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MULTIPLE TIME SCALES SOLUTION OF AN EQUATION WITH QUADRATIC AND CUBIC NONLINEARITIES HAVING FRACTIONAL-ORDER DERIVATIVE

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ABSTRACT- Nonlinear vibrations of quadratic and cubic system are considered. The equation of motion includes fractional order term. Multiple time scales (a perturbation method) solution of the system is developed. Effect of fractional order derivative term is discussed.
Keywords: Fractional differential equation, Caputo fractional derivative, Nonlinear vibrations, Multiple time Scales method

1. INTRODUCTION

Due to rapid development of nonlinear science, many different methods were used to solve nonlinear problems. Perturbation methods are well established and used for over a century to determine approximate analytical solutions for mathematical models. Algebraic equations, integrals, differential equations, difference equations and integro-differential equations can be solved approximately with these techniques. [1-4]. Fractional derivatives appear in different applications such as fluid mechanics, viscoelasticity, biology [5-8]. The asymptotic solution of van der Pol oscillator with small fractional damping was considered by Feng Xie and Xueyuan Lin [9].

Very recently, Pakdemirli *et al.* [10] proposed a new perturbation method to handle strongly nonlinear systems. The method combines Multiple Scales and Lindstedt Poincare method. The new method, namely the Multiple Scales Lindstedt Poincare method (MSLP), is applied to free vibrations of a linear damped oscillator, undamped and damped duffing oscillator. MSLP (a new perturbation solution) was applied to the equation with quadratic and cubic nonlinearities by Pakdemirli and Karahan [11] .

In this paper, multiple time scales method (a perturbation method) is used to solve the equation with quadratic and cubic nonlinearities including fractional-order derivative term. Multiple time scales solution and numerical solutions of the problem are compared.

2. MULTIPLE TIME SCALES (MS) METHOD

The equation of motion is

$$\ddot{x}(t)+\omega_0{}^2 x(t)+\varepsilon^2 D^\alpha x(t)+\varepsilon\alpha_1 x(t)^2+\varepsilon^2\alpha_2 x(t)^3=0 \quad (\varepsilon<<1) \tag{1}$$

with initial conditions

$$x(0)=1,\quad \dot{x}(0)=0 \tag{2}$$

Where the fractional derivative $D^\alpha x$ is in the Caputo sense defined as

$$D^{\alpha}x(t)=\frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{x'(s)ds}{(t-s)^{\alpha}} \quad , \; 0<\alpha<1$$

Fast and slow time scales are

$$T_0=t \;,\quad T_1=\varepsilon t\,,\; T_2=\varepsilon^2 t \tag{3}$$

The time derivatives, dependent variable and fractional derivative are expanded

$$\frac{d}{dt}=D_0+\varepsilon D_1+\varepsilon^2 D_2 \qquad\qquad \frac{d^2}{dt^2}=D_0^2+2\varepsilon D_0D_1+\varepsilon^2\left(D_1^2+2D_0D_2\right)+\ldots \tag{4}$$

$$(\frac{d}{dt})^{\alpha}=D_0^{\alpha}+\varepsilon\alpha D_0^{\alpha-1}D_1+\frac{1}{2}\varepsilon^2\alpha\left[\left(\alpha-1\right)D_0^{\alpha-2}D_1^2+2D_0^{\alpha-1}D_2\right] \tag{5}$$

