

Integral sliding mode control for non-linear systems with mismatched uncertainty based on quadratic sliding mode

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Abstract: In this study, uncertainty and disturbance compensations are addressed for non-linear systems by an integral sliding mode (ISM) design. It is divided into two steps for implementation of the ISM control. First, an ISM switching surface in quadratic form is designed to construct the attractiveness and reachability. Then, the control law design ensures the stability of the sliding mode on the switching surface. The compensation design is applicable for both matched and mismatched perturbations. Preliminary results show that more relaxed design assumptions of this design, compared with other methods. In the end, the effectiveness of the proposed method is demonstrated with simulation results.

1 Introduction

Sliding mode control (SMC) has gained increasing attention from researchers during the past several decades. The main theory of SMC is established in several studies, for example [1–3]. As one of the main approaches for robust control designs for linear and non-linear systems, SMC has many advantages such as robustness, fast response and invariance to system uncertainties and compensation-ability to the external disturbance. At the same time, these are few shortcomings associated with it, for example, control chattering and reaching phase.

Integral sliding mode (ISM) approach was developed by Matthews and DeCarlo [4], Utkin and Shi [5]. It is a neat solution for the problem of reaching phase, by designing appropriate parameters of the ISM [6]. The sliding mode is reached from the initial time instantly and it compensates the perturbations from the beginning of the system running. Furthermore, it could combine another controller for the nominal system and the sliding mode controller [7–9]. Therefore ISM has been widely employed to solve different problems [10–15].

Recently, mismatched uncertainties and perturbations emerge as a challenge while the ISM approach is used for different kinds of systems [16–20]. To minimise the mismatched terms and possibly use less number of conservative nominal system controller are the focuses of this problem [19]. Castaos and Fridman [8] first presented ISM design and its corresponding controller, which consists of a continuous nominal control and a discontinuous control. The former dictates the performance of the nominal system (without uncertainty and disturbance); and the latter is responsible for the compensation of uncertainty and disturbance. In [8], the design of the ‘projection’ matrix of the ISM is the most important part with comprehensive discussion. The authors find that the B^+ with B is the control input matrix and B^+ its left inverse. Rubagotti *et al.* [9] applied this method with predictive control to the sample-data controlled systems. On the basis of their previous studies, Rubagotti *et al.* [21] considered more general non-linear affine systems with B and $B(x)$ instead of B . For both matched and mismatched perturbation, ISM is reachable and the mismatched perturbation is minimised. However, the value of the mismatched perturbation is minimised to its 2-norm from the results (Theorem 1 in [21]). Nevertheless, the ‘projection’ matrix of the ISM should be selected by the Theorem and the assumption for the perturbation is that its 2-norm is less than a constant scalar value. These lead to the conservative controller.

The objective of this paper is to find a more non-conservative way of the ISM approach for reaching phase and compensation for both matched and mismatched perturbations. First, a quadratic ISM surface is designed for a general non-linear affine system, which has some different structures comparing with the ISM in [21]. On the basis of the quadratic structure ISM, the assumption for the mismatched uncertainty could be relaxed and the mismatched uncertainty can be rejected, the validity of which is demonstrated in the results. A corresponding controller design is then proposed. The quadratic ISM is proved to be reachable in finite time and the system stability in the sliding mode is proved to be determined by the nominal system.

The main contribution of this paper is stated as follows. The mismatched uncertainty is rejected or compensated by the quadratic ISM, while it is minimised to be its 2-norm (In [21], the assumption for the mismatched perturbation its 2-norm is less than a constant scalar) and further suppressed by the robust nominal controller. By doing so, both the conservativeness of the nominal controller and the assumption for the uncertainty are reduced.

This paper is organised as follows. Section 2 introduces the formulation of the problem. Then Section 3 designs the quadratic ISM, whereas its corresponding controller design is presented in Section 4. Section 5 gives the proving of reaching condition and Section 6 considers the system stability in the sliding mode. A numerical example is presented in Section 7. Finally, conclusions are made in Section 8.

