces (18) to the simpler system of first-order Bernoulli equations

$$\begin{aligned}\frac{dS_1}{dT} &= (g_1(T) + \frac{f_1'(T)}{f_1(T)} - 2\frac{k'(T)}{k(T)})S_1 - S_1^2 + S_2^2,\\ \frac{dS_2}{dT} &= (g_1(T) + \frac{{f_1}'(T)}{f_1(T)} - 2\frac{k'(T)}{k(T)})S_2 - 2S_1S_2,\end{aligned}$$

which in fact comes from the complex first-order Bernoulli equation (set $S = S_1 + iS_2$)

$$\frac{dS}{dT} = (g_1(T) + \frac{f_1'(T)}{f_1(T)} - 2\frac{k(T)'}{k(T)})S - S^2. \tag{21}$$

It is of importance to stress here that one can use complex nonlocal symmetries to reduce the order of a first and second-order r-CODE which in turn results in the nontrivial reduction of the corresponding system of two real ODEs. It is due to the fact that a complex variable encodes the information of two real variables. Equations considered here are important as they provide us nice examples of systems of those ODEs that can be handled by complex nonlocal symmetries. Hence the analytic continuation of certain ODEs to the restricted complex domain yields nontrivial results for the corresponding systems of ODEs. The above r-CODEs were previously investigated in the real domain in [6]. Here we have extend them to systems by complex splitting

4. NOETHERIAN INTEGRALS AND REDUCTIONS

First integrals can be determined for variational problems for systems of ODEs by the Noether theorem (see, e.g. the books [2, 3, 4, 5] for more details on this theorem). If a local Noether symmetry (symmetry of the local Lagrangian) is known, the explicit formula in Noether's theorem yields a local conservation law. In the study of conservation laws, one may encounter nonlocal Noether symmetries of a nonlocal Lagrangian for the given equations. However, by the use of these nonlocal Noether symmetries and nonlocal Lagrangians in Noether's theorem, one can obtain conservation laws of both local and nonlocal type [17]. A Noether symmetry not only has physical relevance but also provides double reduction of order of the given second-order r-CODE (cf. [19]). Correspondingly, the order of the resulting systems of second-order ODEs obtained from the r-CODE can also be reduced. Consequently, we can apply all these results equally well to nonlocal Noether symmetries to obtain local as well as nonlocal Noetherian integrals for corresponding systems of second-order ODEs.

In this section, we construct first integrals for systems of two second-order ODEs by considering complex nonlocal Noether symmetries of the complex Euler-Lagrange equations. We also invoke Lagrangians of local and nonlocal types. For these systems of ODEs, integrals are obtained from the corresponding r-CODEs.

Some preliminaries are now in order. The authors [16] have stated the Euler-Lagrange equations, the Noether-like operator conditions and expressions of first integrals for systems of two ODEs that arise from r-CODEs. Here we merely state the pertinent results and definitions from [16].

Suppose that $L = L_1 + iL_2$ is a complex Lagrangian of the r-CODE (2) relative to the system (1). Therefore, it satisfies the complex Euler-Lagrange equation. The realification of the Euler-Lagrange equation yields

$$\begin{aligned}\frac{\partial L_1}{\partial f}+\frac{\partial L_2}{\partial g}-\frac{d}{dx}(\frac{\partial L_1}{\partial f'}+\frac{\partial L_2}{\partial g'}) &= 0,\\ \frac{\partial L_2}{\partial f}-\frac{\partial L_1}{\partial g}-\frac{d}{dx}(\frac{\partial L_2}{\partial f'}-\frac{\partial L_1}{\partial g'}) &= 0\end{aligned} \tag{22}$$

for the system (1).

Definition. The operators $\mathbf{X}=\varsigma_1\partial_x+\chi_1\partial_f+\chi_2\partial_g$ and $\mathbf{Y}=\varsigma_2\partial_x+\chi_2\partial_f-\chi_1\partial_g$ are said to be Noether-like operators, of the EL-system (1) which arises from the r-CODE (2), with respect to the Lagrangians L_1 and L_2 if they satisfy

$$\begin{aligned}&\mathbf{X}^{(1)}L_1-\mathbf{Y}^{(1)}L_2+(d_x\varsigma_1)L_1-(d_x\varsigma_2)L_2=d_xA_1,\quad d_x=d/dx\\ &\mathbf{X}^{(1)}L_2+\mathbf{Y}^{(1)}L_1+(d_x\varsigma_1)L_2+(d_x\varsigma_2)L_1=d_xA_2,\end{aligned} \tag{23}$$

for suitable functions A_1 and A_2.