Where $D_n=\dfrac{\partial}{\partial T_n}$. The expansion

$$x=x_o(T_0,T_1,T_2)+\varepsilon x_1(T_0,T_1,T_2)+\varepsilon^2 x_2(T_0,T_1,T_2)+\ldots \tag{6}$$

is substituted into equation (1)

$$\begin{aligned}&\left[D_0^2+2\varepsilon D_0D_1+\varepsilon^2\left(D_1^2+2D_0D_2\right)\right]\left[x_0+\varepsilon x_1+\varepsilon^2x_2\right]+\omega_0{}^2\left[x_0+\varepsilon x_1+\varepsilon^2x_2\right]+\\&\varepsilon^2\left[\;D_0^{\alpha}+\varepsilon\alpha D_0^{\alpha-1}D_1+\frac{1}{2}\varepsilon^2\alpha\left[\left(\alpha-1\right)D_0^{\alpha-2}D_1^2+2D_0^{\alpha-1}D_2\right]\;\right]\left[x_0+\varepsilon x_1+\varepsilon^2x_2\right]+\\&\alpha_1\varepsilon\left[x_0+\varepsilon x_1+\varepsilon^2x_2\right]^2+\alpha_2\varepsilon^2\left[x_0+\varepsilon x_1+\varepsilon^2x_2\right]^3=0\end{aligned} \tag{7}$$

The equations at each order are

O(1) $\quad D_0^2x_0+\omega_0{}^2x_0=0$ (8)

O(ε) $\quad D_0^2\,x_1+\omega_0{}^2x_1=-2D_0D_1x_0-\alpha_1x_0{}^2$ (9)

O(ε^2) $\quad D_0^2\,x_2+\omega_0{}^2x_2=-2D_0D_1x_1-2\alpha_1x_0\;x_1-D_0{}^{\alpha}x_0-\alpha_2x_0{}^3-(D_1{}^2+2D_0D_2)x_0$ (10)

The solution at the first order is

$$x_0=A\left(T_1,T_2\right)e^{i\omega_0T_0}+\overline{A}\left(T_1,T_2\right)e^{-i\omega_0T_0} \tag{11}$$

where A and $\overline{A}$ are complex amplitudes and their conjugates, respectively.
Equation (11) is substituted into (9) and secular terms are eliminated

$$D_0^2\,x_1+\omega_0{}^2x_1=\left(-2i\omega_0D_1A\right)e^{i\omega_0T_0}+cc-\alpha_1\left(A^2e^{2i\omega_0T_0}+2A\overline{A}+cc\right)$$

(12)

$$-2i\omega_0D_1A=0$$

Where $A(T_2)$ is represented the complex amplitudes in polar form

$$A=\frac{1}{2}a(T_2)e^{i\beta(T_2)}$$

Rearranged equation (12)

$$D_0^2\,x_1+\omega_0{}^2x_1=-\alpha_1(A^2e^{2i\omega_0T_0}+2A\overline{A}+cc) \tag{13}$$

Solution of the differential equation (13) is defined as follows

$$x_1=x_{1h}+x_{1p}$$

$x_{1h} = Be^{i\omega_0 T_0}$

The general solution is

$$x_1 = Be^{i\omega_0 T_0} + \frac{\alpha_1}{3{\omega_0}^2} A^2 e^{2i\omega_0 T_0} - \frac{2\alpha_1}{{\omega_0}^2} A\overline{A} + cc \tag{14}$$

Where $B = \frac{1}{2} be^{i\gamma}$

Applying the initial conditions yields

$b(0) = \frac{\alpha_1 c^2}{3{\omega_0}^2}$ and $\gamma(0) = 0 = \beta(0)$

Let us consider Eq. (10) and use formula

$D_0{}^{\alpha} e^{iT_0} = i^{\alpha} e^{iT_0}$ (see[12]).

$$D_0^2\, x_2 + {\omega_0}^2 x_2 = [\left(-2i\omega_0 D_2 A\right) - (i\omega_0)^{\alpha} A + \frac{4{\alpha_1}^2}{{\omega_0}^2} A^2 \overline{A} - \frac{2{\alpha_1}^2}{3{\omega_0}^2} \overline{A} A^2 - 3\alpha_2 A^2 \overline{A}] e^{i\omega_0 T_0} + cc$$

$$-(\frac{2{\alpha_1}^2}{3{\omega_0}^2} A^3 e^{3i\omega_0 T_0} + cc) - (\alpha_2 A^3 e^{3i\omega_0 T_0} + cc) \tag{15}$$

$$A = \frac{1}{2} a(T_2) e^{i\beta(T_2)} \qquad i^{\alpha} = \cos(\frac{\alpha\pi}{2}) + i\sin(\frac{\alpha\pi}{2}) = e^{\frac{\alpha\pi}{2} i} \tag{16}$$

