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## ON PRESERVATION UNDER UNIVARIATE WEIGHTED DISTRIBUTIONS

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*Abstract.* We derive some new results for preservation of various stochastic orders and aging classes under weighted distributions. The corresponding reversed preservation properties as straightforward conclusions of the obtained results for the direct preservation properties, are developed. Damage model of Rao, residual lifetime distribution, proportional hazards and proportional reversed hazards models are discussed as special weighted distributions to try some of our results.

*Keywords:* weighted distribution; preservation; stochastic ordering; aging classes

*MSC 2010:* 60E05, 60E15, 62N05

### 1. INTRODUCTION AND PRELIMINARIES

Weighted distributions are useful to model data in situations where the distribution of the observed data does not coincide with the original distribution of the data. A number of such instances were explained and described by Rao [17] and Patil and Rao [16]. Recently, the study of some reliability aspects of weighted distributions has attracted the attention of many researchers (cf. Kochar and Gupta [10], Nanda and Jain [12], Navarro et al. [14] and Pakes et al. [15] among others). Numerous research works have also been devoted to investigate the properties of weighted distributions in the context of stochastic orderings and aging classes (cf. Bartoszewicz and Skolimowska [4], Misra et al. [11], Błażej [5], Bartoszewicz [3] and Izadkhah et al. [7]). One of the main problems in some of these works was the problem of preservation of stochastic orders and aging classes under univariate weighted distributions. For example, using a representation of weighted distributions Błażej [5] and Bartoszewicz [3] obtained some results for preservation of several stochastic orders and

aging classes. By appealing to some bivariate characterizations of stochastic orders Misra et al. [11] derived a similar kind of results. Also, using some well-known characterizations of aging classes by means of stochastic orders and using the concept of the totally positivity (cf. Karlin [9]), Izadkhah et al. [6] presented some achievements for preservation of a number of aging classes under weighting.

In this paper, using a technical lemma given in Barlow and Proschan [2], we develop a complete study to get the preservation of several univariate stochastic orders and aging classes under weighted distributions. In this context, a new approach will be introduced, although some of the results obtained are similar to the previous results in the literature (cf. Misra et al. [11]). In addition, according to Izadkhah et al. [6], some special weighted distributions are proposed which are applied to some practical situations, and we examine the derived results for those special cases (cf. Ahmad and Kayid [1]). Another direction of this paper, which has not been investigated in the literature before, is to provide the reversed preservation property of weighted distributions. Throughout the paper, examples are also given to explain some useful facts. The notation  $I_A(t)$  stands for the indicator function of any set  $A$  in  $\mathbb{R}$ . It will be also assumed that  $\stackrel{\text{st}}{=}$  denotes the equality of distributions.

Let  $X$  and  $Y$  be two random variables with absolutely continuous cumulative distribution functions (cdf)  $F$  and  $G$ , probability density functions (pdf)  $f$  and  $g$ , and survival functions (sf)  $\bar{F}$  and  $\bar{G}$ , respectively. Assume further that  $u_X = \sup\{x: F(x) < 1\}$  and  $u_Y = \sup\{x: G(x) < 1\}$  are the respective upper bounds of  $X$  and  $Y$ , and  $l_X = \inf\{x: F(x) > 0\}$  and  $l_Y = \inf\{x: G(x) > 0\}$  are their corresponding lower bounds. For two nonnegative weight functions  $w_1$  and  $w_2$ , the random variables  $X_{w_1}$  and  $Y_{w_2}$  are called the weighted random variables associated with  $X$  and  $Y$ , which have probability density functions (cf. Jain et al. [8])

$$f_1(x) = \frac{w_1(x)f(x)}{\eta_1} \quad \text{and} \quad g_1(x) = \frac{w_2(x)g(x)}{\eta_2}, \quad x \in \mathbb{R},$$

respectively, where  $0 < \eta_1 = E(w_1(X)) < \infty$  and  $0 < \eta_2 = E(w_2(Y)) < \infty$ . The distribution functions of  $X_{w_1}$  and  $Y_{w_2}$  are, respectively, obtained as

$$F_1(x) = \frac{A_1(x)F(x)}{\eta_1} \quad \text{and} \quad G_1(x) = \frac{A_2(x)G(x)}{\eta_2}, \quad x \in \mathbb{R},$$

and their corresponding survival functions as

$$\bar{F}_1(x) = \frac{B_1(x)\bar{F}(x)}{\eta_1} \quad \text{and} \quad \bar{G}_1(x) = \frac{B_2(x)\bar{G}(x)}{\eta_2}, \quad x \in \mathbb{R},$$

where  $A_1(x) = E(w_1(X) \mid X \leq x)$ ,  $A_2(x) = E(w_2(Y) \mid Y \leq x)$ ,  $B_1(x) = E(w_1(X) \mid X > x)$ , and  $B_2(x) = E(w_2(Y) \mid Y > x)$ .

The random variable  $X_t = (X - t \mid X > t)$  for  $t < u_X$  is called the residual life of  $X$  having  $\text{sf } \bar{F}_t(x) = \bar{F}(t+x)/\bar{F}(t)$ ,  $x \in (0, \infty)$ . Also, the random variable  $X_{(t)} = (t - X \mid X \leq t)$  for  $t > l_X$  is known as the reversed residual life or the inactivity time of  $X$ , which has  $\text{sf } \bar{F}_{(t)}(x) = F(t-x)/F(t)$ ,  $x \in [0, \infty)$ . The foregoing characteristics are similarly defined for the random variable  $Y$ . The hazard rates (hr) of  $X$  and  $Y$  are, respectively, given by  $r_F(x) = f(x)/\bar{F}(x)$ ,  $x \in (-\infty, u_X)$ , and  $r_G(x) = g(x)/\bar{G}(x)$ ,  $x \in (-\infty, u_Y)$ . The reversed hazard rates (rh) of  $X$  and  $Y$  are defined as  $q_F(x) = f(x)/F(x)$ ,  $x \in (l_X, \infty)$ , and  $q_G(x) = g(x)/G(x)$ ,  $x \in (l_Y, \infty)$ , respectively. The mean of random variables  $X_x$  and  $Y_x$ , called the mean residual lifetimes (mrl) of  $X$  and  $Y$ , are given, respectively, by

$$m_F(x) = \begin{cases} \int_x^\infty \frac{\bar{F}(t)}{\bar{F}(x)} dt, & x < u_X, \\ 0, & x \geq u_X, \end{cases} \quad \text{and} \quad m_G(x) = \begin{cases} \int_x^\infty \frac{\bar{G}(t)}{\bar{G}(x)} dt, & x < u_Y, \\ 0, & x \geq u_Y. \end{cases}$$

