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A GENERALIZED CROSS VALIDATION METHOD FOR THE INVERSE PROBLEM OF 3-D MAXWELL'S EQUATION

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Abstract- The inverse problem of estimation of the electrical conductivity in the Maxwell's equation is considered, which is reformulated as a nonlinear equation. The Generalized Cross Validation is used to estimate the global regularization parameter and the damped Gauss-Newton is applied to impose local regularization. The damped Gauss-Newton method requires no calculation of the Hessian matrix which is expensive for traditional Newton method. GCV method decreases the computational expense and overcomes the influence of nonlinearity and ill-posedness. The results of numerical simulation testify that this method is efficient.
Key Words- Maxwell's Equation, Nonlinear, Ill-posed, Inversion, Damped Gauss-Newton, GCV

1. INTRODUCTION

In this paper, we consider the estimation of the electrical conductivity in Maxwell's equations. This problem can be reformulated as a nonlinear equation, which is ill-posed. Our work is devoted to a new numerical method for the inverse problem. There are two main computational difficulties in the solution of the inverse problem. Firstly, the regularization parameter is unknown, and secondly, a nonlinear functional has to be minimized. These impose special difficulties if the problem is large.

There have been many new methods applied to this domain [1,2]. The first pioneering solution of the fully 3-D Maxwell's equation inverse problem was presented by Eaton[3] more than 15 years ago. Yet in spite of this, until recently, the trial-and-error forward modeling was almost the only available tool to interpret the fully 3-D EM dataset. Today the situation has slightly been improved, and the methods of unconstrained nonlinear optimization [4] are gaining popularity to address the problem. In spite of these successes and the relatively high level of modern computing possibilities, the proper numerical solution of the 3-D inverse problem still remains a very difficult and computationally intense task for the following reasons. (1)It requires a fast, accurate and reliable forward 3-D problem solution. Approximate forward solutions [5,6,7,8,9] may deliver a rapid solution of the inverse problem (especially, for models with low conductivity contrasts), but the general reliability and accuracy of this solution are still open to question.(2)The problem is ill-posed in nature with nonlinear and extremely sensitive solutions. This means that due to the fact that data are limited and contaminated by noise there are many models that can equally fit the data within a given tolerance threshold. (3) We need to solve a new nonlinear problem for every new regularization parameter and therefore the algorithm is computationally expensive. The algorithm is substantially cheaper if we estimate β to be close to the optimal β^*.

Therefore, the shortages of the above mentioned motive us to design a highly efficient, numerically stable and globally convergent algorithm.

As mentioned above, the goal of this paper is to apply the damped Gauss-Newton method coupled with GCV (Generalized Cross-Validation) method to choose an adaptive regularization parameter. The paper is built as follows. We start with a review of the damped Gauss–Newton method as applied to a minimization problem with a constant regularization parameter. We then present the major ideas of our algorithm. Our algorithm uses the Generalized Cross Validation and we review it in section 3. Section 4 gives an example of inverting inductivity from Maxwell's equations.

2. DAMPED GAUSS-NEWTON METHOD

In this section we present damped Gauss-Newton method. Our forward problem time-dependent Maxwell equations can be written as

$$\nabla \times \mathrm{E} + \mu \frac{\partial \mathrm{H}}{\partial t} = 0 \tag{1}$$

$$\nabla \times \mathrm{H} - \sigma \mathrm{E} - \varepsilon \frac{\partial \mathrm{E}}{\partial t} = s_r(t) \tag{2}$$

over a domain $\Omega \times [0,T]$,where E and H are the electric and magnetic fields, σ is the conductivity, ε is the permittivity, μ is the permeability and s_r is a source. The equations are given with boundary and initial conditions:

$$\begin{cases} \mathrm{n} \times \mathrm{H} = 0 \\ \mathrm{H}(\mathrm{x},0) = 0 \\ \mathrm{E}(\mathrm{x},0) = 0 \end{cases} \tag{3}$$