Substituting relationships (16) in Eq. (15), when we separate the real and the imaginary part of the equation

$$(-\frac{da}{dT_2}\omega_0 - \frac{a}{2}{\omega_0}^{\alpha} \sin(\frac{\alpha\pi}{2}) = 0 \tag{17}$$

$$a\omega_0 \frac{d\beta}{dT_2} - \frac{a}{2}{\omega_0}^{\alpha} \cos(\frac{\alpha\pi}{2}) + \frac{10a^3{\alpha_1}^2}{24{\omega_0}^2} - \frac{3\alpha_2 a^3}{8} = 0 \tag{18}$$

We obtain the equations above equations .The solution of Eq. (17) is

$$a(T_2) = ce^{-(0.5{\omega_0}^{\alpha-1} \sin(\frac{\alpha\pi}{2}))T_2} \tag{19}$$

The solution of Eq. (18) is

$$\beta(T_2) = (\frac{1}{2}{\omega_0}^{\alpha-1} \cos(\frac{\alpha\pi}{2}))T_2 + [\ \frac{c^2(10{\alpha_1}^2 - 9\alpha_2{\omega_0}^2)}{24{\omega_0}^{\alpha+2} \sin(\frac{\alpha\pi}{2})}\ ] e^{-{\omega_0}^{\alpha-1}\sin(\frac{\alpha\pi}{2})T_2} + \beta_0 \tag{20}$$

Applying initial conditions yield

c=1 $\beta_0 = \dfrac{-10{\alpha_1}^2 + 9\alpha_2{\omega_0}^2}{24{\omega_0}^{\alpha+2} \sin(\frac{\alpha\pi}{2})}$

Therefore A is defined as follows

$$A = e^{-(0.5{\omega_0}^{\alpha-1} \sin(\frac{\alpha\pi}{2}))T_2} e^{i\left[(\frac{1}{2}{\omega_0}^{\alpha-1} \cos(\frac{\alpha\pi}{2}))T_2 + (\frac{c^2(10{\alpha_1}^2 - 9\alpha_2{\omega_0}^2)}{24{\omega_0}^{\alpha+2}\sin(\frac{\alpha\pi}{2})}) e^{-{\omega_0}^{\alpha-1}\sin(\frac{\alpha\pi}{2})T_2} + \beta\right]} \tag{21}$$

The solution at the first order is

$$x_0 = e^{-(0.5\omega_0^{\alpha-1}\sin(\frac{\alpha\pi}{2}))T_2}\cos(\omega_0 T_0 + \beta(T_2)) \tag{22}$$

The solution at order ε is

$$x_1 = \frac{1}{2}be^{i\gamma}e^{i\omega_0 T_2} + \frac{\alpha_1}{6\omega_0^2}e^{-(\omega_0^{\alpha-1}\sin(\frac{\alpha\pi}{2}))T_2}\cos(2\omega_0 T_0 + 2\beta(T_2)) - \frac{\alpha_1}{2\omega_0^2}e^{-(\omega_0^{\alpha-1}\sin(\frac{\alpha\pi}{2}))T_2} \tag{23}$$

The initial conditions at this order imply

$$b(0) = \frac{\alpha_1 c^2}{3\omega_0^2} \quad , \quad \gamma(0) = 0$$

$$x_1 = \frac{\alpha_1}{3\omega_0^2}e^{-(\omega_0^{\alpha-1}\sin(\frac{\alpha\pi}{2}))T_2}\cos(\omega_0 T_0 + \beta(T_2)) + \frac{\alpha_1}{6\omega_0^2}e^{-(\omega_0^{\alpha-1}\sin(\frac{\alpha\pi}{2}))T_2}\cos(2\omega_0 T_0 + 2\beta(T_2)) - \frac{\alpha_1}{2\omega_0^2}e^{-(\omega_0^{\alpha-1}\sin(\frac{\alpha\pi}{2}))T_2} \tag{24}$$