2 Problem formulation

Consider a class of systems as follows

$$\dot{x}(t) = f(x) + \Delta f(x) + [g(x) + \Delta g(x)]u \quad (1)$$

where $x = x(t) = [x_1 \ x_2 \ \dots \ x_n]^T$ is the state vector, $u \in \mathbf{R}$ is the single control input signal, $f(x) = [f_1(x) \ f_2(x) \ \dots \ f_n(x)]^T$ is the non-linear dynamic vector with $f_i(x): \mathbf{R}^n \rightarrow \mathbf{R}$ and $g(x) = [g_1(x) \ g_2(x) \ \dots \ g_n(x)]^T$ is the non-linear control gain vector with $g_i(x): \mathbf{R}^n \rightarrow \mathbf{R}$ that guarantees the controllability of the system. $\Delta f(x)$ and $\Delta g(x)$ are the corresponding unknown uncertainties of non-linear vector which can be regarded as satisfying the following assumptions.

Assumption 1: $\|\Delta f(x)\| \leq \xi_1 \|x\| + \xi_0$ with the scalars $\xi_0 > 0$, $\xi_1 > 0$.

Assumption 2: $\|\Delta g(x)\| \leq \zeta_1 \|x\| + \zeta_0$ with the scalars $\zeta_0 > 0$, $\zeta_1 > 0$.

Here $|\cdot|$ stands for Euclidean norm.

We first consider the SMC of the system (1) is

$$u = u_0 + u_1$$

where u_1 is the non-linear part of the sliding mode, u_0 is the equivalent control of u , which is responsible for the performance of the nominal system (without uncertainty and disturbance) and satisfies the following inequation

$$u_0 \leq \beta_0 + \beta_1 \|x\| \quad (2)$$

3 Quadratic ISM

Define the sliding mode of system (1) as

$$s = \frac{1}{2} [x^T(t)x(t) - x_0^T x_0] - \int_0^t \{x^T(t)[f(x) + g(x)u] - b(x)u_1\} dt \quad (3)$$

where x_0 is the initial value of the state vector, $b(x) \in R$ are matrices to be designed as

$$b(x) = \sigma + \zeta_0 \|x\| + \zeta_1 \|x\|^2 + \|x^T(t)g(x)\| \quad (4)$$

with an arbitrary scalar $\sigma > 0$. We therefore have

$$\|b(x)\|^{-1} \leq \delta \quad (5)$$

with the definition of scalar δ as

$$\delta = (\sigma + \zeta_0 \|x\| + \zeta_1 \|x\|^2)^{-1} \quad (6)$$

If the sliding mode surface $s=0$ can be reached, then the sliding mode equation becomes

$$\frac{1}{2} [x^T(t)x(t) - x_0^T x_0] - \int_0^t \{x^T(t)[f(x) + g(x)u] - b(x)u_1\} dt = 0 \quad (7)$$

By the system equation (1), it would be followed

$$\begin{aligned} \frac{1}{2} x^T(t)x(t) \Big|_0^t - \int_0^t \{x^T(t)[f(x) + g(x)u] - b(x)u_1\} dt &= 0 \\ \int_0^t \{x^T(t)[\dot{x}(t) - f(x) - g(x)u] + b(x)u_1\} dt &= 0 \\ \int_0^t \{x^T(t)[\Delta f(x) + \Delta g(x)u] + b(x)u_1\} dt &= 0 \end{aligned}$$

And then the sliding mode equation is

$$\int_0^t \{x^T(t)\Delta f(x) + x^T(t)\Delta g(x)u + b(x)u_1\} dt = 0 \quad (8)$$

