



SYMMETRY REDUCTIONS OF A FLOW WITH POWER LAW FLUID AND CONTAMINANT-MODIFIED VISCOSITY

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Abstract- In this study symmetry analysis is carried out for a system dealing with nonreactive pollutant transport along a single channel. Constitutive equations obeying a power law fluid are used in the description of the mathematical problem. We obtain forms of the source term for which the governing system admits extra point symmetries. Invariant solutions which satisfy physical boundary conditions are constructed. In some cases we resort to numerical methods to obtain approximate solutions.

Keywords- Power law fluid, nonlinear system, symmetry reductions, invariant and numerical solutions.

1. INTRODUCTION

The fluid dynamics of pollutants in rivers has a wide potential of applications in many environmental and engineering areas. The river system is complex and involves a number of parameters. Here we focus on macroscopic deterministic models based on local conservation laws. Even in recent times analytical and numerical solutions to the relevant mathematical models of river pollution are of vast interest to researchers. Some previous contributions can be seen through the studies [1, 2, 3, 4, 5, 6, 7].

In all the above mentioned studies, a viscous fluid is considered. However, no information is currently available for a system characterizing pollutant transport of a non-Newtonian fluid in a river. Such consideration in a river seems more realistic. In view of this, our aim is to provide such a study. Hence the power law fluid has been selected. This fluid model is able to describe both shear thinning and shear thickening effects. However, power law fluids cannot predict the normal stress

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effects. The resulting system in the present study is nonlinear and it is linear for a viscous fluid. The governing nonlinear system is solved by symmetry techniques. We employ classical Lie point symmetry analysis. Numerical methods are utilised where necessary. The paper is organized as follows: the mathematical statement of the resulting system in dimensionless variables is presented in Section 2. We provide the classical Lie point symmetry analysis in Section 3. In Section 4, we consider numerical solutions. Discussion of the results and concluding remarks are given in Section 5.

2. GOVERNING EQUATIONS

We consider river pollution which is initially incompressible and fully developed with contaminant-dependent viscosity. An amount of pollutant concentration is injected into the river and allowed to disperse. The Cauchy stress tensor in a power-law fluid is

$$\boldsymbol{\tau} = -p\mathbf{I} + \mu \left(\left| \sqrt{\frac{1}{2} \operatorname{tr} \mathbf{A}_1^2} \right|^{n-1} \right) \mathbf{A}_1, \quad (1)$$

where p denotes the fluid pressure in a river, \mathbf{I} an identity tensor, μ is contaminant-dependent fluid dynamic viscosity, tr is the trace and the first Rivlin-Eriksen tensor \mathbf{A}_1 is given by

$$\mathbf{A}_1 = (\operatorname{grad} \mathbf{V}) + (\operatorname{grad} \mathbf{V})^*, \quad (2)$$

where \mathbf{V} is the fluid velocity and $*$ is the matrix transpose. It should be noted that n is the power law index. With $n = 1$, Eq. (1) represents a viscous fluid. Furthermore, Eq. (1) represents shear thinning ($n < 1$) and shear thickening ($n > 1$) fluids. For unidirectional flow $\mathbf{V} = (u(y, t), 0, 0)$. Here u is velocity of fluid in the x -direction and t is the time. Note that the incompressibility condition is automatically satisfied by the chosen value of \mathbf{V} and momentum equation along with Eqs. (1) and (2) to yield

$$\rho \frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial y} \left(\mu \left| \frac{\partial u}{\partial y} \right|^{n-1} \frac{\partial u}{\partial y} \right), \quad (3)$$

where ρ is the constant density of the fluid. The pollutant concentration equation is given by

$$\rho \frac{\partial c}{\partial t} = \frac{\partial}{\partial y} \left(D \frac{\partial c}{\partial y} \right) + S(y, t). \quad (4)$$

c denotes the pollutant concentration, D the mass diffusion and $S(y, t)$ an external pollutant source term. The river viscosity and the mass diffusivity are assumed to

be pollutant dependent and given by power laws,

$$\frac{\mu}{\mu_0} = \left(\frac{c}{c_0}\right)^\lambda, \quad \frac{D}{D_0} = \left(\frac{c}{c_0}\right)^\lambda. \quad (5)$$