Final solution is obtained as

$$x = x_0 + \varepsilon x_1 + O(\varepsilon^2)$$

$$\beta(T_2) = (\frac{1}{2}\omega_0^{\alpha-1}\cos(\frac{\alpha\pi}{2}))T_2 + (\frac{10\alpha_1^2 - 9\alpha_2\omega_0^2}{24\omega_0^{\alpha+2}\sin(\frac{\alpha\pi}{2})})e^{-\omega_0^{\alpha-1}\sin(\frac{\alpha\pi}{2})T_2} + (\frac{-10\alpha_1^2 + 9\alpha_2\omega_0^2}{24\omega_0^{\alpha+2}\sin(\frac{\alpha\pi}{2})})$$

$$x = e^{-(0.5\omega_0^{\alpha-1}\sin(\frac{\alpha\pi}{2}))T_2}\cos(\omega_0 T_0 + \beta(T_2)) + \varepsilon\Big[\frac{\alpha_1}{3\omega_0^2}e^{-(\omega_0^{\alpha-1}\sin(\frac{\alpha\pi}{2}))T_2}\cos(\omega_0 T_0 + \beta(T_2)) + \frac{\alpha_1}{6\omega_0^2}e^{-(\omega_0^{\alpha-1}\sin(\frac{\alpha\pi}{2}))T_2}\cos(2\omega_0 T_0 + 2\beta(T_2)) - \frac{\alpha_1}{2\omega_0^2}e^{-(\omega_0^{\alpha-1}\sin(\frac{\alpha\pi}{2}))T_2}\Big] \tag{25}$$

Where $T_0 = t$, $T_1 = \varepsilon t$, $T_2 = \varepsilon^2 t$

3. COMPARISONS WITH THE NUMERICAL SOLUTIONS

We consider equation (1) with initial conditions (2). In view of the variational iteration method (VIM), we construct the following iteration formulation:

$$x_{n+1}(t) = x_n(t) + \int_0^t \sin(s-t)[x_n''(s) + \omega_0^2 x_n(s) + \varepsilon^2 D^\alpha x_n(s) + \varepsilon\alpha_1(x_n(s))^2 + \varepsilon^2\alpha_2(x_n(s))^3]ds$$

Where $\omega_0 = 1$, $\varepsilon = 0.1$, $\alpha_1 = \alpha_2 = 1$

If we begin with $x_o(t) = x(0) = 1$, we can obtain a convergent series:

$x_0(t) = 1$

$x_1(t) = -0.11 + 1.11\cos t$

$$x_2(t) = -0.11 + 1.279479162\cos(t) - 0.01754008125t\sin(t) - 0.0034190775\cos(t)^5 - 0.19607393\sin(t)^2 + 0.01004261597t^{\frac{3}{2}}\cos(t)^2 - 0.001331295068t^{\frac{5}{2}}\cos(t)^2 - 0.0005738637697t^{\frac{7}{2}}\cos(t)^2 - 0.11122331\cos(t)^2 + 0.009984713012t^{\frac{1}{2}}\cos(t)^2 + 0.009984713012t^{\frac{1}{2}}\sin(t)^2 + 0.01004261597t^{\frac{3}{2}}\sin(t)^2 - 0.04242531\cos(t)^4 - 0.3419077500\cos(t)^5 - 0.005738637697\sin(t)^2 t^{\frac{7}{2}} - 0.1960739300\sin(t)^2 - 0.01331295068\sin(t)^2 t^{\frac{5}{2}} - 0.1754008125\sin(t)^2\cos(t) - 0.4242531000\sin(t)^2\cos(t)^2 - 0.1887982885\sin(t)\mathrm{FresnelC}(0.7978845608t^{\frac{1}{2}}) + 0.1887982885\cos(t)\mathrm{FresnelS}(0.7978845608t^{\frac{1}{2}}) - 0.1251398197\cos(t)\mathrm{FresnelC}(0.7978845608t^{\frac{1}{2}}) - 0.1251398197\sin(t)\mathrm{FresnelS}(0.7978845608t^{\frac{1}{2}})$$