The reversed mean residual lifetimes (rmr) of  $X$  and  $Y$  are, respectively, defined as the mathematical expectations of  $X_{(x)}$  and  $Y_{(x)}$ , and are given by

$$\alpha_F(x) = \begin{cases} \int_{l_X}^x \frac{F(t)}{F(x)} dt, & x > l_X, \\ 0, & x \leq l_X, \end{cases} \quad \text{and} \quad \alpha_G(x) = \begin{cases} \int_{l_Y}^x \frac{G(t)}{G(x)} dt, & x > l_Y, \\ 0, & x \leq l_Y. \end{cases}$$

According to Shaked and Shanthikumar [18] and Nanda et al. [13], we have the following partial orders to be used throughout the paper. We use the convention that  $a/0 = \infty$  for  $a > 0$ , and also  $0/0 = 0$ . The random variable  $X$  is smaller than  $Y$  in:

- (i) Usual stochastic order ( $X \leq_{\text{st}} Y$ ), if  $\bar{F}(x) \leq \bar{G}(x)$  for all  $x \in \mathbb{R}$ , or equivalently, if  $F(x) \geq G(x)$  for all  $x \in \mathbb{R}$ .
- (ii) Hazard rate order ( $X \leq_{\text{hr}} Y$ ), if  $r_F(x) \geq r_G(x)$  for all  $x \in \mathbb{R}$ , or equivalently, if  $[f(x)\bar{G}(x) - g(x)\bar{F}(x)] \geq 0$  for all  $x \in \mathbb{R}$ .
- (iii) Reversed hazard rate order ( $X \leq_{\text{rh}} Y$ ), if  $q_F(x) \leq q_G(x)$  for all  $x \in \mathbb{R}$ , or equivalently, if  $[g(x)F(x) - f(x)G(x)] \geq 0$  for all  $x \in \mathbb{R}$ .
- (iv) Mean residual life order ( $X \leq_{\text{mrl}} Y$ ), if  $m_F(x) \leq m_G(x)$  for all  $x \in \mathbb{R}$ , or equivalently, if  $[\bar{F}(x) \int_x^\infty \bar{G}(t) dt - \bar{G}(x) \int_x^\infty \bar{F}(t) dt] \geq 0$  for all  $x \in \mathbb{R}$ .
- (v) Reversed mean residual life order ( $X \leq_{\text{rmr}} Y$ ), if  $\alpha_F(x) \geq \alpha_G(x)$  for all  $x \in \mathbb{R}$ , or equivalently, if  $[G(x) \int_0^x F(t) dt - F(x) \int_0^x G(t) dt] \geq 0$  for all  $x \in \mathbb{R}$ .
- (vi) Increasing convex order ( $X \leq_{\text{icx}} Y$ ), if  $\int_x^\infty \bar{F}(t) dt \leq \int_x^\infty \bar{G}(t) dt$  for all  $x \in \mathbb{R}$ .
- (vii) Increasing concave order ( $X \leq_{\text{icv}} Y$ ), if  $\int_{-\infty}^x G(t) dt \leq \int_{-\infty}^x F(t) dt$  for all  $x \in \mathbb{R}$ .

The following aging classes are defined as in Shaked and Shanthikumar [18]. The random variable  $X$  is said to have:

- (i) Decreasing Mean Residual Life (DMRL) property, if the function  $m_F(x)$  is decreasing in  $x \in \mathbb{R}$ , or equivalently, if  $\int_x^\infty \bar{F}(t) dt$  is log-concave on  $S_X$ .
- (ii) Increasing Reversed Mean Residual life (IRMR) property, if the function  $\alpha_F(x)$  is increasing in  $x \in \mathbb{R}$ , or equivalently, if  $\int_{l_X}^x F(t) dt$  is log-concave on  $S_X$ .

The nonnegative random variable  $X$  is said to have:

- (iii) New Better than Used (NBU) property, if  $\bar{F}(x)\bar{F}(y) \geq \bar{F}(x+y)$  for all  $x, y$  on  $[0, \infty)$ .
- (iv) New Better than Used in Expectation (NBUE) property, if  $m_F(0) \geq m_F(x)$  for all  $x \in [0, \infty)$ .
- (v) New Better than Used in Convex order (NBUC) property, if  $X_t \leq_{icx} X$  for all  $t \geq 0$ .

## 2. PRESERVATION OF STOCHASTIC ORDERS

In this section, we establish our main results for preservation of some stochastic orders under weighted distributions. Then, the reversed implication that stochastic orders of weighted distributions imply the stochastic orders of parent distributions, will be discussed. By applying the following lemma and using the fact that the parent distribution can be regarded, at least theoretically, as a weighted version of the weighted distribution, each result for preservation under weighting can be translated to the reversed direction. It is to be mentioned here that the problem of the reversed preservation under weighting is important, because it provides information about the original distribution of a weighted data set via a mathematical implication. Let  $\eta = E(w(X))$ ,  $B(x) = E(w(X) \mid X > x)$ ,  $A(y) = E(w(X) \mid X \leq y)$  and  $C(x, y) = E(w(X) \mid x < X \leq y)$ . Suppose that  $X_w$  is the weighted version of  $X$  with the weight function  $w$  which has pdf  $f_w$  and cdf  $F_w$ .