Here μ_0 , c_0 , D_0 and λ are the river viscosity coefficient, characteristic concentration, mass diffusivity coefficient and constant exponential respectively. Introducing the dimensionless variables

$$u_* = \frac{Lu}{v_0}, \quad y_* = \frac{y}{L}, \quad x_* = \frac{x}{L}, \quad t_* = \frac{v_0 t}{L^2}, \quad P_* = \frac{L^2 P}{\rho v_0}, \quad v_0 = \frac{\mu_0}{\rho}, \quad R = \frac{v_0}{D_0},$$

$$K = -\frac{\partial P_*}{\partial x_*}, \quad S_* = \frac{L^2 S}{v_0 c_0}, \quad c_* = \frac{c}{c_0}, \quad M = \left(\frac{v_0}{L^2}\right)^{n-1},$$

we obtain, after dropping the stars for simplicity, the system of dimensionless PDEs

$$\frac{\partial u}{\partial t} = K + M \frac{\partial}{\partial y} \left(c^\lambda \left| \frac{\partial u}{\partial y} \right|^{n-1} \frac{\partial u}{\partial y} \right), \quad (6)$$

$$\frac{\partial c}{\partial t} = \frac{1}{R} \frac{\partial}{\partial y} \left(c^\lambda \frac{\partial c}{\partial y} \right) + S(y, t).$$

Here R is the Schmidt number and K is the imposed constant pressure axial gradient. We require $n \neq 0$. We assume $u_y > 0$ in the subsequent sections. Note that K may be equated to zero by the transformation $\bar{u} = u + Kt$. However, here we allow K to be nonzero.

3. CLASSICAL LIE POINT SYMMETRY ANALYSIS

The theory and application of symmetry techniques may be found in texts such as [8]. We adopt methods in [8] (which exclude explicit equivalence transformation analysis) to perform group classification. In the initial Lie point symmetry analysis of the system (6) with an arbitrary function S , the admitted generic symmetries are nothing beyond translation in u . The cases for which extra symmetries are obtained are listed in Table 1, wherein the analysis for the viscous fluid has been omitted. Unless stated, all the parameters appearing in system (6) are taken to be arbitrary in Table 1.

3.1. Lie point symmetry reduction: illustrative example

The symmetry generator X_4 , in Table 1, reduces the system (6) with $S = 0$ to the functional form

$$u = Kt + g(y), \quad c = t^{-1/\lambda} h(y),$$

where g and h satisfy the ordinary differential equations (ODEs)

$$\begin{aligned}g' &= \gamma_1 h^{-\lambda/n}, \quad \gamma_1 \in \mathbb{R}, \\h'' + \lambda h^{-1} (h')^2 + \frac{R}{\lambda} h^{1-\lambda} &= 0.\end{aligned}\tag{7}$$

In terms of the original variables we obtain the invariant solution to the system (6) with $\lambda = -1$ and $n = -1$ given by

Table 1: Extra point symmetries admitted by system (6).

