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A PRIORI ERROR ESTIMATES FOR LAGRANGE INTERPOLATION ON TRIANGLES

Kenta Kobayashi, Kunitachi, Takuya Tsuchiya, Matsuyama

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Abstract. We present the error analysis of Lagrange interpolation on triangles. A new a priori error estimate is derived in which the bound is expressed in terms of the diameter and circumradius of a triangle. No geometric conditions on triangles are imposed in order to get this type of error estimates. To derive the new error estimate, we make use of the two key observations. The first is that squeezing a right isosceles triangle perpendicularly does not reduce the approximation property of Lagrange interpolation. An arbitrary triangle is obtained from a squeezed right triangle by a linear transformation. The second key observation is that the ratio of the singular values of the linear transformation is bounded by the circumradius of the target triangle.

Keywords: finite element method; Lagrange interpolation; circumradius condition; minimum angle condition; maximum angle condition

MSC 2010: 65D05, 65N30

1. Introduction

Lagrange interpolation on triangles and the associated error estimates are important subjects in numerical analysis. In particular, they are crucial in the error analysis of finite element methods. It is well known that we must impose some geometric condition on the triangles to obtain an error estimation [2], [14], [18], [19]. In the following, we mention some common estimations.

Let $K \subset \mathbb{R}^2$ be an arbitrary triangle with vertices $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$. Let $\mathcal{P}_1$ be the set of polynomials with two variables whose order is at most 1. For a continuous function $v \in C^0(\overline{K})$, the Lagrange interpolation $\mathcal{I}_K^1 v \in \mathcal{P}_1$ of order 1 is defined by

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$v(\mathbf{x}_i) = (\mathcal{I}_K^1 v)(\mathbf{x}_i)$, $i = 1, 2, 3$. For K, let h_K be the length of its longest edge, and ϱ_K the diameter of its inscribed circle.

The minimum angle condition, Zlámal [19] (1968), Ženíšek [18] (1969)**.** *Let θ_0 $(0 < \theta_0 \leqslant \pi/3)$ be a constant. If any angle θ of K satisfies $\theta \geqslant \theta_0$ and $h_K \leqslant 1$, then there exists a constant $C = C(\theta_0)$ independent of h_K such that*

$$\|v - \mathcal{I}_K^1 v\|_{1,2,K} \leqslant C h_K |v|_{2,2,K} \quad \forall\, v \in H^2(K).$$

Many textbooks on finite element methods, such as those by Ciarlet [6], Brenner-Scott [4], and Ern-Guermond [7], explain the following theorem.

Shape-regularity. *Let $\sigma > 0$ be a constant. If $h_K/\varrho_K \leqslant \sigma$ and $h_K \leqslant 1$, then there exists a constant C that is independent of h_K such that*

$$\|v - \mathcal{I}_K^1 v\|_{1,2,K} \leqslant C \frac{h_K^2}{\varrho_K} |v|_{2,2,K} \leqslant C\sigma h_K |v|_{2,2,K} \quad \forall\, v \in H^2(K). \tag{1.1}$$

It is a simple exercise to show that the minimum angle condition is equivalent to the shape-regularity for triangular elements in $\mathbb{R}^2$. The maximum of the ratio h_K/ϱ_K in a triangulation is called the *chunkiness parameter* [4]. The shape-regularity condition is sometimes called the *inscribed ball condition* as well. On the conditions equivalent to the shape-regularity, see [3]. The minimum angle condition or shape-regularity, however, are not necessarily needed to obtain an error estimate. The following condition is well known.

The maximum angle condition, Babuška-Aziz [2] (1976)**.** *Let θ_1 $(\pi/3 \leqslant \theta_1 < \pi)$ be a constant. If any angle θ of K satisfies $\theta \leqslant \theta_1$ and $h_K \leqslant 1$, then there exists a constant $C = C(\theta_1)$ that is independent of h_K such that*

$$\|v - \mathcal{I}_K^1 v\|_{1,2,K} \leqslant C h_K |v|_{2,2,K} \quad \forall\, v \in H^2(K).$$

Later, Křížek [14] introduced the *semiregularity condition*, which is equivalent to the maximum angle condition (see Section 4.1 (2)). Let R_K be the circumradius of K.

The semiregularity condition, Křížek [14] (1991)**.** *Let $p > 1$ and $\sigma > 0$ be constants. If $R_K/h_K \leqslant \sigma$ and $h_K \leqslant 1$, then there exists a constant $C = C(\sigma)$ that is independent of h_K such that*

$$\|v - \mathcal{I}_K^1 v\|_{1,p,K} \leqslant C h_K |v|_{2,p,K} \quad \forall\, v \in W^{2,p}(K).$$

Since its discovery, the maximum angle condition has been considered the most essential condition for error estimates of Lagrange interpolation on triangular elements. However, Hannukainen, Korotov and Křížek [8] pointed out that *the maximum angle condition is not necessary for convergence of the finite element method* by showing simple numerical examples. Furthermore, the present authors recently reported the following error estimation.

The circumradius condition, Kobayashi-Tsuchiya [11] (2014). *For an arbitrary triangle K with $R_K \leqslant 1$, there exists a constant C_p that is independent of K such that the following estimate holds*:

$$\|v - \mathcal{I}_K^1 v\|_{1,p,K} \leqslant C_p R_K |v|_{2,p,K} \quad \forall\, v \in W^{2,p}(K),\ 1 \leqslant p \leqslant \infty. \tag{1.2}$$

Note that estimate (1.2) follows from

$$B_p^{1,1}(K) := \sup_{v \in \mathcal{T}_p^1(K)} \frac{|v|_{1,p,K}}{|v|_{2,p,K}} \leqslant C_p R_K, \tag{1.3}$$

where the set $\mathcal{T}_p^1(K) \subset W^{2,p}(K)$ is defined by

$$\mathcal{T}_p^1(K) := \{v \in W^{2,p}(K);\ v(\mathbf{x}_i) = 0,\ i = 1,2,3\}.$$