**Lemma 1.** *Let  $T = X_w$  and for  $\nu(x) = 1/w(x)$  let  $C^*(x, y) = E(\nu(T) \mid x < T \leq y)$ . Then  $C^*(x, y) = 1/C(x, y)$  for all  $x \leq y \in \mathbb{R}$ . Furthermore,  $T_\nu =_{st} X$ , where  $T_\nu$  is the weighted version of  $T$  with weight function  $\nu$ .*

**Proof.** Note that

$$C(x, y) = \eta \frac{F_w(y) - F_w(x)}{F(y) - F(x)}.$$

For all  $x \leq y \in \mathbb{R}$ , we get

$$\begin{aligned} C^*(x, y) &= \int_x^y \frac{\nu(t)f_w(t)}{F_w(y) - F_w(x)} dt = \frac{\int_x^y f(t) dt}{\eta(F_w(y) - F_w(x))} \\ &= \frac{1}{\eta} \frac{F(y) - F(x)}{F_w(y) - F_w(x)} = \frac{1}{C(x, y)}. \end{aligned}$$

Denote  $\eta^* = E(\nu(T))$ ,  $B^*(x) = E(\nu(T) \mid T > x)$  and  $A^*(y) = E(\nu(T) \mid T \leq y)$ . Then, using the identity  $C^*(x, y) = 1/C(x, y)$ , by letting  $y \rightarrow \infty$ ,  $x \rightarrow \infty$  and  $(x, y) \rightarrow (-\infty, \infty)$ , one at a time, we obtain, respectively,  $B^*(x) = 1/B(x)$ ,  $A^*(y) = 1/A(y)$ , and  $\eta^* = 1/\eta$ .  $\square$

Now, we will pay our attention to the well-known usual stochastic order. Preservation properties of this order were considered in Theorem 3.1 of Misra et al. [11] and Theorem 11 of Bartoszewicz and Skolimowska [4]. The following result is a characterization of the usual stochastic order.

**Theorem 1.** *Let  $w_i$  be non-increasing (non-decreasing) for some  $i = 1, 2$ ; and let  $w_2(x)/\eta_2 \leq (\geq) w_1(x)/\eta_1$  for all  $x \in \mathbb{R}$ . Then*

$$X \leq_{\text{st}} Y \Leftrightarrow X_{w_1} \leq_{\text{st}} Y_{w_2}.$$

**Proof.** Suppose that  $w_i$  is non-increasing for some  $i = 1, 2$ ; and let  $w_2(t)/\eta_2 \leq w_1(t)/\eta_1$  for all  $t \in \mathbb{R}$ . Then, for all  $x \in \mathbb{R}$ , we get

$$\begin{aligned} F_1(x) - G_1(x) &= \int_{-\infty}^x (f_1(t) - g_1(t)) dt \\ &= \int_{-\infty}^x \left( \frac{w_1(t)}{\eta_1} f(t) - \frac{w_2(t)}{\eta_2} g(t) \right) dt \\ &\geq \int_{-\infty}^x \frac{w_i(t)}{\eta_i} (f(t) - g(t)) dt \\ &= \int_{-\infty}^x h_i(t) dW(t), \end{aligned}$$

where  $h_i(t) = w_i(t)I_{(-\infty, x]}(t)/\eta_i$ , and  $W(t) = F(t) - G(t)$ . Obviously,  $h_i(t)$  is non-negative and by assumption it is non-increasing in  $t$ . We know that  $X \leq_{\text{st}} Y$  gives  $\int_{-\infty}^x dW(t) \geq 0$  for all  $x \in \mathbb{R}$ . Hence, Lemma 7.1 (b) in Barlow and Prochan [2] directly provides the proof.  $\square$

The case when  $w_1 = w_2 = w$  gives the following corollary.

**Corollary 1.** *Let  $w$  be a monotone function for which  $E(w(X)) = E(w(Y))$ . Then  $X \leq_{\text{st}} Y$  if and only if  $X_w \leq_{\text{st}} Y_w$ .*

**Example 1.** Let  $X$  and  $Y$  denote the lifetimes of two devices having cumulative distribution functions  $F$  and  $G$ , respectively. Take  $w(x) = I_{(t_0, \infty)}(x)$ , where  $t_0 \in S_X \cap S_Y$  is a time point at which  $F(t_0) = G(t_0)$ . Then, observe that  $X_w \stackrel{\text{st}}{=} (X \mid X > t_0)$  and  $Y_w \stackrel{\text{st}}{=} (Y \mid Y > t_0)$ . From Corollary 1, by the known properties of the

usual stochastic order,  $X \leq_{\text{st}} Y$  if and only if  $X_{t_0} \leq_{\text{st}} Y_{t_0}$ , where  $X_{t_0}$  and  $Y_{t_0}$  are the residual lifetimes associated with  $X$  and  $Y$ , respectively. Similarly, by taking  $w(x) = I_{(0, t_0]}(x)$  such that  $F(t_0) = G(t_0)$ , we conclude that  $X \leq_{\text{st}} Y$  if and only if  $X_{(t_0)} \geq_{\text{st}} Y_{(t_0)}$ , where  $X_{(t_0)}$  and  $Y_{(t_0)}$  are the reversed residual lifetimes of  $X$  and  $Y$ , respectively.

Now, we discuss the problem of preservation of the hazard rate order and the reversed hazard rate order under weighting. For similar results we refer the readers to Misra et al. [11].

**Theorem 2.** *Let  $w_2/w_1$  be non-decreasing and let  $w_i$  be non-decreasing for some  $i = 1, 2$ . Then  $X \leq_{\text{hr}} Y$  implies  $X_{w_1} \leq_{\text{hr}} Y_{w_2}$ .*

**Proof.** Denote  $dW_x(t) = w(x, t) dt$  with  $w(x, t) = [f(x)g(t) - f(t)g(x)]I_{[x, \infty)}(t)$  for all  $x \in \mathbb{R}$  and for all  $t \in \mathbb{R}$ . The hazard rate order between  $X$  and  $Y$  can be translated to  $\int_y^\infty dW_x(t) \geq 0$  for all  $y \leq x \in \mathbb{R}$ . Note that  $X \leq_{\text{hr}} Y$  is equivalent to  $f(x)/g(x) \geq \bar{F}(x)/\bar{G}(x)$  for all  $x \in \mathbb{R}$ , and on the other hand  $X \leq_{\text{hr}} Y$  provides that  $\bar{F}(x)/\bar{G}(x)$  is non-increasing for  $x \in \mathbb{R}$ . Therefore,  $X \leq_{\text{hr}} Y$  implies that  $f(x)/g(x) \geq \bar{F}(y)/\bar{G}(y)$  for all  $x \leq y \in \mathbb{R}$ . In fact, this means that  $\int_y^\infty dW_x(t) \geq 0$  for all  $x \leq y \in \mathbb{R}$ . As a result,  $X \leq_{\text{hr}} Y$  implies that  $\int_y^\infty dW_x(t) \geq 0$  for all  $x \in \mathbb{R}$ , and for all  $y \in \mathbb{R}$ . The assumption that  $w_2/w_1$  is non-decreasing yields  $w_1(x)w_2(t) \geq w_1(t)w_2(x)$  for all  $t \geq x \in \mathbb{R}$ . Now, one has