$S(y, t)$	Parameters	Symmetries
arbitrary	$R = 0$	$X_2 = \partial_y, X_3 = \left(\frac{1-n}{\lambda}\right) c\partial_c + (u - Kt)\partial_u,$ $X_4 = \left(\frac{n+1}{\lambda} c\partial_c + y\partial_y\right),$ $X_5 = -\frac{h'(t)}{\lambda} c\partial_c + Kh(t)\partial_u + h(t)\partial_t.$
0		$X_2 = \partial_y, X_3 = \partial_t, X_4 = c\partial_c - K\lambda t\partial_u - \lambda t\partial_t$ $X_5 = c\partial_c + Kt\partial_u + y\partial_y + 2t\partial_t$
$(\alpha y + \beta)^p e^{qt}$		$X_2 = \frac{\lambda}{2}(u - Kt)\partial_u + \frac{\lambda}{2\alpha}(\alpha y + \beta)\partial_y$ $+ \frac{1}{2}(2 - \lambda p)\partial_t + c\partial_c$
$e^{py} t^q$		$X_2 = c\partial_c - K\lambda t\partial_u + \frac{\lambda(q+1)}{p}\partial_y - \lambda t\partial_t$
$y^p t^q$	$\lambda = -\frac{1}{q+1},$ $p = -2q - 2$	$X_2 = \left[u - \left(\frac{\lambda p + \lambda q + \lambda + 1}{\lambda q + \lambda + 1}\right) Kt \right] \partial_u +$ $\left(\frac{p+2q+2}{\lambda q + \lambda + 1}\right) \partial_c + y\partial_y + \left(\frac{2-\lambda p}{\lambda q + \lambda + 1}\right) t\partial_t$ $X_2 = (q+1)c\partial_c + Kt\partial_u + t\partial_t$ $X_3 = -2(q+1)c\partial_c + (u - Kt)\partial_u + y\partial_y$
e^{py+qt}		$X_2 = -\frac{q}{p}\partial_y + \partial_t$
Ae^{pt}		$X_2 = \partial_y, X_3 = 2qc\partial_c + \lambda q(u - Kt)\partial_u$ $+ \lambda q y\partial_y + 2\partial_t.$
$y^p f(t)$	$\lambda = \frac{2}{p}$	$X_2 = pc\partial_c + (u - Kt)\partial_u + y\partial_y$
$f(t)$		$X_2 = \partial_y$
t^p		$X_2 = \partial_y, X_3 = 2(p+1)c\partial_c$ $+ (\lambda p + \lambda + 1)(u - Kt)\partial_u$ $+ (\lambda p + \lambda + 1)y\partial_y + 2t\partial_t$
$f(y)$		$X_2 = \partial_t$
e^{py}		$X_2 = c\partial_c - \lambda Kt\partial_u + \left(\frac{\lambda+1}{p}\right) \partial_y - \lambda t\partial_t$
y^p		$X_2 = \partial_t, X_3 = \left(\frac{p+2}{\lambda+1}\right) c\partial_c$ $+ \left[u + \left(\frac{\lambda p + \lambda + 1}{\lambda + 1}\right) Kt \right] \partial_u + y\partial_y + \left(\frac{2-\lambda p}{\lambda + 1}\right) \partial_t$
$\alpha(\beta y - pt)$	$\lambda = \frac{1}{2}$	$X_2 = \frac{p}{\beta}\partial_y + \partial_t$ $X_3 = 2c\partial_c + u\partial_u + y\partial_y + t\partial_t$

$$c = \frac{t}{2\gamma_2 R} \sec^2 \left(\frac{y + c_2}{2c_1} \right), c_1, c_2 \in \mathbb{R}, \gamma_2 = c_1^2$$

$$u = Kt + 4\gamma_1 R \gamma_3 \left\{ \frac{1}{2} \cos \left(\frac{y + c_2}{2c_1} \right) \sin \left(\frac{y + c_2}{2c_1} \right) + \frac{y + c_2}{4c_1} \right\} + c_3, c_3 \in \mathbb{R}, \gamma_3 = c_1^3. \quad (8)$$

In the solution (8)₂, n has been chosen to simplify the calculations. The initial concentration is zero. We impose an instantaneous supply of concentration, $c = 1$, at $y = 0$, with $t = 1$, and a zero flux boundary condition at large distance, say $y = 1$, i.e., $\frac{\partial c}{\partial y} \Big|_{y=1} = 0$. The boundary condition $\frac{\partial c}{\partial y} \Big|_{y=1} = 0$, requires a solution of

$$\sec^2 \left(\frac{1 + c_2}{2c_1} \right) \tan \left(\frac{1 + c_2}{2c_1} \right) = 0.$$