Suppose that $\{\mathcal{T}_h\}_{h>0}$ is a sequence of triangulations of a polygonal domain $\Omega \subset \mathbb{R}^2$ such that

$$\lim_{h \to 0} \max_{K \in \mathcal{T}_h} R_K = 0. \tag{1.4}$$

Let S_h be the set of all piecewise linear functions on $\mathcal{T}_h$, defined by

$$S_h := \{v_h \in H_0^1(\Omega) \cap C(\overline{\Omega});\ v_h|_K \in \mathcal{P}_1\ \forall\, K \in \mathcal{T}_h\},$$

and let $u_h \in S_h$ be the piecewise linear finite element solution on the triangulation $\mathcal{T}_h$ of the Poisson problem

$$-\Delta u = f \text{ in } \Omega, \qquad u = 0 \text{ on } \partial\Omega,$$

for a given $f \in L^2(\Omega)$. Then, Céa's lemma [6], Theorem 2.4.1, claims that, for the exact solution u,

$$\|u - u_h\|_{1,2,\Omega} \leqslant C \inf_{v_h \in S_h} |u - v_h|_{1,2,\Omega} \leqslant C|u - \mathcal{I}_h^1 u|_{1,2,\Omega} \leqslant C\Big(\max_{K \in \mathcal{T}_h} R_K\Big)|u|_{2,2,\Omega},$$

where $\mathcal{I}_h^1 u$ is the global piecewise linear interpolation of u defined by $\mathcal{I}_h^1 u|_K = \mathcal{I}_K^1 u$ for any $K \in \mathcal{T}_h$. Hence, if (1.4) holds and $u \in H^2(\Omega)$, the finite element solutions $\{u_h\}$ converge to u as $h \to 0$. The condition (1.4) is called the *circumradius condition* in [11].

Let $\alpha, \beta \in \mathbb{R}$ be such that $1 < \alpha < \beta < 1 + \alpha$. Consider the triangle K whose vertices are $(0,0)^{\mathrm{T}}$, $(h,0)^{\mathrm{T}}$, and $(h^\alpha, h^\beta)^{\mathrm{T}}$. It is straightforward to see that $\varrho_K = \mathcal{O}(h^\beta)$ and $R_K = \mathcal{O}(h^{1+\alpha-\beta})$. Hence, if $h \to 0$, the convergence rates which (1.1) and (1.2) yield are $\mathcal{O}(h^{2-\beta})$ and $\mathcal{O}(h^{1+\alpha-\beta})$, respectively. Therefore, (1.2) gives a better convergence rate than (1.1). Moreover, if $\beta \geqslant 2$, (1.1) does not yield convergence while (1.2) does. Note that, when $h \to 0$, the maximum angle of K approaches π.

From these facts we can say that the circumradius R_K of K is more important than its minimum and maximum angles (or the chunkiness parameter). It should also be noted that the circumradius condition is closely related to the definition of the surface area [12].

The aim of this paper is to extend (1.2) to higher-order Lagrange interpolation and to prove the following theorem.

Theorem 1.1. *Let K be an arbitrary triangle. Let $1 \leqslant p \leqslant \infty$, and k, m be integers such that $k \geqslant 1$ and $0 \leqslant m \leqslant k$. Then, for the kth-order Lagrange interpolation $\mathcal{I}_K^k$ on K, the following estimation holds:*

$$(1.5) \qquad |v - \mathcal{I}_K^k v|_{m,p,K} \leqslant C\Big(\frac{R_K}{h_K}\Big)^m h_K^{k+1-m} |v|_{k+1,p,K} = C R_K^m h_K^{k+1-2m} |v|_{k+1,p,K}$$

for any $v \in W^{k+1,p}(K)$, where the constant C depends only on k, p and is independent of the geometry of K.

We here emphasize that *no geometric condition on the triangles is imposed* in Theorem 1.1. Therefore, the estimation (1.5) is valid even if the maximum angle condition does not hold.

To prove Theorem 1.1, we make use of two key observations. One of them is that "*squeezing an isosceles right triangle perpendicularly does not reduce the approximation property of Lagrange interpolation,*" which was first noted by Babuška and Aziz [2] for the case $k = 1$ and $p = 2$. This obervation is stated rigorously in Theorem 2.3.

Note that an arbitrary triangle K can be obtained by "folding" or "unfolding" a right triangle. Let A be the 2×2 matrix that defines the linear transformation of "folding" and "unfolding" (see (3.2)). Liu and Kikuchi [15] pointed out that an error estimation of the linear Lagrange interpolation $\mathcal{I}_K^1$ is obtained by considering the eigenvalues of $A^T A$. In Section 3, we rewrite Liu and Kikuchi's proofs using

Kronecker products of matrices, and one of their main results [15], Corollary 1 is immediately obtained (Theorem 3.1). The other key observation is that the upper bound in Theorem 3.1 is closely related to the circumradius R_K of K (Lemma 3.2). Combining Theorem 3.1 and Lemma 3.2, an alternative proof of (1.3) is obtained for the case $p = 2$ (Corollary 3.3).

This method is straightforwardly extended to higher-order Lagrange interpolation in Section 4, and we obtain the main results of Theorem 4.2 that is equivalent to Theorem 1.1.

2. Preliminaries

2.1. Notation. Let $n \geqslant 1$ be a positive integer and $\mathbb{R}^n$ the n-dimensional Euclidean space. Throughout this paper, K is a triangle in $\mathbb{R}^2$. We denote the Euclidean norm of $\mathbf{x} \in \mathbb{R}^n$ by $|\mathbf{x}|$. Let $\mathbb{R}^{n*} := \{l\colon \mathbb{R}^n \to \mathbb{R}\colon\ l \text{ is linear}\}$ be the dual space of $\mathbb{R}^n$. We always regard $\mathbf{x} \in \mathbb{R}^n$ as a column vector and $\mathbf{a} \in \mathbb{R}^{n*}$ as a row vector. For a matrix A and $\mathbf{x} \in \mathbb{R}^n$, A^{T} and $\mathbf{x}^{\mathrm{T}}$ denote their transpositions. For matrices A and B, $A \otimes B$ denotes their Kronecker product. For a differentiable function f with n variables, its gradient $\nabla f = \operatorname{grad} f \in \mathbb{R}^{n*}$ is the row vector

$$\nabla f = \nabla_{\mathbf{x}} f := \Big(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\Big), \qquad \mathbf{x} := (x_1, \ldots, x_n)^{\mathrm{T}}.$$

Let $\mathbb{N}_0$ be the set of nonnegative integers. For $\delta = (\delta_1, \ldots, \delta_n) \in (\mathbb{N}_0)^n$, the multi-index ∂^δ of partial differentiation (in the sense of distribution) is defined by

$$\partial^\delta = \partial^\delta_{\mathbf{x}} := \frac{\partial^{|\delta|}}{\partial x_1^{\delta_1} \ldots \partial x_n^{\delta_n}}, \qquad |\delta| := \delta_1 + \ldots + \delta_n.$$