$$\begin{aligned} f_1(x)\bar{G}_1(x) - g_1(x)\bar{F}_1(x) &= \int_x^\infty [f_1(x)g_1(t) - g_1(x)f_1(t)] dt \\ &= \int_x^\infty \left( \frac{w_1(x)w_2(t)f(x)g(t)}{\eta_1\eta_2} - \frac{w_1(t)w_2(x)g(x)f(t)}{\eta_1\eta_2} \right) dt \\ &\geq \int_x^\infty \frac{w_i(t)w_{3-i}(x)}{\eta_1\eta_2} [f(x)g(t) - f(t)g(x)] dt \\ &= \int_{-\infty}^\infty h_i(t) [f(x)g(t) - f(t)g(x)] I_{[x, \infty)}(t) dt \\ &= \int_{-\infty}^\infty h_i(t) dW_x(t), \end{aligned}$$

where  $h_i(t) = (\eta_1\eta_2)^{-1}w_i(t)w_{3-i}(x)$ . From the assumption of  $w_i$  being non-decreasing, we observe that  $h_i$  is a non-decreasing function. Hence, Lemma 7.1 (a) of Barlow and Proschan [2] completes the proof.  $\square$

By applying Lemma 1 to Theorem 2 we derive the following corollary.

**Corollary 2.** *If  $w_2/w_1$  is non-increasing and at least one of  $w_i$ 's for  $i = 1, 2$  is non-increasing, then  $X_{w_1} \leq_{\text{hr}} Y_{w_2}$  implies  $X \leq_{\text{hr}} Y$ .*

**Example 2** (Damage model of Rao). Suppose that  $Z_1$  and  $Z_2$  are two random variables with density functions  $h_1$  and  $h_2$ , respectively. We assume that  $Z_1$  is independent of  $X$  and that  $Z_2$  is independent of  $Y$ . Visualize the random variable  $X_{w_1}$  as it records the amount of observation  $X$  only if  $Z_1 = X$  (whenever the observed values of  $Z_1$  and of  $X$  are equal) and imagine that  $Y_{w_2}$  records the amount of observation  $Y$  only if  $Z_2 = Y$ . Rao [17] showed that this is a situation, where the weighted distributions can be applied (see also Patil and Rao [16] for more details). Here, the weight functions are  $w_1 = h_1$  and  $w_2 = h_2$ . If at least one of  $Z_i$ 's for  $i = 1, 2$  has a non-increasing density such that  $Z_1 \leq_{\text{lr}} Z_2$ , then according to Corollary 2,  $X_{w_1} \leq_{\text{hr}} Y_{w_2}$  implies  $X \leq_{\text{hr}} Y$ .

The following theorem is analogously derived.

**Theorem 3.** *Let  $w_2/w_1$  be non-decreasing and also let  $w_i$  be non-increasing for some  $i = 1, 2$ . Then  $X \leq_{\text{rh}} Y$  implies  $X_{w_1} \leq_{\text{rh}} Y_{w_2}$ .*

**Proof.** As in the proof of Theorem 2, by appealing to the fact that  $X \leq_{\text{rh}} Y$  is equivalent to  $\int_{-\infty}^x [f(t)g(x) - f(x)g(t)] dt \geq 0$  for all  $x \in \mathbb{R}$ , and further because it is equivalent to  $G(x)/F(x)$  being non-decreasing in  $x$ , we obtain that  $\int_{-\infty}^y dW_x(t) \geq 0$  for all  $x, y \in \mathbb{R}$ , where  $dW_x(t) = w(x, t) dt$  with  $w(x, t) = [f(t)g(x) - f(x)g(t)]I_{(-\infty, x)}(t)$ . The result will be obtained by using Lemma 7.1 (b) of Barlow and Proschan [2].  $\square$

**Corollary 3.** *If  $w_2/w_1$  is non-increasing and at least one of  $w_i$ 's for  $i = 1, 2$  is non-decreasing, then  $X_{w_1} \leq_{\text{rh}} Y_{w_2}$  implies  $X \leq_{\text{rh}} Y$ .*

We now concentrate on the preservation of the mean residual life order and the reversed mean residual life order under weighted distributions (see also Theorem 2.2 in Izadkhah et al. [7] for similar results).

**Theorem 4.** *Let  $B_2/B_1$  be non-decreasing such that  $B_i$  is non-decreasing for some  $i = 1, 2$ . Let  $X \leq_{\text{mrl}} Y$ . Then  $X_{w_1} \leq_{\text{mrl}} Y_{w_2}$ .*

**Proof.** To start, we have

$$\begin{aligned} X \leq_{\text{mrl}} Y &\Leftrightarrow \int_x^\infty [\overline{G}(t)\overline{F}(x) - \overline{G}(x)\overline{F}(t)] dt \geq 0 \quad \forall x \in \mathbb{R}, \\ &\Leftrightarrow \int_{-\infty}^\infty [\overline{G}(t)\overline{F}(x) - \overline{G}(x)\overline{F}(t)] I_{[x, \infty)}(t) dt \geq 0 \quad \forall x \in \mathbb{R}, \\ &\Leftrightarrow \int_y^\infty dW_x(t) \geq 0 \quad \forall y \leq x \in \mathbb{R}, \end{aligned}$$



where  $dW_x(t) = [\overline{G}(t)\overline{F}(x) - \overline{G}(x)\overline{F}(t)]I_{[x,\infty)}(t) dt$ . Note that

$$\begin{aligned} X \leq_{\text{mrl}} Y &\Leftrightarrow \frac{\int_x^\infty \overline{F}(t) dt}{\overline{F}(x)} \leq \frac{\int_x^\infty \overline{G}(t) dt}{\overline{G}(x)} \quad \forall x \in \mathbb{R}, \\ &\Leftrightarrow \frac{\int_x^\infty \overline{F}(t) dt}{\int_x^\infty \overline{G}(t) dt} \leq \frac{\overline{F}(x)}{\overline{G}(x)} \quad \forall x \in \mathbb{R}. \end{aligned}$$