This implies that $\tan \left(\frac{1 + c_2}{2c_1} \right) = 0$. Thus $\left(\frac{1 + c_2}{2c_1} \right) = m\pi$, $m = 0, 1, 2, \dots$. Choosing the smallest value of m namely $m = 0$, implies $c_2 = -1$ explicitly. Thus the invariant solution (8)₁ becomes

$$c = \frac{t}{2R c_1^2} \sec^2 \left(\frac{y - 1}{2c_1} \right). \quad (9)$$

It is worth noting that for $m > 0$, c_2 is dependent on c_1 . With the boundary condition $c = 1$ at $y = 0$, with $t = 1$, c_1 satisfies the nonlinear equation

$$\sec^2 \left(\frac{1}{2c_1} \right) = 2R c_1^2. \quad (10)$$

Eq. (10) is solved numerically for c_1 (choosing $R = 0.1, 0.25, 0.5$, respectively) by utilising the Newton-Raphson method. The solution is shown in Table 2.

Furthermore the invariant solution (8)₂ then is

Table 2: Chosen R values and corresponding c_1 values obtained by solving Eq. (10) numerically.

R	0.10	0.25	0.5
c_1	2.2904	1.4964	1.1107

$$u = Kt + c_3 + 4\gamma_1 R c_1^3 \left(\frac{1}{2} \cos \left(\frac{y-1}{2c_1} \right) \sin \left(\frac{y-1}{2c_1} \right) + \frac{y-1}{4c_1} \right).$$

With $t = 1$, $y = 1$, $u = 0$, so that $c_3 = -K$. Thus solution (8)₂ becomes

$$u = K(t-1) + 4\gamma_1 R c_1^3 \left(\frac{1}{2} \cos \left(\frac{y-1}{2c_1} \right) \sin \left(\frac{y-1}{2c_1} \right) + \frac{y-1}{4c_1} \right). \quad (11)$$

We observe that for $\lambda \neq -1$, the change of variables $h = w^{1/(1+\lambda)}$, $w = w(y)$ transforms Eq. (7)₂ into

$$w'' + \frac{(1+\lambda)R}{\lambda} w^{1/(1+\lambda)} = 0. \quad (12)$$

The ODE (12) is known as the Emden-Fowler type equation (see e.g. [9], [10]). We omit the full analysis of Eq. (12), but provide a solution for a special case. Given $\lambda = 1/3$ we obtain the implicit solution

$$\pm \sqrt{-\frac{5}{c_1}} w {}_2F_1 \left(\frac{2}{5}, \frac{1}{2}; \frac{7}{5}; \frac{8R}{c_1} w^{5/2} \right) - y - c_2 = 0, \quad (13)$$

where ${}_2F_1$ is the hypergeometric function [11]. Due to the implicit nature of the solution (13), it is difficult to construct the solution to the original system (6). However, we resort to numerical solutions.

4. NUMERICAL SOLUTIONS

For $\lambda \neq -1$, system (7) is rewritten as

$$g' = \gamma_1 w^{-\lambda/n(1+\lambda)}, \quad w'' + \frac{(1+\lambda)R}{\lambda} w^{1/(\lambda+1)} = 0. \quad (14)$$

Imposing the zero contaminant flux flow condition and $u = K(t-1)$ at $y = 1$, as well as the instantaneous supply of contaminant at $y = 0$, necessitates a solution to the system of equations in (14) subject to the conditions

$$w'(1) = 0, \quad w(0) = 1, \quad g(1) = 0.$$

We compute the approximate concentration $w(y)$ and the flux $w'(y)$ numerically using `bvp4c` in Matlab[®]. The approximate velocity $g(y)$ is dependent on $w(y)$; $g(y)$ is computed using a second order Taylor series approximation. Higher order Taylor series approximations could improve the accuracy of the velocity profiles g .