4 SMC control law design

Design the following SMC law

$$u = u_0 - b^{-1}(x)[(\lambda_0 + \lambda_1 \|x\|)s + (\eta_0 + \eta_1 \|x\|) \operatorname{sgn}(s)] \quad (9)$$

where $\lambda_0, \lambda_1, \eta_0, \eta_1 \in R$ are scalars defined as follows

$$\lambda_0 \geq \frac{\varepsilon_1 (\zeta_1 \|x\|^2 + \zeta_0 \|x\| + \sigma)}{\sigma}, \quad \lambda_1 \geq 0 \quad (10)$$

$$\eta_0 \geq \frac{\varepsilon_2 (\zeta_1 \|x\|^2 + \zeta_0 \|x\| + \sigma)}{\sigma} \quad (11)$$

$$\eta_1 \geq (\sigma\delta)^{-1} [\xi_0 + \zeta_0\beta_0 + (\xi_1 + \zeta_0\beta_1 + \zeta_1\beta_0)\|x\| + \zeta_1\beta_1\|x\|^2] \quad (12)$$

with the two approaching parameters ε_1 and ε_2 are two arbitrary positive scalars.

If $s=0$ is reached and remains there, then the SMC control input u is equal to its equivalent control input, that is, $u = u_{\text{eq}}$ which can be written as $u = u_0$. The design of the equivalent control u_0 will be considered after the reachability of the sliding mode and the stability of the system in the sliding mode are achieved.

5 Reachability of the sliding mode

After designing the sliding mode and its corresponding SMC input, stability of the state variables of the system (1) is obtained as follows.

Theorem 1: For the uncertain non-linear system (1), the system state variable reaches to the sliding mode $s(x)=0$ in finite time and remains there, if the SMC is designed as (9)–(12).

Proof: Consider the time derivative of the sliding mode (3) along the system state (1), wherefore

$$\begin{aligned} \dot{s} &= x^T(t)\Delta f(x) + x^T(t)\Delta g(x)u + b(x)u_1 \\ &= x^T(t)\Delta f(x) + x^T(t)\Delta g(x)u_0 + (x^T(t)\Delta g(x) + b(x))u_1 \end{aligned}$$

Substitute the control law (9) and consider Assumptions 1 and 2 and (2)

$$\begin{aligned} s\dot{s} &= x^T(t)\Delta f(x)s + sx^T(t)\Delta g(x)u - (\lambda_0 + \lambda_1 \|x\|)s^2 - (\eta_0 + \eta_1 \|x\|)|s| \\ &\leq \|x\| \cdot \|\Delta f(x)\| \cdot |s| - sx^T(t)\Delta g(x)[\beta_0 + \beta_1 \|x\|] \\ &\quad - (1 + x^T(t)\Delta g(x)b^{-1}(x))[(\lambda_0 + \lambda_1 \|x\|)s^2 + (\eta_0 + \eta_1 \|x\|)|s|] \\ &\leq (\xi_0 + \xi_1 \|x\|)\|x\| \cdot |s| + \|x\|(\zeta_0 + \zeta_1 \|x\|)(\beta_0 + \beta_1 \|x\|)|s| \\ &\quad + (\lambda_0 + \lambda_1 \|x\|)(\zeta_0 + \zeta_1 \|x\|)\|x\|b^{-1}(x)s^2 \\ &\quad + (\eta_0 + \eta_1 \|x\|)(\zeta_0 + \zeta_1 \|x\|)\|x\|b^{-1}(x)|s| \\ &\quad - (\lambda_0 + \lambda_1 \|x\|)s^2 - (\eta_0 + \eta_1 \|x\|)|s| \\ &\leq \{(\xi_0 + \xi_1 \|x\|) + (\zeta_0 + \zeta_1 \|x\|)(\beta_0 + \beta_1 \|x\|)\}\|x\| \cdot |s| \\ &\quad - (\lambda_0 + \lambda_1 \|x\|)[1 - (\zeta_0 + \zeta_1 \|x\|)\|x\|b^{-1}(x)]s^2 \\ &\quad - (\eta_0 + \eta_1 \|x\|)[1 - (\zeta_0 + \zeta_1 \|x\|)\|x\|b^{-1}(x)]|s| \end{aligned}$$