In addition,

$$X \leq_{\text{mrl}} Y \Leftrightarrow \frac{\int_x^\infty \overline{F}(t) dt}{\int_x^\infty \overline{G}(t) dt} \leq \frac{\int_y^\infty \overline{F}(t) dt}{\int_y^\infty \overline{G}(t) dt} \quad \forall x \leq y \in \mathbb{R}.$$

Therefore,  $X \leq_{\text{mrl}} Y$  yields

$$\frac{\int_y^\infty \overline{F}(t) dt}{\int_y^\infty \overline{G}(t) dt} \leq \frac{\overline{F}(x)}{\overline{G}(x)} \quad \forall x \leq y \in \mathbb{R}.$$

The above inequality holds if and only if  $\int_y^\infty dW_x(t) \geq 0$  for all  $x \leq y \in \mathbb{R}$ . Thus,  $X \leq_{\text{mrl}} Y$  implies that  $\int_y^\infty dW_x(t) \geq 0$  for all  $x \in \mathbb{R}$  and for all  $y \in \mathbb{R}$ . Because  $B_2/B_1$  is non-decreasing, we get for all  $x \in \mathbb{R}$ ,

$$\begin{aligned} &\int_x^\infty [\overline{G}_1(t)\overline{F}_1(x) - \overline{G}_1(x)\overline{F}_1(t)] dt \\ &= \int_x^\infty \left( \frac{B_2(t)B_1(x)}{\eta_1\eta_2} \overline{G}(t)\overline{F}(x) - \frac{B_2(x)B_1(t)}{\eta_1\eta_2} \overline{G}(x)\overline{F}(t) \right) dt \\ &\geq \int_x^\infty \frac{B_{3-i}(x)B_i(t)}{\eta_1\eta_2} [\overline{G}(t)\overline{F}(x) - \overline{G}(x)\overline{F}(t)] dt \\ &= \int_{-\infty}^\infty \frac{B_{3-i}(x)B_i(t)}{\eta_1\eta_2} [\overline{G}(t)\overline{F}(x) - \overline{G}(x)\overline{F}(t)] I_{[x,\infty)}(t) dt \\ &= \int_{-\infty}^\infty h_i(t) dW_x(t), \end{aligned}$$

where  $h_i(t) = (\eta_1\eta_2)^{-1}B_{3-i}(x)B_i(t)$ , by the second assumption, is non-decreasing in  $t$ . By Lemma 7.1 (a) of Barlow and Proschan [2] we deduce that  $\int_{-\infty}^\infty h_i(t) \times dW_x(t) \geq 0$  for all  $x \in \mathbb{R}$ .  $\square$

**Corollary 4.** *If  $B_2/B_1$  is non-increasing and if  $B_i$  is non-increasing for some  $i = 1, 2$ , then  $X_{w_1} \leq_{\text{mrl}} Y_{w_2}$  implies  $X \leq_{\text{mrl}} Y$ .*

**Theorem 5.** Let  $A_i$  be non-increasing for some  $i = 1, 2$  and let  $A_2/A_1$  be non-decreasing. Then  $X \leq_{\text{rmr}} Y$  implies  $X_{w_1} \leq_{\text{rmr}} Y_{w_2}$ .

**Proof.** We know that  $X \leq_{\text{rmr}} Y$  yields  $\int_{-\infty}^x [F(t)G(x) - F(x)G(t)] dt \geq 0$  for all  $x \in \mathbb{R}$ . Also,  $X \leq_{\text{rmr}} Y$  is equivalent to  $\int_{-\infty}^x F(t) dt / \int_{-\infty}^x G(t) dt$  being non-increasing in  $x \in \mathbb{R}$ . Therefore, in a manner similar to the discussion made in the proof of Theorem 4, we get  $\int_{-\infty}^y dW(t) \geq 0$  for all  $x \in \mathbb{R}$  and for all  $y \in \mathbb{R}$ , where  $dW(t) = [F(t)G(x) - F(x)G(t)]I_{(-\infty, x]}(t) dt$ . Again, Lemma 7.1 (b) of Barlow and Proschan [2] completes the proof.  $\square$

**Corollary 5.** If  $A_i$  is non-decreasing for some  $i = 1, 2$  such that  $A_2/A_1$  is non-increasing, then  $X_{w_1} \leq_{\text{rmr}} Y_{w_2}$  implies  $X \leq_{\text{rmr}} Y$ .

We now study conditions under which the increasing convex order and the increasing concave order are preserved by weighting.

**Theorem 6.** Let  $B_i$  be non-decreasing for some  $i = 1, 2$  and let  $B_2(x)/\eta_2 \geq B_1(x)/\eta_1$  for all  $x \in \mathbb{R}$ . Then  $X \leq_{\text{icx}} Y$  implies  $X_{w_1} \leq_{\text{icx}} Y_{w_2}$ .

**Proof.** By imposing the second assumption, we get, for all  $x \in \mathbb{R}$ ,

$$\begin{aligned} \int_x^\infty [\overline{G}_1(t) - \overline{F}_1(t)] dt &= \int_x^\infty \left( \frac{B_2(t)}{\eta_2} \overline{G}(t) - \frac{B_1(t)}{\eta_1} \overline{F}(t) \right) dt \\ &\geq \int_x^\infty \frac{B_i(t)}{\eta_i} [\overline{G}(t) - \overline{F}(t)] dt \\ &= \int_{-\infty}^\infty \frac{B_i(t)I_{[x, \infty)}(t)}{\eta_i} [\overline{G}(t) - \overline{F}(t)] dt. \end{aligned}$$

Now, take  $h_i(t) = \eta_i^{-1} B_i(t) I_{[x, \infty)}(t)$ , which by assumption is increasing in  $t$  for all  $x$ . Because of  $X \leq_{\text{icx}} Y$ , we have that  $\int_x^\infty [\overline{G}(t) - \overline{F}(t)] dt \geq 0$  for all  $x \in \mathbb{R}$ . Lemma 7.1 (a) of Barlow and Proschan [2] is again applicable and gives the proof.  $\square$

**Corollary 6.** If  $A_i$  is non-increasing for some  $i = 1, 2$  such that  $B_2(x)/\eta_2 \leq B_1(x)/\eta_1$  for all  $x \in \mathbb{R}$ , then  $X_{w_1} \leq_{\text{icx}} Y_{w_2}$  implies  $X \leq_{\text{icx}} Y$ .