Let $\Omega \subset \mathbb{R}^n$ be a (bounded) domain. The usual Lebesgue space is denoted by $L^p(\Omega)$ for $1 \leqslant p \leqslant \infty$. For a positive integer k, the Sobolev space $W^{k,p}(\Omega)$ is defined by $W^{k,p}(\Omega) := \{v \in L^p(\Omega);\ \partial^\delta v \in L^p(\Omega),\ |\delta| \leqslant k\}$. The norm and semi-norm of $W^{k,p}(\Omega)$ are defined, for $1 \leqslant p < \infty$, by

$$|v|_{k,p,\Omega} := \Big(\sum_{|\delta|=k} |\partial^\delta v|^p_{0,p,\Omega}\Big)^{1/p}, \quad \|v\|_{k,p,\Omega} := \Big(\sum_{0 \leqslant m \leqslant k} |v|^p_{m,p,\Omega}\Big)^{1/p},$$

and $|v|_{k,\infty,\Omega} := \max_{|\delta|=k} \big\{\operatorname{ess\,sup}_{\mathbf{x}\in\Omega} |\partial^\delta v(\mathbf{x})|\big\}$, $\|v\|_{k,\infty,\Omega} := \max_{0 \leqslant m \leqslant k} \{|v|_{m,\infty,\Omega}\}$.

2.2. Preliminaries from matrix analysis. We introduce some facts from the theory of matrix analysis. For their proofs, readers are referred to textbooks on matrix analysis such as [9] and [17].

Let $n \geqslant 2$ be an integer and A an $n \times n$ regular matrix. Let $B := A^{-1}$. Then $A^{\mathrm{T}}A$ is symmetric positive-definite and has n positive eigenvalues. Let $0 < \mu_m \leqslant \mu_M$ be the minimum and maximum eigenvalues. Then we have

$$\mu_m|\mathbf{x}|^2 \leqslant |A\mathbf{x}|^2 \leqslant \mu_M|\mathbf{x}|^2, \quad \mu_M^{-1}|\mathbf{x}|^2 \leqslant |B^{\mathrm{T}}\mathbf{x}|^2 \leqslant \mu_m^{-1}|\mathbf{x}|^2 \quad \forall\, \mathbf{x} \in \mathbb{R}^n.$$

Then the minimum and maximum eigenvalues of $(A^{\mathrm{T}}A)\otimes(A^{\mathrm{T}}A) = (A\otimes A)^{\mathrm{T}}(A\otimes A)$ are $0 < \mu_m^2 \leqslant \mu_M^2$. Hence, for any $\mathbf{w} \in \mathbb{R}^{n^2}$, we have

$$\mu_m^2|\mathbf{w}|^2 \leqslant |(A\otimes A)\mathbf{w}|^2 \leqslant \mu_M^2|\mathbf{w}|^2, \quad \mu_M^{-2}|\mathbf{w}|^2 \leqslant |(B\otimes B)^{\mathrm{T}}\mathbf{w}|^2 \leqslant \mu_m^{-2}|\mathbf{w}|^2.$$

The above facts can be straightforwardly extended to the case of the higher-order Kronecker product $A\otimes\ldots\otimes A$. For $A\otimes\ldots\otimes A$, $B\otimes\ldots\otimes B$ (the kth Kronecker products), we have, for $\mathbf{w} \in \mathbb{R}^{n^k}$,

$$\mu_m^k|\mathbf{w}|^2 \leqslant |(A\otimes\ldots\otimes A)\mathbf{w}|^2 \leqslant \mu_M^k|\mathbf{w}|^2,\ \mu_M^{-k}|\mathbf{w}|^2 \leqslant |(B\otimes\ldots\otimes B)^{\mathrm{T}}\mathbf{w}|^2 \leqslant \mu_m^{-k}|\mathbf{w}|^2.$$

2.3. The affine transformation defined by a regular matrix. Let A be an $n\times n$ matrix with $\det A > 0$. We consider the affine transformation $\varphi(\mathbf{x})$ defined by $\mathbf{y} = \varphi(\mathbf{x}) := A\mathbf{x} + \mathbf{b}$ for $\mathbf{x} = (x_1, \ldots, x_n)^{\mathrm{T}}$, $\mathbf{y} = (y_1, \ldots, y_n)^{\mathrm{T}}$ with $\mathbf{b} \in \mathbb{R}^n$. Suppose that a reference region $\widehat{\Omega} \subset \mathbb{R}^n$ is transformed to a domain Ω by φ; $\Omega := \varphi(\widehat{\Omega})$. Then a function $v(\mathbf{y})$ defined on Ω is pulled-back to the function $\hat{v}(\mathbf{x})$ on $\widehat{\Omega}$ as $\hat{v}(\mathbf{x}) := v(\varphi(\mathbf{x})) = v(\mathbf{y})$. Then we have $\nabla_{\mathbf{x}}\hat{v} = (\nabla_{\mathbf{y}}v)A$, $\nabla_{\mathbf{y}}v = (\nabla_{\mathbf{x}}\hat{v})B$, and $|\nabla_{\mathbf{y}}v|^2 = |(\nabla_{\mathbf{x}}\hat{v})B|^2 = (\nabla_{\mathbf{x}}\hat{v})BB^{\mathrm{T}}(\nabla_{\mathbf{x}}\hat{v})^{\mathrm{T}}$.

The Kronecker product $\nabla\otimes\nabla$ of the gradient ∇ is defined by

$$\nabla\otimes\nabla := \Big(\frac{\partial}{\partial x_1}\nabla, \ldots, \frac{\partial}{\partial x_n}\nabla\Big) = \Big(\frac{\partial^2}{\partial x_1^2}, \frac{\partial^2}{\partial x_1 \partial x_2}, \ldots, \frac{\partial^2}{\partial x_{n-1}\partial x_n}, \frac{\partial^2}{\partial x_n^2}\Big).$$