Then the following would be true by (3)

$$\begin{aligned} s\dot{s} &\leq \{\xi_0 + \zeta_0\beta_0 + (\xi_1 + \zeta_0\beta_1 + \zeta_1\beta_0)\|x\| + \zeta_1\beta_1\|x\|^2\}\|x\| \cdot |s| \\ &\quad - \frac{\sigma + \|x^T g\|}{b(x)}(\lambda_0 + \lambda_1 \|x\|)s^2 - \frac{\sigma + \|x^T g\|}{b(x)}(\eta_0 + \eta_1 \|x\|)|s| \end{aligned}$$

According to the inequalities (10)–(12), one can obtain

$$s\dot{s} \leq -\varepsilon_1 s^2 - \varepsilon_2 |s|$$

Therefore the sliding mode $s = 0$ can be reached in finite time and the system state remains there. \square

6 System stability of the sliding mode

When the sliding mode $s = 0$ was reached and $s = 0$ stay at stable state, its derivative for time $\dot{s} = 0$ by (7), which implies that

$$x^T(t)\Delta f(x) + x^T(t)\Delta g(x)u + b(x)u_1 = 0 \quad (13)$$

Then the stability of the system in the sliding mode is elaborated as the following theorem.

Theorem 2: For the uncertain non-linear system (1) in the sliding mode $s = 0$, the stability of the system state is determined by

$$\dot{x}(t) = f(x) + g(x)u_0 \quad (14)$$

Proof: Select a Lyapunov function

$$V(x) = \frac{1}{2}x^T(t)x(t) \quad (15)$$

its total time derivative along the state (1) is

$$\begin{aligned} \dot{V}(x) &= x^T(t)\dot{x}(t) \\ &= x^T(t)[f(x) + g(x)u + \Delta f(x) + \Delta g(x)u] \end{aligned}$$

Using (13), one can obtain

$$\dot{V}(x) = x^T(t)[f(x) + g(x)u - b(x)u_1]$$

In the sliding mode of the system, $u_1 = 0$. Therefore the following can be obtained

$$\dot{V}(x) = x^T(t)f(x) + x^T(t)g(x)u_0 \quad (16)$$

It can be seen that a positive definite Lyapunov function (15) gives negative definite derivative with respect to time along the state (14), if the nominal system (14) is stabilised by u_0 . Therefore the stability of the system state is determined by (14).

By Theorem 2, u_0 can be designed according to any suitable design method, such as dynamic feedback linearisation method [9], robust H_∞ control method [21] etc., which determines the performance of the nominal system (14). \square

7 Application example

Consider a system formulated as

$$\begin{aligned} \dot{x}_1 &= x_2 + 0.6x_1^3 + 1.1 \sin 6\pi t \\ \dot{x}_2 &= 1.3x_2^2 + u + 0.7 \sin 4\pi t \end{aligned} \quad (17)$$

where in (1) $f(x) = [x_2 \ 0]^T$, $\Delta f(x) = [0.6x_1^3 \ 1.3x_2^2]^T$, $g(x) = [0 \ 1]^T$, $\Delta g(x) = 0$. The external disturbance $d(x, t) = [1.1 \sin 6\pi t \ 0.7 \sin 4\pi t]^T$. According to Assumptions 1 and 2, choose $\zeta_0 = \zeta_1 = 0$, $\xi_0 = 0$ and $\xi_1 = 1.69$.

First we designed the controller as (9), and $u_0 = -20x_1 - 5x_2$ with $\beta_0 = 0$ and $\beta_1 = 20$. On the basis of the inequation (12), β_0 and β_1 mainly affect the selection of η_1 . Fortunately, because $\zeta_0 = \zeta_1 = 0$, they do not affect η_1 any longer. Furthermore, we selected $\sigma = 1$, $\delta = 1$ by (6) and $\lambda_0 = 20$, $\lambda_1 = 0$ and $\eta_0 = 2$ by (10)–(12).