Parallely with the result of Theorem 6 we have the following result.

**Theorem 7.** Let  $A_i$  be non-increasing for some  $i = 1, 2$  and let  $A_1(x)/\eta_1 \geq A_2(x)/\eta_2$  for all  $x \in \mathbb{R}$ . Then  $X \leq_{\text{icv}} Y$  implies  $X_{w_1} \leq_{\text{icv}} Y_{w_2}$ .

**Proof.** The proof is obtained as the proof of Theorem 6, by knowing that  $X \leq_{\text{icv}} Y$  is equivalent to  $\int_{-\infty}^x [F(t) - G(t)] dt \geq 0$  for all  $x \in \mathbb{R}$ , and then applying Lemma 7.1 (b) of Barlow and Proschan [2].  $\square$

**Corollary 7.** *If  $A_i$  is non-decreasing for some  $i = 1, 2$  and if  $A_1(x)/\eta_1 \leq A_2(x)/\eta_2$  for all  $x \in \mathbb{R}$ , then  $X_{w_1} \leq_{\text{icv}} Y_{w_2}$  gives  $X \leq_{\text{icv}} Y$ .*

In order for conditions of Theorems 6 and 7 to be well satisfied, it is noticeable here that a sufficient condition to get  $B_2/\eta_2 \geq B_1/\eta_1$  is that the function  $B_2/B_1$  is non-decreasing and also a sufficient condition for  $A_1/\eta_1 \geq A_2/\eta_2$  to be valid is that  $A_2/A_1$  is non-decreasing. Besides, if  $w_1$  and  $w_2$  are, respectively, non-decreasing and non-increasing, then  $B_1$  and  $A_2$  have the same monotonic properties accordingly (see Remark 2.2 in Misra et al. [11] for more detailed discussions).

### 3. PRESERVATION OF AGING CLASSES

In this section, using some representations of aging classes via stochastic orders we develop some results for preservation of several aging classes under weighting. Parallely, some relevant characterizations are given. We will focus only on positive aging classes. Results for negative aging classes which indeed are dual classes for the classes that were defined in Section 1.1 can be similarly derived.

**Theorem 8.** *Let  $w$  be non-increasing (non-decreasing) and let  $w(x)/\eta \leq (\geq) w(x+t)/B(t)$  for all  $x \geq 0$  and for all  $t \geq 0$ . Then  $X$  is NBU if and only if  $X_w$  is NBU.*

**Proof.** We know that  $X$  is NBU if and only if  $X_t \leq_{\text{st}} X$  for all  $t > 0$ . From Lemma 2.1 (ii) in Izadkhah et al. [6],  $E(w(X_t + t)) = B(t)$ . Taking  $w_1(x) = w(x+t)$  and  $w_2(x) = w(x)$ , under the assumptions  $X_t \leq_{\text{st}} X$  is equivalent to  $(X_t)_{w_1} \leq_{\text{st}} X_{w_2}$  for all  $t > 0$ . By Lemma 2.2 of Izadkhah et al. [6],  $(X_t)_{w_1} \stackrel{\text{st}}{=} (X_w - t \mid X_w > t)$  for all  $t > 0$ . Thus, it follows that  $(X_w - t \mid X_w > t) \leq_{\text{st}} X_w$  for all  $t > 0$ , which means that  $X_w$  is NBU.  $\square$

It is to be mentioned that when  $l_X = 0$ , if  $w(t+x)/B(t)$  is increasing in  $t$  for all  $x \geq 0$ , then  $w(x)/\eta \leq w(x+t)/B(t)$  holds true for all  $x \geq 0$  and for all  $t \geq 0$ .

**Theorem 9.** *Let  $B$  be non-decreasing and let  $w(x)E(X_w) \geq B(x)E(X)$  for all  $x > 0$ . Then  $X \in \text{NBUE}$  implies  $X_w \in \text{NBUE}$ .*

**Proof.** First, notice that  $X \in \text{NBUE}$  if and only if  $\int_x^\infty [f(t)E(X) - \bar{F}(t)] dt \geq 0$  for all  $t > 0$ . We have, for all  $x > 0$ , that

$$\int_x^\infty [f_w(t)E(X_w) - \bar{F}_w(t)] dt = \int_x^\infty \left( \frac{w(t)E(X_w)}{\eta} f(t) - \frac{B(t)\bar{F}(t)}{\eta} \right) dt$$

$$\geq \int_x^\infty \frac{B(t)}{\eta} [E(X)f(t) - \bar{F}(t)] dt = \int_0^\infty \frac{B(t)I_{[x,\infty)}(t)}{\eta} [E(X)f(t) - \bar{F}(t)] dt,$$

where the inequality follows from the second assumption. Set  $h(t) = \eta^{-1}B(t)I_{[x,\infty)}(t)$ . From the first assumption,  $h$  is non-decreasing in  $t$  for all  $x \geq 0$ . Lemma 7.1 (a) of Barlow and Proschan [2] is applicable providing the proof of the theorem.  $\square$

**Corollary 8.** *If  $B$  is non-increasing and if  $w(x)E(X_w) \leq B(x)E(X)$  for all  $x > 0$ , then  $X_w \in \text{NBUE}$  implies  $X \in \text{NBUE}$ .*

**Theorem 10.** *Let  $X$  be DMRL. If  $B$  is non-decreasing and  $w/B$  is non-decreasing, then  $X_w$  is DMRL.*

**Proof.** We know that  $X$  is DMRL if and only if the mean residual life of  $X$  i.e.,  $m_F(x) = \int_x^\infty \bar{F}(t) dt / \bar{F}(x)$  is non-increasing in  $x$ . We get

$$\frac{d}{dx} m_F(x) = \frac{f(x)}{\bar{F}(x)} \frac{\int_x^\infty \bar{F}(t) dt}{\bar{F}(x)} - 1.$$