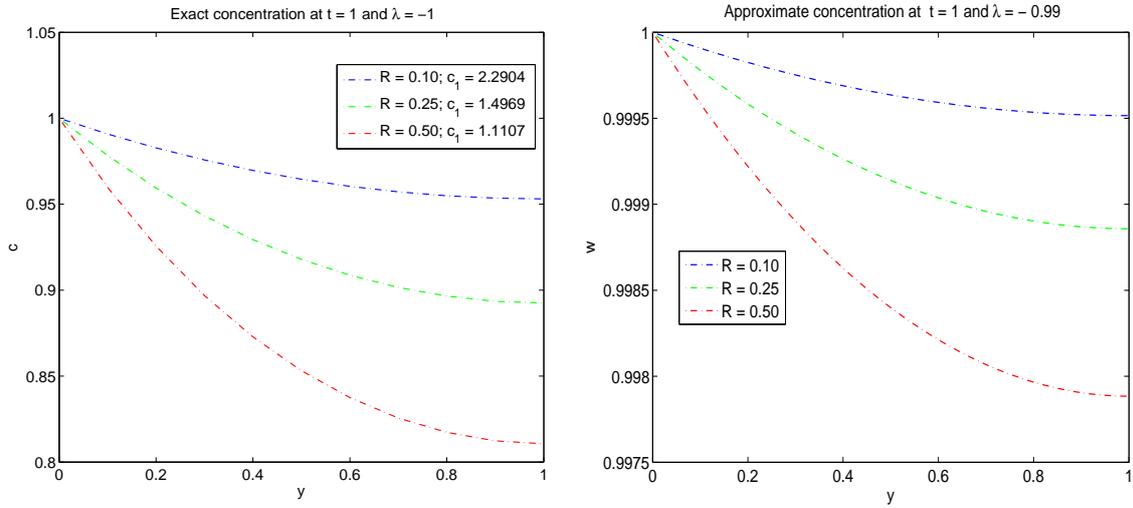
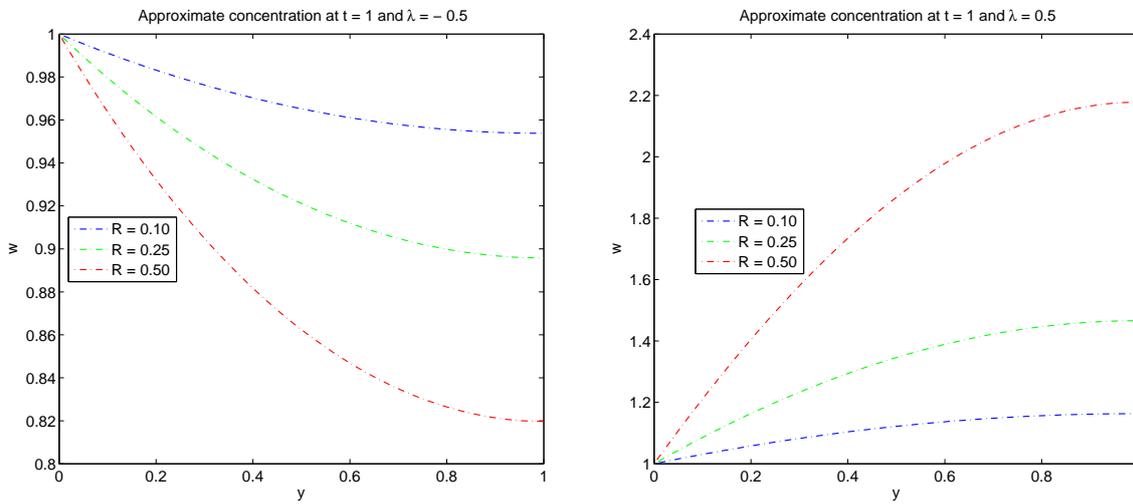


Figure 1: Comparison at $t = 1$ of the variation of the exact concentration profiles in Eq. (9) and of the approximate concentration profiles in Eq. (14).