By [10], we know an inverse electromagnetic problem can be formulated as the following nonlinear operator equation:

$$F(\sigma) = d^{\delta}, \tag{4}$$

where F is a nonlinear operator, d^{δ} is the observed data. The goal of this paper is to reconstruct the inductivity from the observed data with noise. The optimal distribution of conductivity minimized the functional

$$\phi = \left\| F(\sigma) - d^{\delta} \right\|^2 + \beta \left\| W\left(\sigma - \sigma_{ref}\right) \right\|^2, \tag{5}$$

where $\|\cdot\|$ denotes 2-norm, W is positive weight function, σ_{ref} is reference model, β is regularization parameter. If β is fixed, we must solve the unconstrained optimization problem (5). This leads to the Euler-Lagrange system

$$\frac{\partial \phi}{\partial \sigma} = g(\sigma) = \beta W^T W\left(\sigma - \sigma_{ref}\right) + J(\sigma)^T \left(F(\sigma) - d^{\delta}\right) = 0 \tag{6}$$

$g(\sigma)$ is the gradient of (5), $J(\sigma)$ is sensitive matrix

$$J(\sigma) = \frac{\partial F}{\partial \sigma}. \tag{7}$$

In order to avoid the difficulty of calculating Hessian matrix, which is second derivative of F, we linearize the operator F

$$F\left(\sigma+\delta\sigma\right)=F\left(\sigma\right)+J\left(\sigma\right)\delta\sigma+R\left(\sigma,\delta\sigma\right) \tag{8}$$

where $R\left(\sigma,\delta\sigma\right)=o\left(\delta\sigma^2\right)$. From (6) and (8), we can obtain Gauss-Newton equation

$$\beta W^T W\left(\sigma+\delta\sigma-\sigma_{ref}\right)+J\left(\sigma\right)^T\left(F\left(\sigma\right)+J\left(\sigma\right)\delta\sigma-d^\delta\right)=0 \tag{9}$$

The minimization problem is solved by the iterative format as follows, at the kth step

$$\left(J\left(\sigma_k\right)^T J\left(\sigma_k\right)+\beta W^T W\right)\delta\sigma=J\left(\sigma_k\right)^T\left(d^\delta-F\left(\sigma_k\right)\right)-\beta W^T W\left(\sigma_k-\sigma_{ref}\right) \tag{10}$$

Denoted $\sigma_{k+1}=\sigma_k+\delta\sigma$, (10) can be written as

$$\left(J\left(\sigma_k\right)^T J\left(\sigma_k\right)+\beta W^T W\right)\sigma_{k+1}=J\left(\sigma_k\right)^T\left(d^\delta-F\left(\sigma_k\right)+J\left(\sigma_k\right)\sigma_k\right)-\beta W^T W\sigma_{ref} \tag{11}$$

The iterative solution at kth step is the minimum norm solution of Tikhonov functional

$$\phi=\left\|F\left(\sigma_k\right)+J\left(\sigma_k\right)\left(\sigma_{k+1}-\sigma_k\right)-d^\delta\right\|^2+\beta\left\|W\left(\sigma_{k+1}-\sigma_{ref}\right)\right\|^2. \tag{12}$$

If the initial value is close to true inductivity, the algorithm is fast convergent. If the initial value is undesirable, the search step must become bigger. Consequently, the $R\left(\sigma,\delta\sigma\right)$ what we removed in the linear process became larger, this will affect accuracy. In order to keep the Gauss-Newton iteration in the descent direction and the iterative step is small enough, we adopt armijo method to amend the iterative step. If the iterative direction is no longer descent, we amend the step $\delta\sigma$ to $\omega\delta\sigma$, where $0.1<\omega<0.5$. As discussed above, with small enough step, the nonlinear functional is always in the descent direction

$$\phi\left(\beta,\sigma_{k+1}\right)<\phi\left(\beta,\sigma_k\right) \tag{13}$$

What we discussed above is in the case of regularization fixed. Next we introduce how to adaptively select regularization parameter, stopping criteria for iteration and general sketch of algorithm.