We regard $\nabla\otimes\nabla$ as a row vector. From this definition, it follows that

$$\sum_{|\delta|=2}(\partial^\delta v)^2 = \sum_{i,j=1}^{n}\Big(\frac{\partial^2 v}{\partial x_i \partial x_j}\Big)^2 = |(\nabla\otimes\nabla)v|^2$$

and $(\nabla_{\mathbf{x}}\otimes\nabla_{\mathbf{x}})\hat{v} = ((\nabla_{\mathbf{y}}\otimes\nabla_{\mathbf{y}})v)(A\otimes A)$, $(\nabla_{\mathbf{y}}\otimes\nabla_{\mathbf{y}})v = ((\nabla_{\mathbf{x}}\otimes\nabla_{\mathbf{x}})\hat{v})(B\otimes B)$. Suppose that the minimum and maximum eigenvalues of BB^{T} are $0 < \lambda_m \leqslant \lambda_M$. Then we have $\lambda_m|\nabla_{\mathbf{x}}\hat{v}|^2 \leqslant |\nabla_{\mathbf{y}}v|^2 \leqslant \lambda_M|\nabla_{\mathbf{x}}\hat{v}|^2$ and

$$\begin{aligned}\sum_{|\delta|=2}(\partial_{\mathbf{y}}v)^2 &= |(\nabla_{\mathbf{y}}\otimes\nabla_{\mathbf{y}})v|^2 = ((\nabla_{\mathbf{x}}\otimes\nabla_{\mathbf{x}})\hat{v})(B\otimes B)(B\otimes B)^{\mathrm{T}}((\nabla_{\mathbf{x}}\otimes\nabla_{\mathbf{x}})\hat{v})^{\mathrm{T}}\\ &= ((\nabla_{\mathbf{x}}\otimes\nabla_{\mathbf{x}})\hat{v})(BB^{\mathrm{T}}\otimes BB^{\mathrm{T}})((\nabla_{\mathbf{x}}\otimes\nabla_{\mathbf{x}})\hat{v})^{\mathrm{T}},\end{aligned}$$

$$\lambda_m^2\sum_{|\delta|=2}(\partial_{\mathbf{x}}^\delta\hat{v})^2 \leqslant \sum_{|\delta|=2}(\partial_{\mathbf{y}}^\delta v)^2 \leqslant \lambda_M^2\sum_{|\delta|=2}(\partial_{\mathbf{x}}^\delta\hat{v})^2.$$

The above inequalities can be easily extended to higher-order derivatives giving the following inequalities:

$$\lambda_m^k \sum_{|\delta|=k} (\partial_{\mathbf{x}}^\delta \hat{v})^2 \leqslant \sum_{|\delta|=k} (\partial_{\mathbf{y}}^\delta v)^2 \leqslant \lambda_M^k \sum_{|\delta|=k} (\partial_{\mathbf{x}}^\delta \hat{v})^2, \quad k \geqslant 1. \tag{2.1}$$

2.4. Useful inequalities. For N positive real numbers $U_1, \ldots, U_N$, the following inequalities hold:

$$\sum_{k=1}^N U_k^p \leqslant N^{\tau(p)} \Big(\sum_{k=1}^N U_k^2\Big)^{p/2}, \quad \tau(p) := \begin{cases} 1 - p/2, & 1 \leqslant p \leqslant 2, \\ 0, & 2 \leqslant p < \infty, \end{cases} \tag{2.2}$$

$$\Big(\sum_{k=1}^N U_k^2\Big)^{p/2} \leqslant N^{\gamma(p)} \sum_{k=1}^N U_k^p, \quad \gamma(p) := \begin{cases} 0, & 1 \leqslant p \leqslant 2, \\ p/2 - 1, & 2 \leqslant p < \infty. \end{cases} \tag{2.3}$$

2.5. The Sobolev imbedding theorems. If $1 < p < \infty$, Sobolev's Imbedding Theorem and Morry's inequality imply that

$$W^{2,p}(K) \subset C^{1,1-2/p}(\overline{K}), \quad p > 2,$$
$$H^2(K) \subset W^{1,q}(K) \subset C^{0,1-2/q}(\overline{K}) \quad \forall\, q > 2,$$
$$W^{2,p}(K) \subset W^{1,2p/(2-p)}(K) \subset C^{0,2(p-1)/p}(\overline{K}), \quad 1 < p < 2.$$

For proofs of the Sobolev imbedding theorems, see [1] and [5]. For the case $p = 1$, we still have the continuous imbedding $W^{2,1}(K) \subset C^0(\overline{K})$. For the proof of the critical imbedding, see [1], Theorem 4.12 and [4], Lemma 4.3.4.

2.6. Lagrange interpolation on triangles and their estimations. Let K be a triangle with vertices $\mathbf{x}_i$, $i = 1, 2, 3$, and let $(\lambda_1, \lambda_2, \lambda_3)$ be its barycentric coordinates with respect to $\mathbf{x}_i$. By definition, we have $0 \leqslant \lambda_i \leqslant 1$, $\lambda_1 + \lambda_2 + \lambda_3 = 1$. For a positive integer $k \geqslant 1$, the set $\Sigma^k(K)$ of points on K is defined by

$$\Sigma^k(K) := \Big\{\Big(\frac{a_1}{k}, \frac{a_2}{k}, \frac{a_3}{k}\Big) \in K;\ a_i \in \mathbb{N}_0,\ 0 \leqslant a_i \leqslant k,\ a_1 + a_2 + a_3 = k\Big\}. \tag{2.4}$$

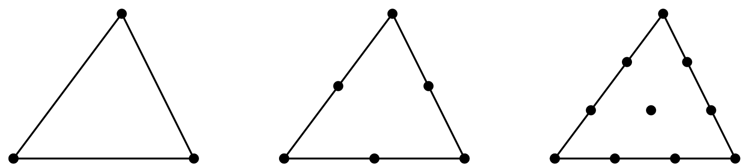

Figure 1. The set $\Sigma^k(K)$, $k = 1$, $k = 2$, $k = 3$.

For a triangle K, a positive integer k, and $1 \leqslant p \leqslant \infty$, we define the subset $\mathcal{T}_p^k(K) \subset W^{k+1,p}(K)$ by

$$\mathcal{T}_p^k(K) := \{v \in W^{k+1,p}(K);\ v(\mathbf{x}) = 0 \quad \forall\, \mathbf{x} \in \Sigma^k(K)\}. \tag{2.5}$$

Let $\mathcal{P}_k$ be the set of polynomials with two variables whose degree is at most k. For a continuous function $v \in C(\overline{K})$, the kth-order Lagrange interpolation $\mathcal{I}_K^k v \in \mathcal{P}_k$ of v is defined by $v(\mathbf{x}) = (\mathcal{I}_K^k v)(\mathbf{x})$ for any $\mathbf{x} \in \Sigma^k(K)$. From this definition, it is clear that $v - \mathcal{I}_K^k v \in \mathcal{T}_p^k(K)$ for any $v \in W^{k+1,p}(K)$.