Here, we specially point out that $|\Delta f(x)| \leq 0.36|x_1|^3 + 1.69|x_2|^2$ and $|d(x, t)| \leq 2$, which should be considered in η_1 . By (12)

$$\eta_1 = 2 + \|x\| + 2\|x\|^2 + 0.4\|x\|^3$$

is defined. Accordingly $b(x) = 1 + x^T g(x) = 1 + x_2$, and the quadratic ISM can be represented as follows

$$s = 0.5[x^T(t)x(t) - x^T(0)x(0)] - \int_0^t \{x_1x_2 + x_2u - (1 + x_2)u_1\} dt$$

Set the initial value of the state $x = [-0.6 \ 1.2]^T$ and $d(x, t) = 0$ (i.e. the external disturbance is not considered). Another important thing was to improve the sign function in discontinuous controller. In (9), we replaced the sign function with saturation function with

$$\text{sat}\left(\frac{s}{\varphi}\right) = \begin{cases} \text{sgn}(s), & \text{if } |s| \geq \varphi \\ s/\varphi, & \text{if } 0 \leq |s| < \varphi \end{cases}$$

where φ is the boundary layer, which is selected as 0.001 in first.

The configuration of the system is now complete. The simulation results are shown in Figs. 1–3. The quadratic ISM value is shown in