Figure1 , Figure2 and Figure3 show multiple time scales solution of the system for $\alpha = \frac{1}{2}$ $\varepsilon = 0.1$, $\alpha = \frac{1}{2}$ $\varepsilon = 0.5$ and $\alpha = \frac{1}{2}$ $\varepsilon = 1$ respectively. Figure4 shows comparison of approximate analytical and (VIM) numerical solutions for $\alpha = \frac{1}{2}$, $\varepsilon = 0.1$, $\omega_0 = 1$, $\alpha_1 = \alpha_2 = 1$. As seen from figure4 , MS method is more suitable than VIM method. As time increases, VIM method fails in our problem. Figure 5 shows comparison of approximate analytical and finite difference method numerical solutions for $\alpha = \frac{1}{2}$, $\varepsilon = 0.1$, $\omega_0 = 1$, $\alpha_1 = \alpha_2 = 1$.

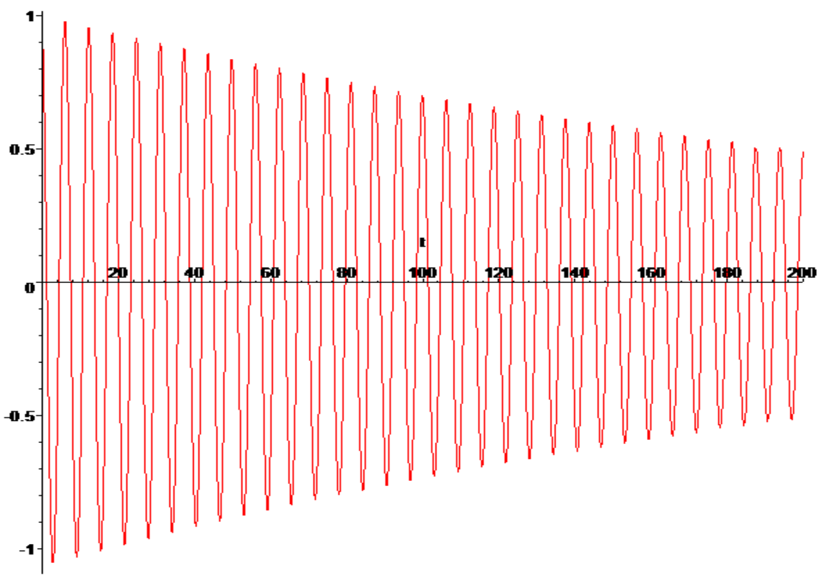

Figure1. Approximate analytical solutions (MS) for $\alpha = \frac{1}{2}$, $\varepsilon = 0.1$, $\omega_0 = 1$, $\alpha_1 = \alpha_2 = 1$