For an integer m such that $0 \leqslant m \leqslant k$, $B_p^{m,k}(K)$ is defined by

$$B_p^{m,k}(K) := \sup_{v \in \mathcal{T}_p^k(K)} \frac{|v|_{m,p,K}}{|v|_{k+1,p,K}}.$$

Note that we have

$$B_p^{m,k}(K) = \inf\{C;\ |v - \mathcal{I}_K^k v|_{m,p,K} \leqslant C|v|_{k+1,p,K} \quad \forall\, v \in W^{k+1,p}(K)\}. \tag{2.6}$$

For an error estimate of Lagrange interpolation, standard textbooks such as [6] and [4] explain the following theorem. Recall that ϱ_K is the diameter of its inscribed circle of K.

Theorem 2.1. *Let $1 \leqslant p \leqslant \infty$, and let $k \geqslant 1$ be an integer. Let $\sigma > 0$ be a positive constant. Then, for a triangle K that satisfies $h_K/\varrho_K \leqslant \sigma$, the following estimate holds:*

$$|v - \mathcal{I}_K^k v|_{m,p,K} \leqslant C h_K^{k+1-m} |v|_{k+1,p,K} \quad \forall\, v \in W^{k+1,p}(K), \tag{2.7}$$

where $m = 0, 1, \ldots, k$, and the constant C depends on k, p, and σ.

Jamet presented an improved estimation, which does not require the shape-regularity condition [10], Théorème 3.1.

Theorem 2.2 (Jamet). *Let $1 \leqslant p \leqslant \infty$. Let $m \geqslant 0$, $k \geqslant 1$ be integers such that $k + 1 - m > 2/p$ $(1 < p \leqslant \infty)$ or $k - m \geqslant 1$ $(p = 1)$.*[1] *Then the following estimate holds:*

$$|v - \mathcal{I}_K^k v|_{m,p,K} \leqslant C \frac{h_K^{k+1-m}}{\cos^m(\theta_K/2)} |v|_{k+1,p,K} \quad \forall\, v \in W^{k+1,p}(K), \tag{2.8}$$

where $\theta_K \geqslant \pi/3$ is the maximum angle of K, and C depends only on k, p.

[1] Note that in [10], Théorème 3.1 the case $p = 1$ is not mentioned explicitly but clearly holds for triangles.

Note that, if $m = k \geqslant 1$ and $1 \leqslant p \leqslant 2$, estimate (2.8) cannot be applied. As will be noted in Section 4.1 (2), Theorem 1.1 includes Theorem 2.2 as a special case.

Let K_α be the right triangle with vertices $(0,0)^{\mathrm{T}}$, $(1,0)^{\mathrm{T}}$, and $(0,\alpha)^{\mathrm{T}}$ $(0 < \alpha \leqslant 1)$ that is obtained by squeezing $\widehat{K}$. As is stated in Section 1, squeezing a right triangle perpendicularly does not deteriorate the approximation property of Lagrange interpolation. We have the following theorem:

Theorem 2.3. *There exists a constant $C_{k,p}$ that depends only on k and p $(1 \leqslant p \leqslant \infty)$ and is independent of α $(0 < \alpha \leqslant 1)$ such that*

$$B_p^{m,k}(K_\alpha) := \sup_{v \in \mathcal{T}_p^k(K_\alpha)} \frac{|v|_{m,p,K_\alpha}}{|v|_{k+1,p,K_\alpha}} \leqslant C_{k,p}, \qquad m = 0, 1, \ldots, k. \tag{2.9}$$

Note that Theorem 2.3 is not a totally new result. For the case $m = k = 1$ and $p = 2$, (2.9) was proved by Babuška and Aziz in [2]. Kobayashi and Tsuchiya [11] proved (2.9) with $m = k = 1$ and any p $(1 \leqslant p \leqslant \infty)$. For the case $k \geqslant 1$ with $p = 2$ and $m = 0$, 1, (2.9) was proved by Shenk [16]. By (2.8), estimate (2.9) holds if $k + 1 - m > 2/p$ $(1 < p \leqslant \infty)$ or $k - m \geqslant 1$ $(p = 1)$. Hence, it seems that (2.9) with $k = m \geqslant 2$ and $1 \leqslant p \leqslant 2$ has not yet been proved. A proof of Theorem 2.3 by the Babuška-Aziz type technique will be given in [13].

3. Liu and Kikuchi's method

In this section, we give an alternative proof of (1.3) for the case $p = 2$ using the Liu and Kikuchi's method. To this end, we rewrite their proof using the Kronecker product of matrices.

For s, t, and α with $s^2 + t^2 = 1$, $t > 0$, $0 < \alpha \leqslant 1$, we consider the vector $(\alpha s, \alpha t)^{\mathrm{T}} \in \mathbb{R}^2$. Let K be the triangle with vertices $\mathbf{x}_1 := (0,0)^{\mathrm{T}}$, $\mathbf{x}_2 := (1,0)^{\mathrm{T}}$, and $\mathbf{x}_3 := (\alpha s, \alpha t)^{\mathrm{T}}$. Let e_1, e_2, e_3 be the three edges of K, as depicted in Figure 2. Without loss of generality, we assume that e_2 is the longest edge of K. Let θ be the angle between e_1 and e_3. Then $s = \cos\theta$, $t = \sin\theta$, and the assumption that e_2 is the longest yields

$$s = \cos\theta \leqslant \frac{\alpha}{2} \leqslant \frac{1}{2}, \quad \frac{\pi}{3} \leqslant \theta < \pi. \tag{3.1}$$

Note that an arbitrary triangle in $\mathbb{R}^2$ can be transformed to K by a sequence of scaling, translation, rotation, and mirror imaging.

We define the 2×2 matrices as

$$A := \begin{pmatrix} 1 & s \\ 0 & t \end{pmatrix}, \qquad B := A^{-1} = \begin{pmatrix} 1 & -st^{-1} \\ 0 & t^{-1} \end{pmatrix}. \tag{3.2}$$

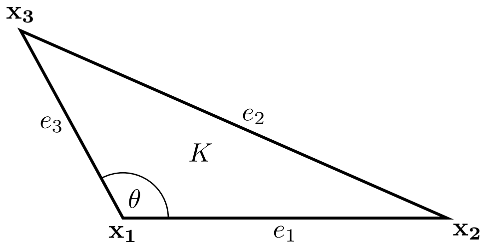

Figure 2. The triangle under consideration. The vertices are $\mathbf{x_1} = (0,0)^{\mathrm{T}}$, $\mathbf{x_2} = (1,0)^{\mathrm{T}}$, and $\mathbf{x_3} = (\alpha s, \alpha t)^{\mathrm{T}}$, where $s^2 + t^2 = 1$, $t > 0$, and $0 < \alpha \leqslant 1$. We assume that $|e_1| = 1 \leqslant |e_2|$.