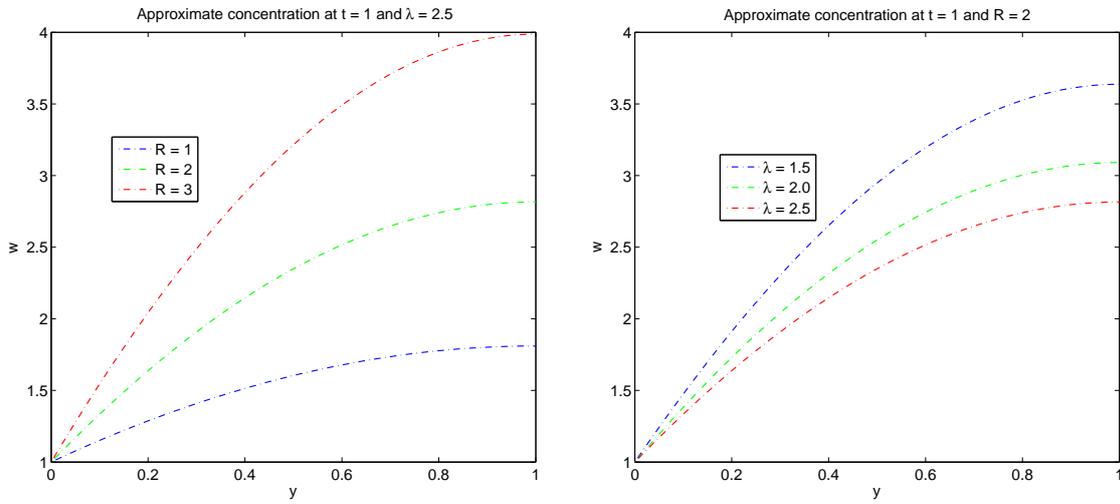


Figure 2: Variation at $t = 1$ of the approximate pollutant concentration profiles in Eq. (14) as the pollutant disperses down the river.

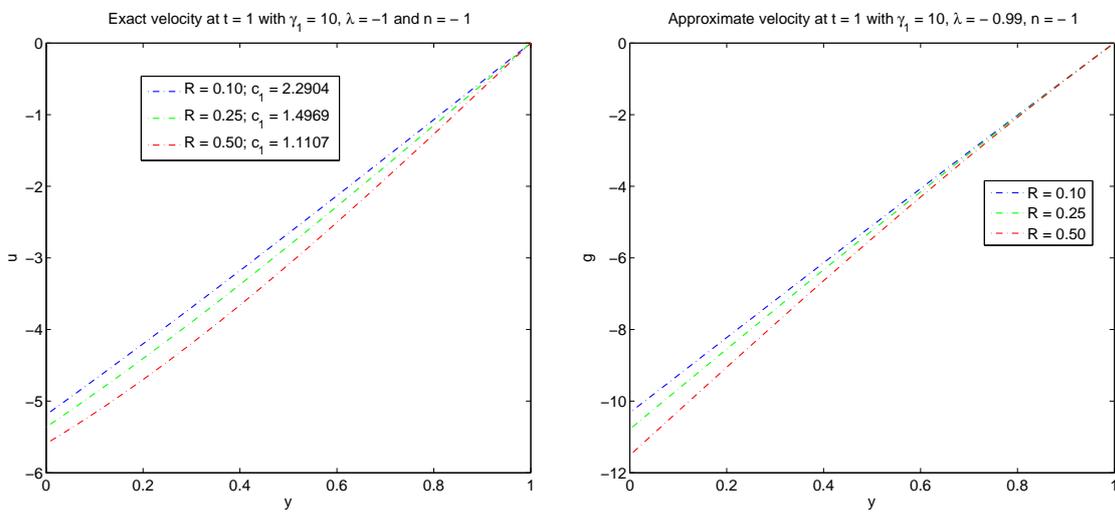


Figure 3: Variation at $t = 1$ of the exact fluid velocity profiles in Eq. (9) and of the approximate fluid velocity profiles in Eq. (14).