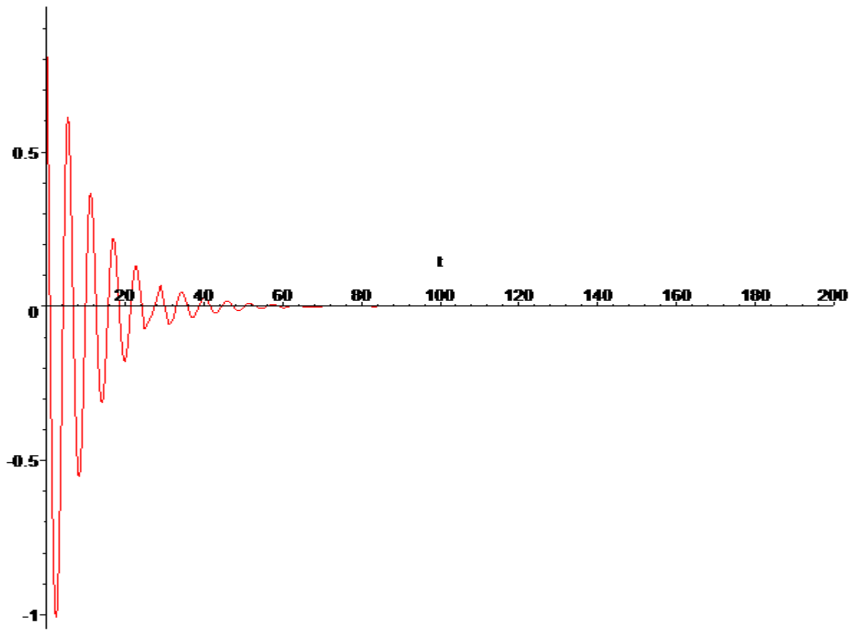

Figure2. Approximate analytical solutions (MS) for $\alpha = \frac{1}{2}$, $\varepsilon = 0.5$, $\omega_0 = 1$, $\alpha_1 = \alpha_2 = 1$

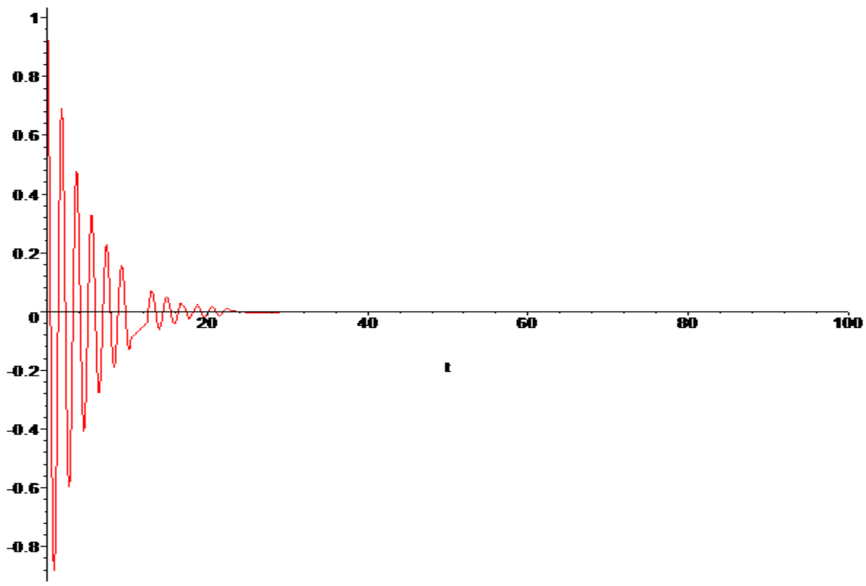

Figure3. Approximate analytical solutions (MS) for $\alpha = \frac{1}{2}$, $\varepsilon = 1$, $\omega_0 = \pi$, $\alpha_1 = \alpha_2 = 1$