Thus,  $X$  is DMRL if and only if for all  $x \in \mathbb{R}$ ,

$$(3.1) \quad \int_x^\infty [\bar{F}(x)f(t) - \bar{F}(t)f(x)] dt \geq 0,$$

which is equivalent to  $\int_y^\infty dW_x(t) \geq 0$  for all  $y \leq x \in \mathbb{R}$ , where  $dW_x(t) = w(x, t) dt$  where  $w(x, t) = [\bar{F}(x)f(t) - \bar{F}(t)f(x)]I_{[x,\infty)}(t)$ . Note that, by using (3.1) and by definition, we have

$$\begin{aligned} X \text{ is DMRL} &\Leftrightarrow \frac{\int_x^\infty \bar{F}(t) dt}{\bar{F}(x)} \geq \frac{\int_y^\infty \bar{F}(t) dt}{\bar{F}(y)} \quad \forall x \leq y \in \mathbb{R}, \\ &\Leftrightarrow \frac{\bar{F}(x)}{f(x)} \geq \frac{\int_x^\infty \bar{F}(t) dt}{\bar{F}(x)} \quad \forall x \in \mathbb{R}. \end{aligned}$$

Using the above inequalities, we obtain

$$\begin{aligned} X \text{ is DMRL} &\Rightarrow \frac{\bar{F}(x)}{f(x)} \geq \frac{\int_y^\infty \bar{F}(t) dt}{\bar{F}(y)} \quad \forall x \leq y \in \mathbb{R}, \\ &\Leftrightarrow \int_y^\infty [\bar{F}(x)f(t) - \bar{F}(t)f(x)] dt \geq 0 \quad \forall x \leq y \in \mathbb{R}, \\ &\Leftrightarrow \int_y^\infty dW_x(t) \geq 0 \quad \forall x \leq y \in \mathbb{R}. \end{aligned}$$

Therefore,  $X \in \text{DMRL}$  provides that  $\int_y^\infty dW_x(t) \geq 0$  for all  $x \in \mathbb{R}$  and for all  $y \in \mathbb{R}$ . It can be here written by assumption that

$$\begin{aligned} & \int_x^\infty [\bar{F}_w(x)f_w(t) - \bar{F}_w(t)f_w(x)] dt \\ &= \int_x^\infty \left( \frac{B(x)w(t)f(t)\bar{F}(x)}{\eta^2} - \frac{B(t)w(x)f(x)\bar{F}(t)}{\eta^2} \right) dt \\ &\geq \int_x^\infty \frac{B(t)w(x)}{\eta^2} [\bar{F}(x)f(t) - \bar{F}(t)f(x)] dt \\ &= \int_{-\infty}^\infty \frac{B(t)w(x)}{\eta^2} [\bar{F}(x)f(t) - \bar{F}(t)f(x)] I_{[x,\infty)}(t) dt = \int_{-\infty}^\infty h(t) dW_x(t), \end{aligned}$$

where  $h(t) = \eta^{-2}B(t)w(x)$ , which is non-decreasing by assumption. Hence, Lemma 7.1 (a) of Barlow and Proschan [2] can be applied to obtain the proof.  $\square$

**Corollary 9.** *If  $B$  is non-increasing and if  $w/B$  is non-decreasing, then  $X_w \in \text{DMRL}$  implies  $X \in \text{DMRL}$ .*

**Example 3** (Proportional hazards model). Consider the model  $\bar{G}(x) = [\bar{F}(x)]^\theta$ ,  $\theta > 0$ . This model is referred to as the PHR model in the literature. The cdf  $G$  is easily shown to be a weighted version of  $F$  induced by the weight function  $w(x) = [\bar{F}(x)]^{\theta-1}$  from which we get  $B(x) = \theta^{-1}[\bar{F}(x)]^{\theta-1}$ . It can be readily seen that if  $\theta \in (0, 1]$ , then the assumptions of Theorem 10 hold. Thus, if  $F$  has the DMRL property then  $G$  has the DMRL property.

**Theorem 11.** *Let  $X$  be IRMR, let  $A$  be non-increasing and also let  $A/w$  be non-decreasing. Then  $X_w$  is IRMR.*

**Proof.** We know that  $X$  is IRMR if and only if the reversed mean residual life of  $X$  i.e., the function  $\alpha$  given by  $\alpha(x) = \int_{-\infty}^x F(t) dt / F(x)$  is non-decreasing in  $x$ . It is obvious that

$$\frac{d}{dx} \alpha(x) = 1 - \frac{f(x)}{F(x)} \frac{\int_{-\infty}^x F(t) dt}{F(x)}.$$

So,  $X$  is IRMR if and only if

$$(3.2) \quad \int_{-\infty}^x [F(x)f(t) - F(t)f(x)] dt \geq 0 \quad \forall x \in \mathbb{R},$$

which means that  $\int_{-\infty}^y dW_x(t) \geq 0$  for all  $x \leq y \in \mathbb{R}$ , where  $dW_x(t) = [F(x)f(t) - F(t)f(x)] I_{(-\infty, x]}(t) dt$ . Furthermore,

$$X \text{ is IRMR} \Leftrightarrow \frac{\int_{-\infty}^y F(t) dt}{F(y)} \leq \frac{\int_{-\infty}^x F(t) dt}{F(x)} \quad \forall y \leq x \in \mathbb{R}.$$

Therefore, using the above equivalence relation and via (3.2), if  $X$  is IRMR then

$$\frac{\int_{-\infty}^y F(t) dt}{F(y)} \leq \frac{F(x)}{f(x)} \quad \forall y \leq x \in \mathbb{R},$$

which is equivalent to  $\int_{-\infty}^y dW_x(t) \geq 0$  for all  $y \leq x \in \mathbb{R}$ . That is  $\int_{-\infty}^y dW_x(t) \geq 0$  for all  $x \in \mathbb{R}$  and for all  $x \in \mathbb{R}$ . In other direction, we deduce by assumption that, for all  $x \in \mathbb{R}$ ,

$$\begin{aligned} & \int_{-\infty}^x [F_w(x)f_w(t) - F_w(t)f_w(x)] dt \\ &= \int_{-\infty}^x \left( \frac{A(x)w(t)F(x)f(t)}{\eta^2} - \frac{A(t)w(x)f(x)F(t)}{\eta^2} \right) dt \\ &\geq \int_{-\infty}^x \frac{A(t)w(x)}{\eta^2} [F(x)f(t) - F(t)f(x)] dt \\ &= \int_{-\infty}^{\infty} \frac{A(t)w(x)}{\eta^2} [F(x)f(t) - F(t)f(x)] I_{(-\infty, x]}(t) dt = \int_{-\infty}^{\infty} h(t) dW_x(t), \end{aligned}$$

where  $h(t) = \eta^{-2}A(t)w(x)$ , which is non-increasing by assumption. At the end, by Lemma 7.1 (b) of Barlow and Proschan [2] it follows that  $\int_{-\infty}^{\infty} h(t) dW_x(t) \geq 0$  for all  $x \in \mathbb{R}$ , which completes the proof.  $\square$