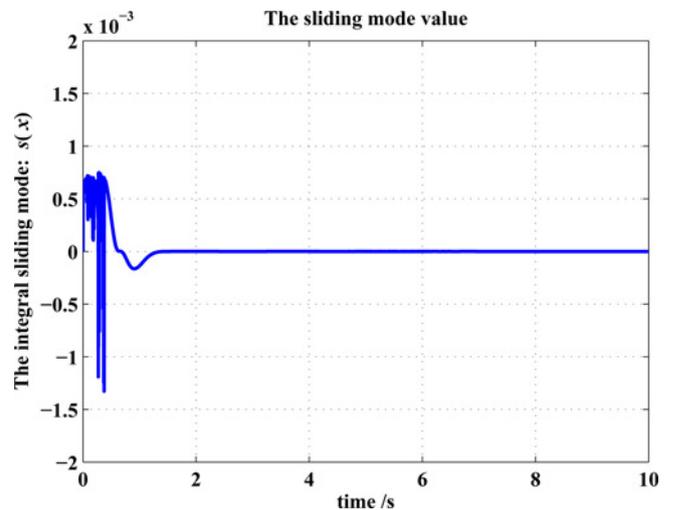


Fig. 1 Quadratic ISM curve when $d(x, t) = 0$

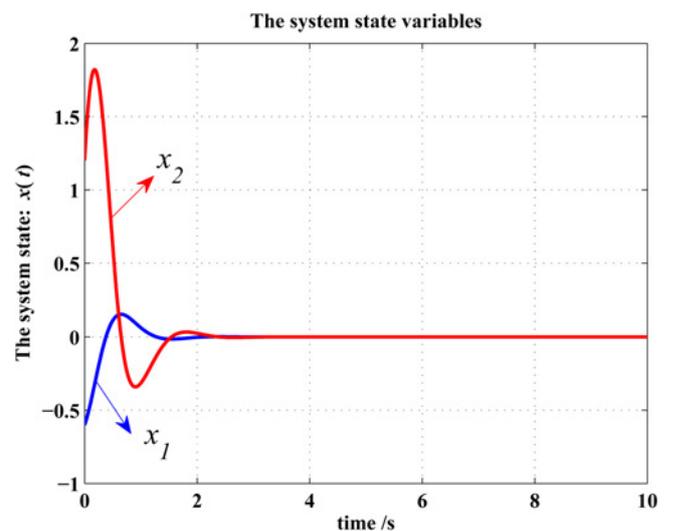


Fig. 2 System state curves when $d(x, t) = 0$

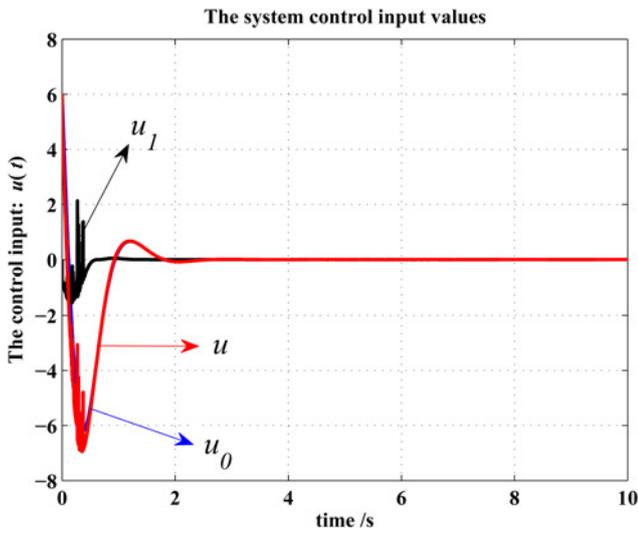


Fig. 3 Control signals when $d(x, t) = 0$

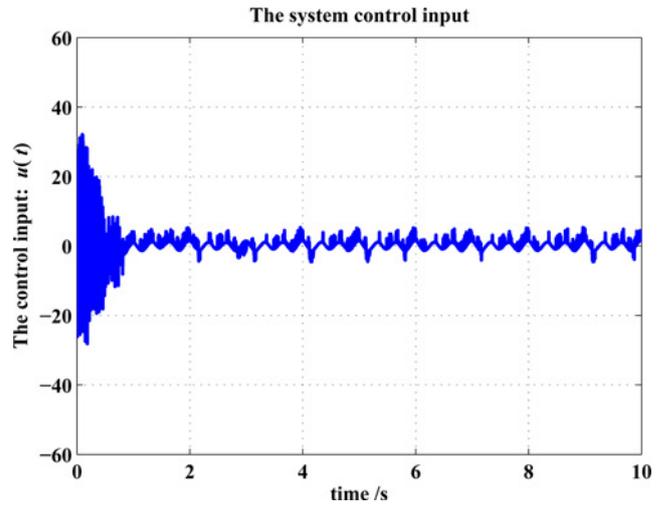


Fig. 6 System control input u when $d(x, t) \neq 0$

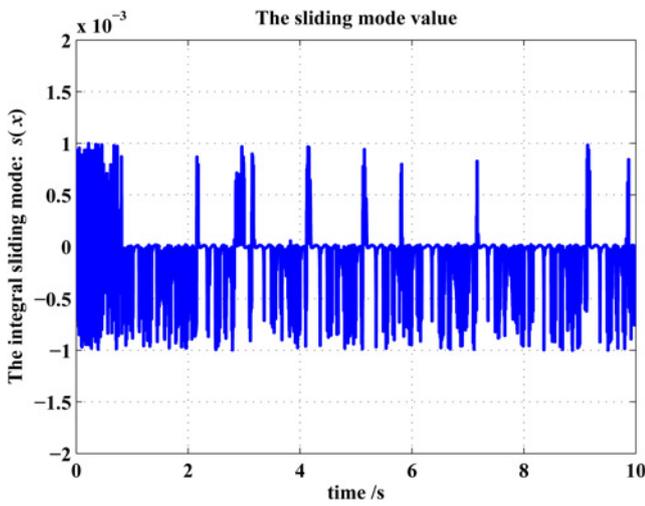


Fig. 4 Quadratic ISM curve when $d(x, t) \neq 0$

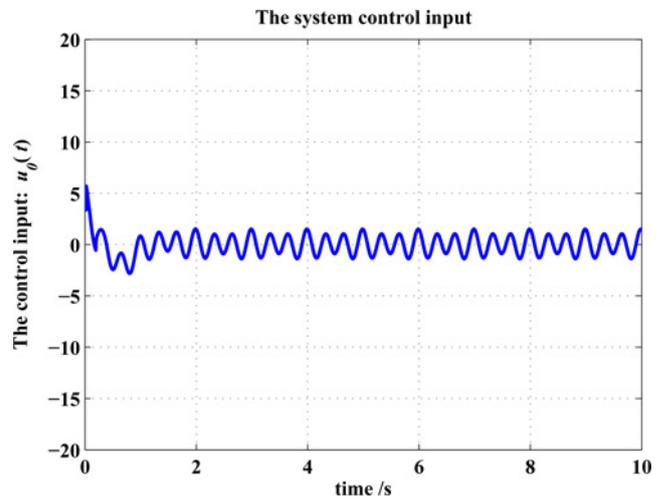


Fig. 7 System control input u_0 when $d(x, t) \neq 0$

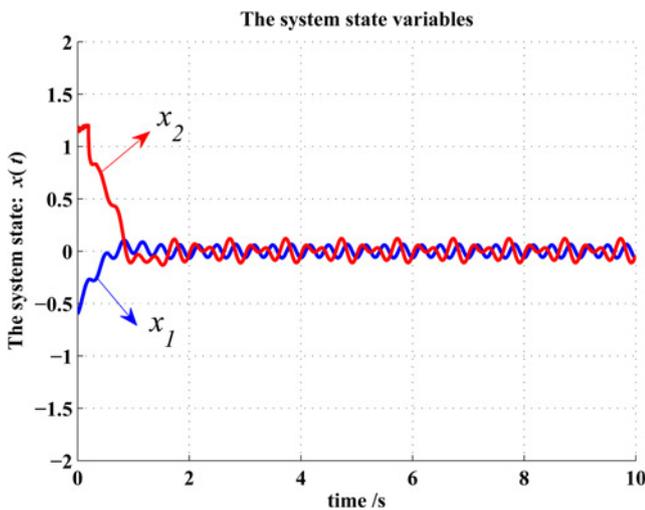


Fig. 5 System state curves when $d(x, t) \neq 0$

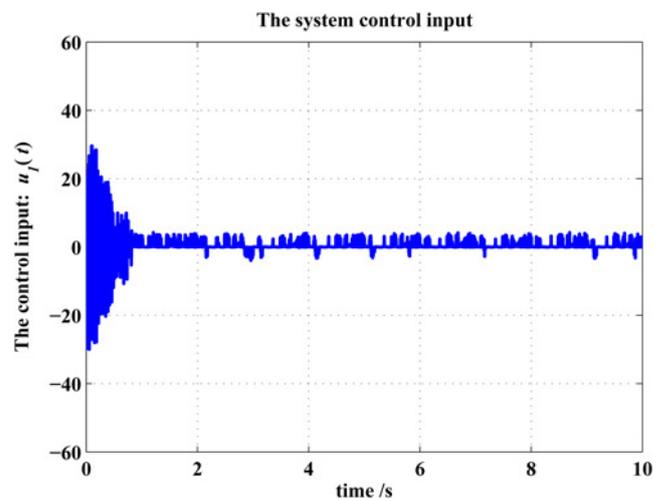


Fig. 8 System control input u_1 when $d(x, t) \neq 0$

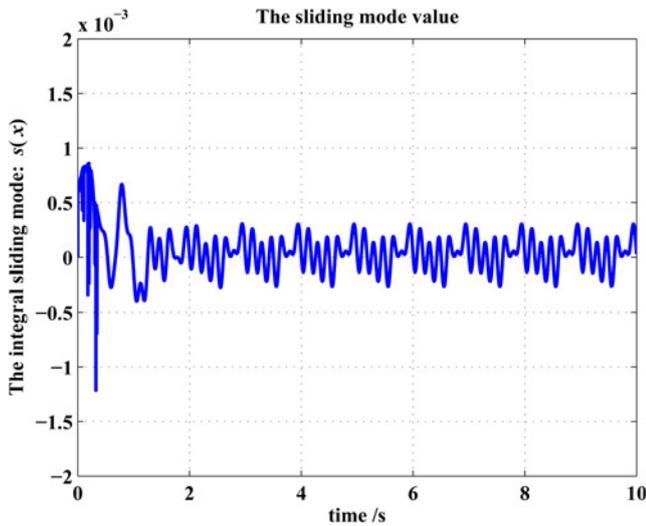


Fig. 9 Quadratic ISM curve when $\phi = 0.01$ and $d(x, t) \neq 0$

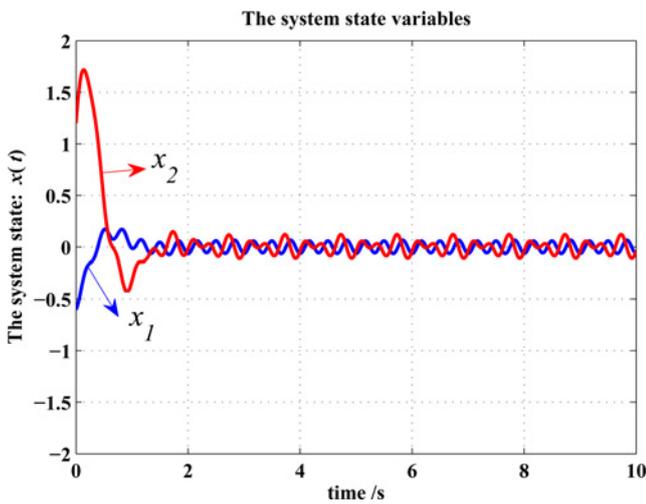


Fig. 10 System state curves when $\phi = 0.01$ and $d(x, t) \neq 0$

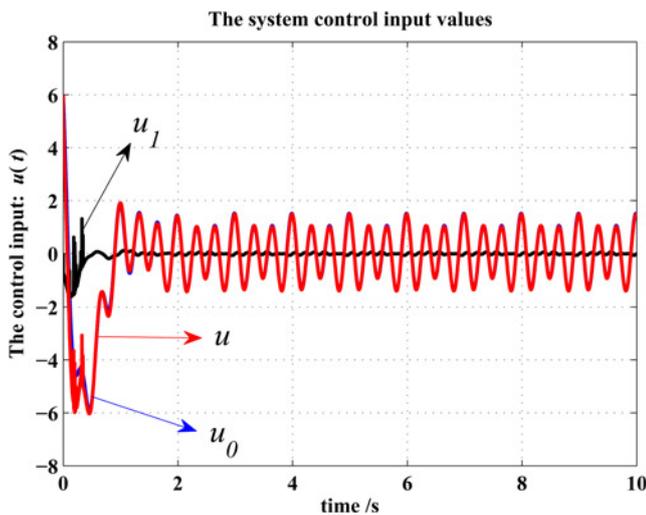