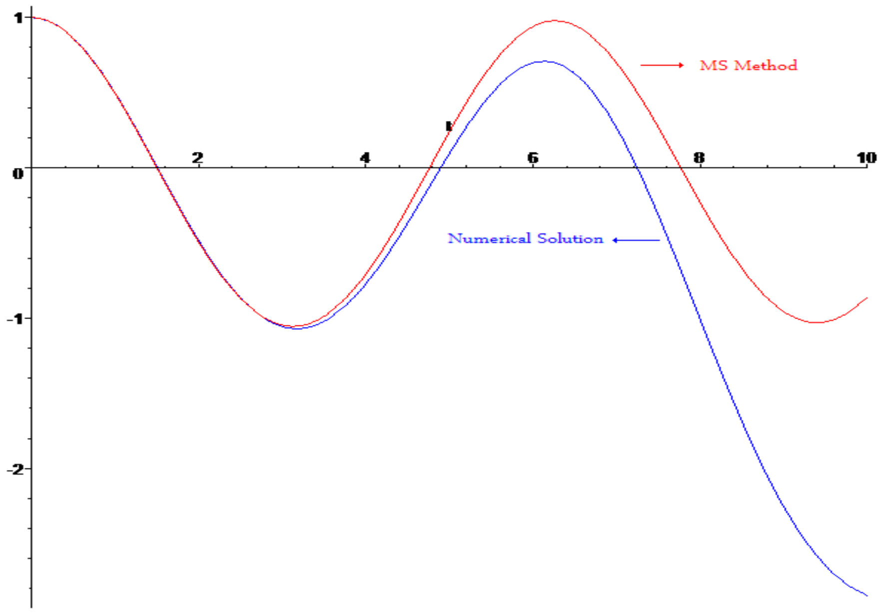

Figure4. Comparison of approximate analytical (MS) and VIM numerical solutions for $\alpha = \frac{1}{2}$, $\varepsilon = 0.1$, $\omega_0 = 1$, $\alpha_1 = \alpha_2 = 1$

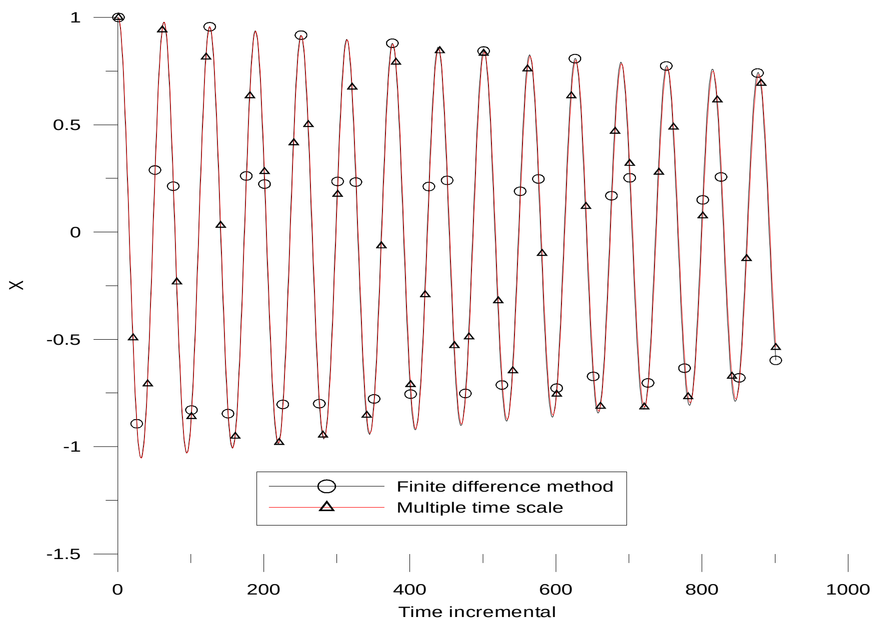

Figure5. Comparison of approximate analytical (MS) and finite difference method numerical solutions for $\alpha = \frac{1}{2}$, $\varepsilon = 0.1$, $\omega_0 = 1$, $\alpha_1 = \alpha_2 = 1$

4. CONCLUDING REMARKS

In this paper, multiple time scales method is successfully applied to find the solution of the equation with quadratic and cubic nonlinearities having fractional order derivative which corresponds to unharmonic nonlinear oscillator. The fractional derivative is considered in the Caputo sense which is more physical than other derivatives [13]. The solution of equation is made by using variational iteration method (VIM) and Multiple scale method (MS). The solutions of VIM, finite difference method and MS methods are compared. MS method produced solutions with good agreement with the numerical solutions. It is concluded that fractional derivative term effects as damping due to fraction . That is, the amplitudes decrease by increasing time.

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