Then K_α can be transformed to K by the transformation $\mathbf{y} = A\mathbf{x}$. Moreover, $\mathcal{T}_p^k(K)$ is pulled-back to $\mathcal{T}_p^k(K_\alpha)$ as $\mathcal{T}_p^k(K) \ni v \mapsto \hat{v} := v \circ A \in \mathcal{T}_p^k(K_\alpha)$. A simple computation yields that $A^{\mathrm{T}}A$ has eigenvalues $1 \pm |s|$, and BB^{T} has eigenvalues $(1 \mp |s|)/t^2$. It follows from (2.1) that $(1-|s|)t^{-2}|\nabla_{\mathbf{x}}\hat{v}|^2 \leqslant |\nabla_{\mathbf{y}} v|^2 \leqslant (1+|s|)t^{-2}|\nabla_{\mathbf{x}}\hat{v}|^2$ and

$$\frac{(1-|s|)^2}{t^4} \sum_{|\delta|=2} (\partial_{\mathbf{x}}^\delta \hat{v})^2 \leqslant \sum_{|\delta|=2} (\partial_{\mathbf{y}}^\delta v)^2 \leqslant \frac{(1+|s|)^2}{t^4} \sum_{|\delta|=2} (\partial_{\mathbf{x}}^\delta \hat{v})^2. \tag{3.3}$$

Furthermore, because the determinant of A is t, we have

$$|v|_{1,2,K}^2 \leqslant \frac{1+|s|}{t} |\hat{v}|_{1,2,K_\alpha}^2, \qquad \frac{(1-|s|)^2}{t^3} |\hat{v}|_{2,2,K_\alpha}^2 \leqslant |v|_{2,2,K}^2,$$
$$\frac{|v|_{1,2,K}^2}{|v|_{2,2,K}^2} \leqslant \frac{t^2(1+|s|)|\hat{v}|_{1,2,K_\alpha}^2}{(1-|s|)^2|\hat{v}|_{2,2,K_\alpha}^2} = \frac{(1+|s|)^2|\hat{v}|_{1,2,K_\alpha}^2}{(1-|s|)|\hat{v}|_{2,2,K_\alpha}^2}.$$

Combining this estimate and (2.9) with $m = k = 1$ and $p = 2$, we obtain the following theorem [15], Corollary 1:

Theorem 3.1 (Liu-Kikuchi). *For $0 < \alpha \leqslant 1$ we have the estimate*

$$B_2^{1,1}(K) \leqslant \frac{1+|s|}{\sqrt{1-|s|}} B_2^{1,1}(K_\alpha) \leqslant \frac{2C_{1,2}}{\sqrt{1-|s|}}.$$

The following is the key lemma.

Lemma 3.2. *Let R_K be the circumradius of K. For the triangle K considered in this section, the following inequality holds:*

$$\frac{1}{\sqrt{1-|s|}} \leqslant 2\sqrt{2}\, R_K.$$

P r o o f. Recall from (3.1) that $s = \cos\theta$, $t = \sin\theta$, and $\pi/3 \leqslant \theta < \pi$. A straightforward computation implies that

$$\sqrt{1+|s|} \leqslant \sqrt{2}\sqrt{1+\alpha^2 - 2\alpha s} \quad \forall\, \alpha \in (0,1],\ -1 < s \leqslant \frac{\alpha}{2}.$$

From the cosine and sine laws, we have $|e_2|^2 = 1 + \alpha^2 - 2\alpha s = 4R_K^2 t^2$. Therefore, we obtain

$$\frac{1}{\sqrt{1-|s|}} = \frac{\sqrt{1+|s|}}{t} \leqslant \frac{\sqrt{2}}{t}\sqrt{1+\alpha^2-2\alpha s} = \frac{\sqrt{2}}{t}\sqrt{4R_K^2 t^2} = 2\sqrt{2}\, R_K.$$

□

Combining Theorem 3.1 and Lemma 3.2, we have obtained an alternative proof of (1.3) for the triangle depicted in Figure 2 with $p = 2$.

Corollary 3.3. *Let K be the triangle depicted in Figure* 2. *Then we have*

$$B_2^{1,1}(K) := \sup_{v \in \mathcal{T}_2^1(K)} \frac{|v|_{1,2,K}}{|v|_{2,2,K}} \leqslant 4\sqrt{2}\, C_{1,2} R_K.$$

4. Main results and their proofs

The method explained so far can be immediately extended to higher-order Lagrange interpolation. Inequality (3.3) is extended to the case of arbitrary k as follows:

$$\frac{(1-|s|)^k}{t^{2k}} \sum_{|\delta|=k} (\partial_{\mathbf{x}}^{\delta}\hat{v})^2 \leqslant \sum_{|\delta|=k} (\partial_{\mathbf{y}}^{\delta} v)^2 \leqslant \frac{(1+|s|)^k}{t^{2k}} \sum_{|\delta|=k} (\partial_{\mathbf{x}}^{\delta}\hat{v})^2.$$

Let $1 \leqslant p < \infty$. Then inequalities (2.2) and (2.3) yield

$$\begin{aligned}
|v|^p_{m,p,K} &= \int_K \sum_{|\delta|=m} |\partial^\delta_{\mathbf{y}} v(\mathbf{y})|^p \,\mathrm{d}\mathbf{y} \leqslant 2^{m\tau(p)} \int_K \Big(\sum_{|\delta|=m} |\partial^\delta_{\mathbf{y}} v(\mathbf{y})|^2 \Big)^{p/2} \,\mathrm{d}\mathbf{y} \\
&\leqslant 2^{m\tau(p)} \Big(\frac{1+|s|}{t^2} \Big)^{mp/2} \int_K \Big(\sum_{|\delta|=m} |\partial^\delta_{\mathbf{x}} \hat{v}(\mathbf{x})|^2 \Big)^{p/2} \,\mathrm{d}\mathbf{y} \\
&= 2^{m\tau(p)} \Big(\frac{1+|s|}{t^2} \Big)^{mp/2} t \int_{K_\alpha} \Big(\sum_{|\delta|=m} |\partial^\delta_{\mathbf{x}} \hat{v}(\mathbf{x})|^2 \Big)^{p/2} \,\mathrm{d}\mathbf{x} \\
&\leqslant 2^{m(\tau(p)+\gamma(p))} \Big(\frac{1+|s|}{t^2} \Big)^{mp/2} t \int_{K_\alpha} \sum_{|\delta|=m} |\partial^\delta_{\mathbf{x}} \hat{v}(\mathbf{x})|^p \,\mathrm{d}\mathbf{x} \\
&= 2^{m(\tau(p)+\gamma(p))} \Big(\frac{1+|s|}{t^2} \Big)^{mp/2} t |\hat{v}|^p_{m,p,K_\alpha}
\end{aligned}$$