If we take

$$\varsigma=\varsigma_1+i\varsigma_2,\, A=A_1+iA_2,\, \mathbf{Z}=\mathbf{X}+i\mathbf{Y} \tag{24}$$

and further let $\chi=\chi_1+i\chi_2$ and $\chi^{(1)}=\chi_1^{(1)}+i\chi_2^{(1)}$ in the complex symmetry operator

$$\mathbf{Z}^{(1)}=\varsigma\frac{\partial}{\partial x}+\chi\frac{\partial}{\partial u}+\chi^{(1)}\frac{\partial}{\partial u'},$$

then $\mathbf{X}^{(1)}$ and $\mathbf{Y}^{(1)}$ are

$$\begin{aligned}2\mathbf{X}^{(1)} &= 2\varsigma_1\partial_x+\chi_1\partial_f+\chi_2\partial_g+\chi_1^{(1)}\partial_{f'}+\chi_2^{(1)}\partial_{g'},\\ 2\mathbf{Y}^{(1)} &= 2\varsigma_2\partial_x+\chi_2\partial_f-\chi_1\partial_g+\chi_2^{(1)}\partial_{f'}-\chi_1^{(1)}\partial_{g'}.\end{aligned} \tag{25}$$

These are the first prolongations of the Noether-like operators $\mathbf{X}$ and $\mathbf{Y}$.

Noether-like Theorem. $\mathbf{X}$ and $\mathbf{Y}$ are two Noether-like operators of system (1) with respect to the Lagrangians L_1 and L_2, then (1) admits two first integrals

$$\begin{aligned}&I_1=\varsigma_1L_1-\varsigma_2L_2+\partial_{f'}L_1(\chi_1-f'\varsigma_1-g'\varsigma_2)-\partial_{f'}L_2(\chi_2-f'\varsigma_2-g'\varsigma_1)-A_1,\\ &I_2=\varsigma_1L_2+\varsigma_2L_1+\partial_{f'}L_2(\chi_1-f'\varsigma_1-g'\varsigma_2)+\partial_{f'}L_1(\chi_2-f'\varsigma_2-g'\varsigma_1)-A_2.\end{aligned} \tag{26}$$

These formulae for first integrals are different from the usual Noether first integrals for systems. We now discuss few examples which illustrate our approach. Some of these r-CODEs have been studied in the real domain [17].

Example 1. We investigate the free particle equations

$$\begin{aligned}f'' &= 0,\\ g'' &= 0.\end{aligned} \tag{27}$$

from an unusual viewpoint, viz. nonlocal as well as having non-standard Lagrangians. This system corresponds to the restricted complexified free particle equation

$$u'' = 0. \tag{28}$$

The standard complex Lagrangian admitted by (28) is

$$L = \frac{u'^2}{2}. \tag{29}$$

The real Lagrangians admitted by the above system (27) of free particle equations are as a consequence of (29) (note although derived from a standard Lagrangian, these are not the standard Lagrangians of the free particle system)

$$\begin{aligned} L_1 &= \frac{1}{2}(f'^2 - g'^2), \\ L_2 &= f'g'. \end{aligned} \tag{30}$$

The Lagrangian (29) has many complex nonlocal Noether symmetries [17]. One of these nonlocal symmetries is [17]

$$\mathbf{Z} = \int u dx \partial/\partial x + \frac{1}{2}u^2 \partial/\partial u. \tag{31}$$

By complex splitting this gives us the following two real nonlocal Noether-like operators

$$\begin{aligned} \mathbf{X}_1 &= \int f dx \frac{\partial}{\partial x} + \frac{1}{2}[(f^2 - g^2)\frac{\partial}{\partial f} + 2fg\frac{\partial}{\partial g}], \\ \mathbf{Y}_1 &= \int g dx \frac{\partial}{\partial x} + \frac{1}{2}[2fg\frac{\partial}{\partial f} - (f^2 - g^2)\frac{\partial}{\partial g}], \end{aligned} \tag{32}$$

for the system of free particle equations (27). As both the Lagrangian and Noether symmetry are known, we require a first integral of the free particle equation (28). Application of Noether's theorem to (31) results in the first integral (see [17])

$$I = \frac{1}{2}u'^2 \int u dx - \frac{1}{4}u^2 u', \tag{33}$$

for the equation (28) corresponding to the complex nonlocal Noether symmetry (31). This first integral is complex and yields two real nonlocal first integrals