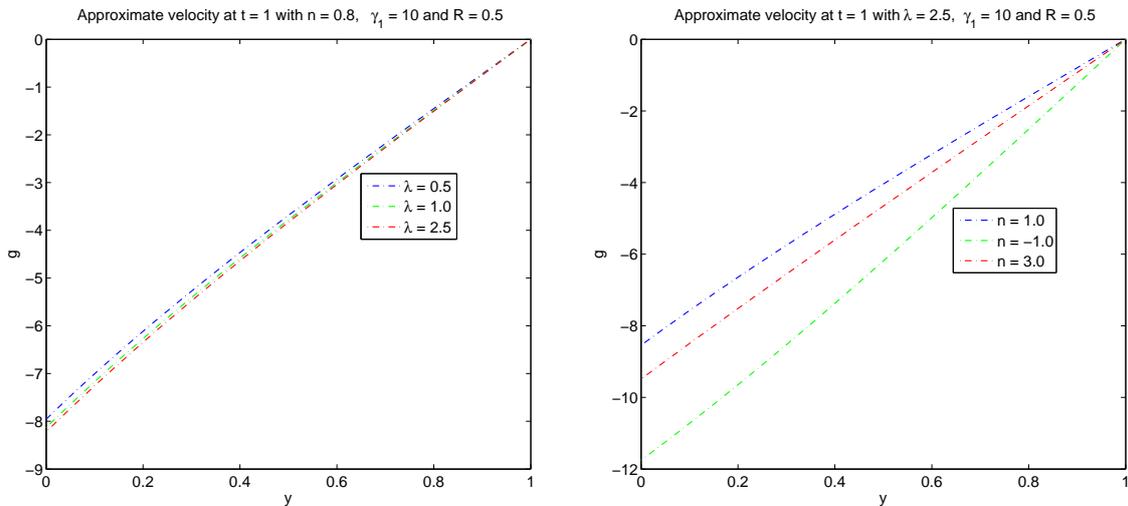


Figure 4: Variation at $t = 1$ of the approximate fluid velocity profiles in Eq. (14) for both the shear-thinning and shear-thickening cases.

5. DISCUSSION AND CONCLUDING REMARKS

Nonreactive pollutants are introduced into a river or an open channel at a point $y = 0$ and transported to some distance, say $y = 1$. After some time another quantity of pollutants is introduced and the process is repeated. In Fig. 1 we compare the exact concentration profiles at $t = 1$ with the corresponding approximate concentration profiles for specific values of the exponential constant λ . The exact concentration solution is determined for $\lambda = -1$, while in the approximate case $\lambda \neq -1$. Thus in the approximate case we choose $\lambda = -0.99$. We observe that the concentration saturates at $y = 1$ in both cases. The Schmidt number R is inversely proportional to the concentration. The variation of the approximate pollutant concentration is depicted in Fig. 2 for different values of λ and R respectively, as the pollutant disperses down the river or open channel. We notice that for $\lambda > 0$ the concentration is inversely proportional λ and directly proportional to R , while for $\lambda < 0$ the concentration is inversely proportional to R . In Fig. 3 the variation at $t = 1$ for the exact- and approximate fluid velocity profiles is shown for specific values of λ and n . We observe that the velocity is inversely proportional to R in both these cases. The exact velocity solution is obtained for $n = -1$ and $\lambda = -1$, while in the approximate case $\lambda \neq -1$. Here we choose $\lambda = -0.99$. The approximate fluid velocity $g(y)$ is dependent on n and λ . The variation at $t = 1$ for the approximate fluid velocity profiles is illustrated in Fig. 4 for the shear thinning case with different values of λ . Similar results are observed for the shear thickening case. We observe

that the velocity decreases with increasing λ . The approximate velocity profiles are also depicted for the Newtonian, shear thinning and shear thickening cases for fixed λ . The velocity is least for the shear thinning case. In both Figs. 3 and 4 we notice that the relative fluid velocity is increasing and slows at large distance. This is due to the high pollutant concentration at this distance. At this large distance ($y = 1$), we also observe the interesting phenomenon of fluid back flow. The fluid is very dense here and creeps backward slowly. A system of PDEs describing nonreactive contaminant transport in a river (or along a single channel) and power-law fluid has been analyzed. We employed classical Lie point symmetry techniques. Numerical techniques were utilised where appropriate. We made use of invariant boundary conditions throughout so that pertinent comparisons could be made.

The problem considered in this paper is applicable in industrial fluids such as water-alcohol mixtures and so on.

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