and

$$\begin{aligned}
|v|^p_{k+1,p,K} &= \int_K \sum_{|\delta|=k+1} |\partial^\delta_{\mathbf{y}} v(\mathbf{y})|^p \,\mathrm{d}\mathbf{y} \\
&\geqslant 2^{-(k+1)\gamma(p)} \int_K \Big(\sum_{|\delta|=k+1} |\partial^\delta_{\mathbf{y}} v(\mathbf{y})|^2 \Big)^{p/2} \,\mathrm{d}\mathbf{y} \\
&\geqslant 2^{-(k+1)\gamma(p)} \Big(\frac{1-|s|}{t^2} \Big)^{(k+1)p/2} \int_K \Big(\sum_{|\delta|=k+1} |\partial^\delta_{\mathbf{x}} \hat{v}(\mathbf{x})|^2 \Big)^{p/2} \,\mathrm{d}\mathbf{y} \\
&= 2^{-(k+1)\gamma(p)} \Big(\frac{1-|s|}{t^2} \Big)^{(k+1)p/2} t \int_{K_\alpha} \Big(\sum_{|\delta|=k+1} |\partial^\delta_{\mathbf{x}} \hat{v}(\mathbf{x})|^2 \Big)^{p/2} \,\mathrm{d}\mathbf{x} \\
&\geqslant 2^{-(k+1)(\tau(p)+\gamma(p))} \Big(\frac{1-|s|}{t^2} \Big)^{(k+1)p/2} t \int_{K_\alpha} \sum_{|\delta|=k+1} |\partial^\delta_{\mathbf{x}} \hat{v}(\mathbf{x})|^p \,\mathrm{d}\mathbf{x} \\
&= 2^{-(k+1)(\tau(p)+\gamma(p))} \Big(\frac{1-|s|}{t^2} \Big)^{(k+1)p/2} t |\hat{v}|^p_{k+1,p,K_\alpha}.
\end{aligned}$$

The two inequalities and Theorem 2.3, Lemma 3.2 imply

$$\begin{aligned}
\frac{|v|^p_{m,p,K}}{|v|^p_{k+1,p,K}} &\leqslant \tilde{c}^p_{k,m,p} \frac{t^{p(k+1-m)}(1+|s|)^{mp/2} |\hat{v}|^p_{m,p,K_\alpha}}{(1-|s|)^{(k+1)p/2} |\hat{v}|^p_{k+1,p,K_\alpha}} \\
&= \tilde{c}^p_{k,m,p} \frac{(1+|s|)^{(k+1+m)p/2} |\hat{v}|^p_{m,p,K_\alpha}}{t^{pm} |\hat{v}|^p_{k+1,p,K_\alpha}}, \\
\frac{|v|_{m,p,K}}{|v|_{k+1,p,K}} &\leqslant \tilde{c}_{k,m,p} \frac{(1+|s|)^{(k+1+m)/2} |\hat{v}|_{m,p,K_\alpha}}{t^m |\hat{v}|_{k+1,p,K_\alpha}} \leqslant c_{k,p} C_{k,p} R^m_K,
\end{aligned}$$

where $\tilde{c}_{k,m,p} := 2^{(k+1+m)(\tau(p)+\gamma(p))/p}$ and the constant $c_{k,p}$ depends only on k, p. If $p = \infty$, the same estimation is obtained by letting $p \to \infty$ in the above inequalities. Thus, denoting $c_{k,p}C_{k,p}$ by $C_{k,p}$, the following theorem has been proved.

Theorem 4.1. *Let K be the triangle depicted in Figure 2. Then the estimate*

$$B_p^{m,k}(K) := \sup_{v \in \mathcal{T}_p^k(K)} \frac{|v|_{m,p,K}}{|v|_{k+1,p,K}} \leqslant C_{k,p} R_K^m \quad \forall\, p,\ 1 \leqslant p \leqslant \infty$$

holds, where R_K is the circumradius of K and the constant $C_{k,p}$ depends only on k and p.

Now, let K be an arbitrary triangle. Theorem 4.1 and Corollary 3.3 can be extended to K. A similar transformation G_Y for a positive $Y \in \mathbb{R}$ is defined by $G_Y\colon \mathbb{R}^2 \to \mathbb{R}^2$, $G_Y(\mathbf{x}) := Y\mathbf{x}$. Let K_1 be defined by $K_1 = G_Y(K)$. A function $u \in W^{k,p}(K)$ on K is pulled-back to $v(\mathbf{x}) := u(G_Y^{-1}(\mathbf{x})) = u(G_{1/Y}(\mathbf{x}))$ on K_1. Then for a nonnegative integer k and any p ($1 \leqslant p \leqslant \infty$) we have

$$|v|_{k,p,K_1} = Y^{2/p-k}|u|_{k,p,K} \quad \forall\, u \in W^{p,k}(K).$$

Let $h_K \geqslant h_2 \geqslant h_1$ be the lengths of the three edges of K. Suppose that the second longest edge of K is parallel to the x- or y-axis. Then, by a translation, a mirror imaging, and G_{1/h_2}, K can be transformed to the triangle $\widetilde{K}$ depicted in Figure 2. Hence, we may apply Theorem 4.1 to $\widetilde{K}$, and obtain

$$\sup_{u \in \mathcal{T}_p^k(K)} \frac{h_2^m |u|_{m,2,K}}{h_2^{k+1}|u|_{k+1,2,K}} = \sup_{v \in \mathcal{T}_p^k(\widetilde{K})} \frac{|v|_{m,p,\widetilde{K}}}{|v|_{k+1,p,\widetilde{K}}} \leqslant C_{k,p} R_{\widetilde{K}}^m$$

and

$$\sup_{u \in \mathcal{T}_p^k(K)} \frac{|u|_{m,p,K}}{|u|_{k+1,p,K}} \leqslant C_{k,p} R_{\widetilde{K}}^m h_2^{k+1-m} \leqslant C_{k,p} R_K^m h_K^{k+1-2m}.$$

Here we use the fact that $R_{\widetilde{K}} h_2 = R_K$ and $h_K/2 < h_2 \leqslant h_K$. The constant $C_{k,p}$ can be modified up to a constant multiple. Note that if $p \neq 2$, the Sobolev norms are modified by a rotation. Therefore, we have shown the following theorem, which is equivalent to Theorem 1.1 because of (2.6).