3. GCV FUNCTION AND INVERSION ALGORITHM

Generalized Cross Validation (GCV) method is a deformation of Unbiased Predictive Risk Estimator (UPRE) method [11]. Even if we don't know the variance of noise, this method can give the adaptive regularization parameter and distinguish the noise and nonlinearity. We denote $\sigma=\sigma_{k+1}-\sigma_{ref}$, $r=r\left(\sigma_k\right)-J\sigma_{ref}$, $J=J\left(\sigma_k\right)$,

$$\phi=\left\|J\left(\sigma\right)-r\right\|^2+\beta\left\|W\sigma\right\|^2 . \tag{14}$$

And let $r\left(\beta\right)=J\sigma\left(\beta\right)$, $\sigma\left(\beta\right)$ is the solution of (number), the GCV function

$$\mathrm{GCV}\left(\beta\right)=\frac{\left\|r\left(\beta\right)-r\right\|}{\mathrm{trace}\left(I-C\left(\beta\right)\right)^2} \tag{15}$$

where $C\left(\beta\right)=J\left(J^T J+\beta W^T W\right)^{-1}J^T$, I is unit matrix. With regularization parameter β which minimizes the GCV function, we denote the regularization parameter β_k of the kth iteration. Next we introduce some necessary condition and stopping criteria of damped Gauss-Newton method with adaptive regularization parameter. If we got β_k and

σ_k at the k[th] iteration, we can obtain σ_{k+1} by(number). In the case of β fixed, $\phi(\beta,\sigma_{k+1}) < \phi(\beta,\sigma_k)$ at every iterative step. But with the parameter changing, we demand

$$\phi(\beta_{k+1},\sigma_{k+1}) < \phi(\beta_{k+1},\sigma_k), \tag{16}$$

namely

$$\left\|F(\sigma_{k+1}) - d^\delta\right\|^2 + \beta_{k+1}\left\|W(\sigma_{k+1} - \sigma_{ref})\right\|^2 < \left\|F(\sigma_k) - d^\delta\right\|^2 + \beta_{k+1}\left\|W(\sigma_k - \sigma_{ref})\right\|^2 \tag{17}$$

We select

$$\frac{\|\sigma_{k+1} - \sigma_k\|}{\max(\|\sigma_{k+1}\|, \|\sigma_k\|)} < \delta \tag{18}$$

as stopping criteria.

The damped Gauss-Newton algorithm on basis of GCV function:

Select initial value σ_0 and reference model σ_{ref}, as $k = 1, 2\cdots$

i) Calculate $J(\sigma_k)$ and $r(\sigma_k)$.

ii) Calculate σ_{k+1} by (11) and evaluate β_k by GCV function.

iii) Use the regularization parameter to calculate $\delta\sigma = \sigma_{k+1} - \sigma_k$, $\phi(\beta_{k+1}, m_{k+1})$, $\phi(\beta_{k+1}, m_k)$.

iv) If $\phi(\beta_{k+1}, m_{k+1}) \le \phi(\beta_{k+1}, m_k)$, let $m_k = m_{k+1}$ and go to i).

v) If $\phi(\beta_{k+1}, m_{k+1}) > \phi(\beta_{k+1}, m_k)$, let $m_{k+1} = m_k + \omega\delta m$ and go to iii).

vi) If stopping criteria $\dfrac{\|m_{k+1} - m_k\|}{\max(\|m_{k+1}\|, \|m_k\|)} < \delta$ is satisfied, stop iteration.

4. NUMERICAL SINMULATIONS

In this section, we compare the efficiency of traditional damped Gauss-Newton method and GCV method which is based on the damped Gauss-Newton method in two experiments.The physical domain is Ω. For simplicity the inverse problem is considered on a uniform, $34\times35\times16$ grid. The source function is $f(t) = t^2 e^{-\alpha t}\sin(\omega_0 t)$, where $\alpha = \dfrac{\omega_0}{\sqrt{3}}$, $\omega_0 = 2\pi f_0$, $f_0 = 100\text{MHz}$ is the centre frequency.