**Corollary 10.** *If  $A$  is non-decreasing and if  $A/w$  is non-increasing, then  $X_w \in \text{IRMR}$  implies  $X \in \text{IRMR}$ .*

**Example 4** (Proportional reversed hazards model). The model of  $G(x) = [F(x)]^\theta$ ,  $\theta > 0$ , is well-known in the literature as the PRHR model. The distribution function  $G$  is a weighted version of  $F$  with the weight  $w(x) = [F(x)]^{\theta-1}$ , which gives  $A(x) = \theta^{-1}[F(x)]^{\theta-1}$ . It can be readily seen that if  $\theta \in (0, 1]$ , then the assumptions of Theorem 11 hold. As a result, if  $F$  is IRMR then  $G$  is IRMR.

**Theorem 12.** *Let  $X \in \text{NBUC}$ . If  $B$  is non-decreasing in  $x$ , and  $B(x)/\eta \geq B(t+x)/B(t)$  for all  $x \geq 0$  and for all  $t \geq 0$ . Then  $X_w \in \text{NBUC}$ .*

**Proof.** Let  $t \geq 0$  be fixed. Then,  $X \in \text{NBUC}$  implies that  $X_t \leq_{\text{icx}} X$  for all  $t \geq 0$ . By an application of Lemma 2.1 in Izadkhah et al. [6], for  $w_1(x) = w(t+x)$  and  $w_2(x) = w(x)$ , we have  $B_1(x) = B(t+x)$ ,  $\eta_1 = B(t)$  and also we know that  $B_2(x) = B(x)$  and that  $\eta_2 = \eta$ . Now, Theorem 6 gives  $X_w \geq_{\text{icx}} (X_t)_{w_1}$ . From Lemma 2.2 of Izadkhah et al. [6],  $(X_t)_{w_1} \stackrel{\text{st}}{=} (X_w - t \mid X_w > t)$  for all  $t \geq 0$ . Hence,  $X_w \geq_{\text{icx}} (X_w - t \mid X_w > t)$ , for all  $t \geq 0$ , which provides the proof directly.  $\square$

**Corollary 11.** *If  $B$  is non-increasing and  $B(x)/\eta \leq B(t+x)/B(t)$  for all  $t \geq 0$  and for all  $x \geq 0$ , then  $X_w \in \text{NBUC}$  yields  $X \in \text{NBUC}$ .*

Assume that  $l_X = 0$  and that  $B$  is log-concave on  $(0, \infty)$ . Then  $B(x)/\eta \geq B(t+x)/B(t)$  for all  $t \geq 0$  and for all  $x \geq 0$ . It is to be mentioned here that if  $w$  is non-decreasing then  $B$  is also non-decreasing as discussed after Corollary 7.

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### References

- [1] *I. Ahmad, M. Kayid:* Reversed preservation of stochastic orders for random minima and maxima with applications. *Stat. Pap.* 48 (2007), 283–293.
- [2] *R. E. Barlow, F. Proschan:* Statistical Theory of Reliability and Life Testing. International Series in Decision Processes, Holt, Rinehart and Winston, New York, 1975.
- [3] *J. Bartoszewicz:* On a representation of weighted distributions. *Stat. Probab. Lett.* 79 (2009), 1690–1694.
- [4] *J. Bartoszewicz, M. Skolimowska:* Preservation of classes of life distributions and stochastic orders under weighting. *Stat. Probab. Lett.* 76 (2006), 587–596.
- [5] *P. Błażej:* Preservation of classes of life distributions under weighting with a general weight function. *Stat. Probab. Lett.* 78 (2008), 3056–3061.
- [6] *S. Izadkhah, A. H. Rezaei, M. Amini, G. R. Mohtashami Borzadaran:* A general approach for preservation of some aging classes under weighting. *Commun. Stat., Theory Methods* 42 (2013), 1899–1909.
- [7] *S. Izadkhah, A. H. Rezaei Roknabadi, G. R. Mohtashami Borzadaran:* On properties of reversed mean residual life order for weighted distributions. *Commun. Stat., Theory Methods* 42 (2013), 838–851.
- [8] *K. Jain, H. Singh, I. Bagai:* Relations for reliability measures of weighted distributions. *Commun. Stat., Theory Methods* 18 (1989), 4393–4412.
- [9] *S. Karlin:* Total Positivity. Vol. I. Stanford University Press, Stanford, California, 1968.
- [10] *S. C. Kocher, R. P. Gupta:* Some results on weighted distributions for positive-valued random variables. *Probab. Eng. Inf. Sci.* 1 (1987), 417–423.
- [11] *N. Misra, N. Gupta, I. D. Dhariyal:* Preservation of some aging properties and stochastic orders by weighted distributions. *Commun. Stat., Theory Methods* 37 (2008), 627–644.
- [12] *A. K. Nanda, K. Jain:* Some weighted distribution results on univariate and bivariate cases. *J. Stat. Plann. Inference* 77 (1999), 169–180.
- [13] *A. K. Nanda, H. Singh, N. Misra, P. Paul:* Reliability properties of reversed residual lifetime. *Commun. Stat., Theory Methods* 32 (2003), 2031–2042.
- [14] *J. Navarro, Y. del Aguila, J. M. Ruiz:* Characterizations through reliability measures from weighted distributions. *Stat. Pap.* 42 (2001), 395–402.
- [15] *A. G. Pakes, J. Navarro, J. M. Ruiz, Y. del Aguila:* Characterizations using weighted distributions. *J. Stat. Plann. Inference* 116 (2003), 389–420.
- [16] *G. P. Patil, C. R. Rao:* Weighted distributions and size-biased sampling with applications to wildlife populations and human families. *Biometrics* 34 (1978), 179–189.
- [17] *C. R. Rao:* On discrete distributions arising out of methods of ascertainment. *Sankhyā, Ser. A* 27 (1965), 311–324.

- [18] *Shaked, J. G. Shanthikumar*: Stochastic Orders. Springer Series in Statistics, Springer, New York, 2007.

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