Fig. 11 System control signals when $\phi = 0.01$ and $d(x, t) \neq 0$

Fig. 1, which is always of small value 0.001, and finally asymptotically equals to zero. The system state variables are shown in Fig. 2, and the system control input signals are shown in Fig. 3. All these three figures show that the system was robustly stabilised better by the quadratic ISM and the controller, although there was the strong uncertainty $\Delta f(x)$.

$d(x, t) = [1.1 \sin 6\pi t \ 0.7 \sin 4\pi t]^T$ was then considered. The minor change of η_1 in (17) was considered as $\eta_1 = 2 + |x| + 2|x|^2 + 0.4|x|^3$. In Fig. 4, the quadratic ISM value is within 0.001, but it has chattering. The system state variables are shown in Fig. 5, which have weak fluctuations in steady state. This is because the sliding mode has the chattering and boundary layer. The control input signals are shown in Figs. 6–8. These figures illustrate that the system state is robustly stabilised by the quadratic ISM and the controller, although the external disturbance exists.

Figs. 6–8 show the control signals have heavy chattering. We can impair chattering simply by increasing the boundary layer $\phi = 0.01$. Fig. 9 shows the sliding mode value is also within 0.001, but its chattering is weakening. Fig. 10 clearly shows that the system state is still robustly stabilised better. Additionally, the chattering of control input signals is dispelled well, as indicated by Fig. 11.

8 Conclusions

The design of ISM control is comprised of a quadratic-type sliding surface and its corresponding controller. The quadratic ISM can reject mismatched uncertainty well, and reduce the conservativeness of the nominal controller. Furthermore, the constraints for the uncertainty and disturbance could be relaxed. In the proposed approach, the ‘projection’ matrix is not required to minimise the mismatched perturbation as the quadratic ISM has good rejection and compensation performance. A numerical example with lots of uncertainty is presented to demonstrate the approach in this paper.

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