$$\begin{aligned} I_1 &= \frac{1}{2}[(f'^2 - g'^2)\int f dx - 2f'g' \int g dx] - \frac{1}{4}[f'(f^2 - g^2) - 2fgg'], \\ I_2 &= \frac{1}{2}[(f'^2 - g'^2)\int g dx + 2f'g' \int f dx] - \frac{1}{4}[g'(f^2 - g^2) + 2f'fg], \end{aligned} \tag{34}$$

for the above system of free particle equations (27) corresponding to the nonlocal Noether-like operators $\mathbf{X}_1$ and $\mathbf{Y}_1$. These two operators satisfy the Noether-like condition of the Definition. Alternatively one can determine the first integrals (34) from the Noether-like theorem.

Example 2. The general linear second order ODE in the restricted complex domain is

$$u'' + a(x)u' + b(x)u = 0. \tag{35}$$

After use of an appropriate transformation, the above r-CODE (35) can be reduced to the free particle equation $U'' = 0$. By performing the same analysis as for the free particle equations we can obtain the complex nonlocal symmetries as well as the complex nonlocal first integrals for the general linear r-CODE (35). Hence it enables us to analyze the nonlocal properties of the system of two second-order ODEs corresponding to the general r-CODE (35). For example one can deduce intriguing nonlocal Noether-like operators and nonlocal first integrals for the two-dimensional harmonic oscillator system.

Similarly the complex nonlocal Noether symmetries and corresponding complex first integrals for nonlinear r-CODEs which are linearizable by some point transformation can also be derived. For instance, consider the familiar nonlinear r-CODE

$$u'' + 3uu' + u^3 = 0. \tag{36}$$

Utilizing the change of variables $X = x - 1/u$, $U = x^2/2 - x/u$ (see e.g. [20]), the above r-CODE (36) can be reduced to the free-particle equation $d^2U/dX^2 = 0$. For the latter equation our procedure can be used to obtain nonlocal properties for the system of two ODEs which corresponds to (36).

Example 3. Consider the analytic continuation of the nonlinear ODE [17, 21] to the restricted complex domain

$$u'' = \frac{u'^2}{u} + a(x)uu' + a'(x)u^2. \tag{37}$$

The complex nonlocal Lagrangian associated with the above r-CODE (37) is given by [17],

$$L = \frac{1}{2}(\frac{u'}{u} - au)^2 \exp(-\int audx). \tag{38}$$

The complex nonlocal Noether symmetry corresponding to the above complex nonlocal Lagrangian is [17],

$$\mathbf{Z} = u \exp(\int a(x)udx)\frac{\partial}{\partial u}. \tag{39}$$

Since $\mathbf{Z}$ is a complex nonlocal Noether symmetry for the nonlocal Lagrangian, we may construct a first integral for the r-CODE (37). This example illustrates that not only Lie point symmetries are necessarily useful in order to reduce a given equation to quadrature but nonlocal symmetries can also do this very elegantly. Further double reduction in the order of the corresponding system of ODEs is carried out easily. The complex variables approach helps us to write the following system of ODEs corresponding to (37) as

$$\begin{aligned} ff''-gg'' &= f'^2-g'^2+a(x)\{(f^2-g^2)f'-2fgg'\}+a'(x)(f^3-3fg^2), \\ f''g+fg'' &= 2f'g'+a(x)\{2ff'g+(f^2-g^2)g'\}+a'(x)(3f^2g-g^3). \end{aligned} \tag{40}$$

The two real nonlocal Lagrangians associated with the above system of ODEs (40) are

$$\begin{aligned} L_1 &= \frac{1}{2}\exp(-a\int fdx)[\cos\int agdx(\frac{\{(f'^2-g'^2)(f^2-g^2)+4ff'gg'\}}{(f^2+g^2)^2} \\ &\quad +a^2(f^2-g^2)-2af')+2\sin\int agdx(\frac{\{(f^2-g^2)f'g'-(f'^2-g'^2)fg\}}{(f^2+g^2)^2} \\ &\quad +a^2fg-ag')], \\ L_2 &= \frac{1}{2}\exp(-a\int fdx)[2\cos a\int gdx(\frac{\{(f^2-g^2)f'g'-(f'^2-g'^2)fg\}}{(f^2+g^2)^2} \\ &\quad +a^2fg-ag')-(\sin\int agdx\frac{\{(f'^2-g'^2)(f^2-g^2)+4f'gg'\}}{(f^2+g^2)^2} \\ &\quad +a^2(f^2-g^2)-2af')]. \end{aligned} \tag{41}$$