4.1. Experiment 1

In this section we suppose there is a cavity in the homogeneous background, where permittivity $\varepsilon = 56.64\times10^{-12}\,\text{F/m}$, permeability $\mu = 0$. The relative conductivity of cavity and background are -3.2775 and -5.2983 respectively. Both methods select the same relative initial value $\sigma = -1.5$. The inversion took approximately 20 minutes by GCV method against to 30 minutes by traditional damped Gauss-Newton method run on a 3.0 Ghz Pentium Dual-Core with 2Gb of RAM. The relative error

$\|\sigma_{inv} - \sigma_{true}\| / \|\sigma_{true}\|$ of a and b in Fig.1 is 35.76%, a and c is 36.75%, where σ_{inv} is iterative solution, σ_{true} is the true model.

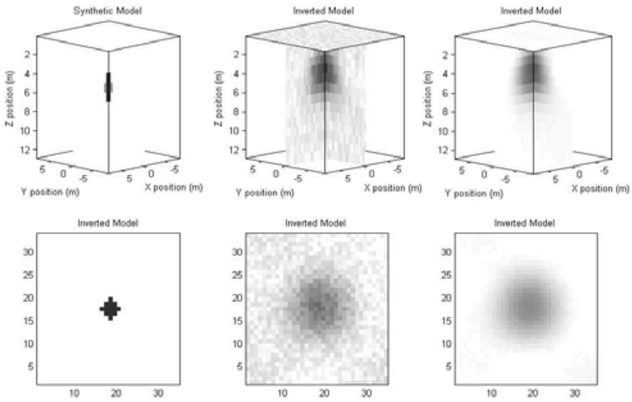

Fig.1 a)True model; b)Results of damped Gauss-Newton method; c)Results of GCV method; d)Cross-section of a at x=0; e) Cross-section of b at x=0; f) Cross-section of c at x=0

4.2. Experiment 2

This model has two cavities in the homogeneous background, the relative conductivity of cavity and background are -3.2775 and -5.2983 respectively, other parameters are same as the experiment 1. The observed data were generated using the forward modeling algorithm (Finite Difference in Time Domain), these data were corrupted with 5% uniformly distributed random noise. The relative error of a and b in Fig.2 is 157.76%, a and c is 45.75%.

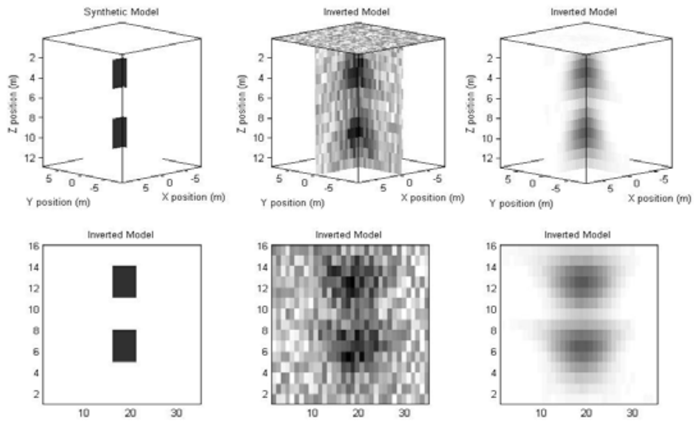

Fig.2 a)True model; b)Results of damped Gauss-Newton method; c)Results of GCV method; d)Cross-section of a at Z=8; e) Cross-section of b at Z=8; f) Cross-section of c at Z=8

5. CONCLUDING REMARKS

This paper has addressed the GCV method for solving inverse Maxwell's equations. GCV method coupled with traditional damped Gauss-Newton methods have been used in the experiments. Results show that this new method is more available than damped Gauss-Newton method. It is a widely and fast convergent method even if the initial guess value is far away from the true coefficient.

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