Theorem 4.2. *Let K be an arbitrary triangle. Let R_K be its circumradius and h_K the length of its longest edge. Let $1 \leqslant p \leqslant \infty$, and let m, k be integers such that $0 \leqslant m \leqslant k$. Then there exists a positive constant C that depends only on k, p such that the following estimation holds:*

$$B_p^{m,k}(K) := \sup_{u \in \mathcal{T}_p^k(K)} \frac{|u|_{m,p,K}}{|u|_{k+1,p,K}} \leqslant C\Big(\frac{R_K}{h_K}\Big)^m h_K^{k+1-m} = CR_K^m h_K^{k+1-2m}.$$

4.1. Concluding remarks. Here we compare the newly obtained estimate (1.5) with known results such as (2.7), (2.8), and (1.2).

(1) For an error analysis of the finite element method, the cases $m = 0, 1$ are the most important. In these cases, the estimates obtained from (1.5) can be written, for any $v \in W^{k+1,p}(K)$, as

$$|v - \mathcal{I}_K^k v|_{1,p,K} \leqslant CR_K h_K^{k-1} |v|_{k+1,p,K}, \quad |v - \mathcal{I}_K^k v|_{0,p,K} \leqslant Ch_K^{k+1} |v|_{k+1,p,K}.$$

They are extensions of (1.2). Recall that the constant C is independent of the geometry of K.

(2) Recall that $h_1 \leqslant h_2 \leqslant h_K$ are the lengths of the three edges of K. Let θ_K be the maximum angle of K and S_K the area of K. Then, from the formulas $S_K = \frac{1}{2} h_1 h_2 \sin\theta_K$ and $R_K = h_1 h_2 h_K/(4S_K)$, we have

$$\frac{R_K}{h_K} = \frac{1}{2\sin\theta_K}, \qquad \frac{\pi}{3} \leqslant \theta_K < \pi.$$

Thus, it is clear that *the boundedness of R_K/h_K, which is the semiregularity of K defined by Křížek, is equivalent to the maximum angle condition* $\theta_K \leqslant \theta_1 < \pi$ with a fixed constant θ_1. If this is the case, the estimate from (1.5) becomes

$$|v - \mathcal{I}_K^k v|_{m,p,K} \leqslant \frac{C}{(2\sin\theta_1)^m} h_K^{k+1-m} |v|_{k+1,p,K} \quad \forall\, v \in W^{k+1,p}(K)$$

for $m = 0, 1, \ldots, k$, which is an extension of Jamet's result of (2.8).

References

[1] *R. A. Adams, J. J. F. Fournier*: Sobolev Spaces. Pure and Applied Mathematics 140, Academic Press, New York, 2003.

[2] *I. Babuška, A. K. Aziz*: On the angle condition in the finite element method. SIAM J. Numer. Anal. *13* (1976), 214–226.

[3] *J. Brandts, S. Korotov, M. Křížek*: On the equivalence of regularity criteria for triangular and tetrahedral finite element partitions. Comput. Math. Appl. *55* (2008), 2227–2233.

[4] *S. C. Brenner, L. R. Scott*: The Mathematical Theory of Finite Element Methods. Texts in Applied Mathematics 15, Springer, New York, 2008.

[5] *H. Brezis*: Functional Analysis, Sobolev Spaces and Partial Differential Equations. Universitext, Springer, New York, 2011.

[6] *P. G. Ciarlet*: The Finite Element Method for Elliptic Problems. Classics in Applied Mathematics 40, SIAM, Philadelphia, 2002, Repr., unabridged republ. of the orig. 1978.

[7] *A. Ern, J.-L. Guermond*: Theory and Practice of Finite Elements. Applied Mathematical Sciences 159, Springer, New York, 2004.

[8] *A. Hannukainen, S. Korotov, M. Křížek*: The maximum angle condition is not necessary for convergence of the finite element method. Numer. Math. *120* (2012), 79–88.

[9] *R. A. Horn, C. R. Johnson*: Topics in Matrix Analysis. Cambridge University Press, Cambridge, 1991.

[10] *P. Jamet*: Estimations d'erreur pour des éléments finis droits presque dégénérés. Rev. Franc. Automat. Inform. Rech. Operat., R *10* (1976), 43–60. (In French.)

[11] *K. Kobayashi, T. Tsuchiya*: A Babuška-Aziz type proof of the circumradius condition. Japan J. Ind. Appl. Math. *31* (2014), 193–210.

[12] *K. Kobayashi, T. Tsuchiya*: On the circumradius condition for piecewise linear triangular elements. Japan J. Ind. Appl. Math. *32* (2015), 65–76.

[13] *K. Kobayashi, T. Tsuchiya*: An extension of Babuška-Aziz's theorem to higher order Lagrange interpolation. ArXiv:1508.00119 (2015).

[14] *M. Křížek*: On semiregular families of triangulations and linear interpolation. Appl. Math., Praha *36* (1991), 223–232.

[15] *X. Liu, F. Kikuchi*: Analysis and estimation of error constants for P_0 and P_1 interpolations over triangular finite elements. J. Math. Sci., Tokyo *17* (2010), 27–78.

[16] *N. A. Shenk*: Uniform error estimates for certain narrow Lagrange finite elements. Math. Comput. *63* (1994), 105–119.

[17] *T. Yamamoto*: Elements of Matrix Analysis. Saiensu-sha, 2010. (In Japanese.)

[18] *A. Ženíšek*: The convergence of the finite element method for boundary value problems of the system of elliptic equations. Apl. Mat. *14* (1969), 355–376. (In Czech.)

[19] *M. Zlámal*: On the finite element method. Numer. Math. *12* (1968), 394–409.

Authors' addresses: *Kenta Kobayashi*, Graduate School of Commerce and Management, Hitotsubashi University, Kunitachi, 186-8601, Japan, e-mail: `kenta.k@r.hit-u.ac.jp`; *Takuya Tsuchiya*, Graduate School of Science and Engineering, Ehime University, Matsuyama, 790-8577, Japan, e-mail: `tsuchiya@math.sci.ehime-u.ac.jp`.