The nonlocal Noether symmetry (39) is complex and it gives rise to two real nonlocal Noether-like operators

$$\begin{aligned} \mathbf{X}_1 &= \exp(\int afdx)[\{f\cos(\int agdx)-g\sin(\int agdx)\}\frac{\partial}{\partial f} \\ &\quad +\{g\cos(\int agdx+f\sin\int agdx\}\frac{\partial}{\partial g}] \\ \mathbf{Y}_1 &= \exp(a\int fdx)[\{g\cos(\int agdx+f\sin\int agdx\}\frac{\partial}{\partial f} \\ &\quad -\{f\cos(\int agdx)-g\sin(\int agdx)\}\frac{\partial}{\partial g}]. \end{aligned} \tag{42}$$

with respect to the real Lagrangians (41). Once the Noether symmetry (39) is known, our next step is to write the first integral of equation (37). Hence the application of Noether's theorem to (38) results in the local first integral

$$I=\frac{u'}{u}-au, \tag{43}$$

which corresponds to the nonlocal Noether symmetry (39). This first integral (43) splits into two real first integrals

$$\begin{aligned} I_1 &= \frac{ff' + gg'}{f^2 + g^2} - af, \\ I_2 &= \frac{fg' - f'g}{f^2 + g^2} - ag, \end{aligned} \tag{44}$$

for the system of ODEs (40) associated with the nonlocal Noether-like operators (42). Note that these satisfy the Noether-like theorem. Alternatively one can obtain these from from the Noether-like theorem with respect to the Lagrangians L_1 and L_2 given in (42). As I satisfies $\mathbf{Z}^{[1]}I = 0$, we have

$$u' - au^2 - ku = 0, \quad k \quad \text{is some constant.} \tag{45}$$

The above r-CODE (45) is a Bernoulli equation and it yields a Bernoulli system in two dimensions

$$\begin{aligned} f' - a(f^2 - g^2) - kf &= 0, \\ g' - 2afg - kg &= 0. \end{aligned} \tag{46}$$

Note that we can compare (45) with the general form of the Riccati equation of the previous section if we set $f_1 = a$, $g_1 = k$ and $h = 0$. The r-CODE (45) is solvable by quadrature and consequently the system of second-order ODEs (46) can also be reduced to quadratures via nonlocal symmetries. Hence, a complex exponential nonlocal operator is useful in converting the given system of ODEs (46) to quadratures.

5. CONCLUSION AND DISCUSSIONS

The main purpose of this paper was to exhibit how a complex nonlocal symmetry approach that works for r-CODEs can be implemented to reduce the order of the corresponding systems of ODEs to quadratures. There is extensive literature on the use of Lie point symmetries [2, 3, 4, 5] whereas the same is not the case for nonlocal symmetries although progress has been made [6, 7, 8, 9, 10, 11, 12, 13, 17, 22]. We have examined a second-order r-CODE admitting a two-dimensional connected complex algebra and reduced the equation via one symmetry - the wrong one. The other symmetry is lost as a Lie point symmetry and becomes a complex nonlocal symmetry of the reduced equation and with the help of real transformations, we have obtained the corresponding reduced system of first-order ODEs which are solvable by quadratures. We also showed that the use of exponential complex nonlocal symmetry facilitates the reduction of the given system to quadratures.

We also studied the use of nonlocal Noether symmetry. A complex nonlocal Noether symmetry of a complex local or nonlocal Lagrangian can be used to obtain two first integrals for the corresponding system of two second-order ODEs. In some cases, the corresponding system of ODEs can also be converted to quadratures as we have seen. However, the picture can be quite different in other cases where use of a complex nonlocal Noether symmetry to construct first integrals for the corresponding system of ODEs enables one reduction. Though all the first integrals in these cases satisfy the requirement of first integrals, occurrence of nonlocal terms in the integrals does create difficulties in the utility of these integrals for further reduction.

In the last part we discussed an example of a nonlinear ODE admitting a complex nonlocal Noether symmetry which yields a complex first integral in the form of a first-order Bernoulli equation. This Bernoulli equation has an exponential nonlocal symmetry. The use of the invariants of this complex symmetry not only reduces the first-order Bernoulli equation but also the resulting system of two first-order Bernoulli equations to quadratures. Ultimately, we arrive at the double reduction of that system of second-order ODEs which arises due to our nonlocal complex variable approach.

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