

## SIMILARITY SOLUTIONS FOR BOUNDARY LAYER EQUATIONS OF A POWEL-EYRING FLUID

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**Abstract-** Boundary layer equations are derived for the first time for the Powel-Eyring fluid model, a non-Newtonian model proposed for pseudoplastic behavior. Using a scaling symmetry of the equations, partial differential system is transferred to an ordinary differential system. Resulting equations are numerically solved using a finite difference algorithm. Effects of non-Newtonian parameters on the solutions are discussed.

**Key Words-** Non-Newtonian Fluid, Powel-Eyring Fluid, Boundary Layer Theory, Similarity Transformations

### 1. INTRODUCTION

Many fluids like biological fluids, shampoo, yoghurt, tomato sauce, paints, lubricants, polymeric fluids do not obey the linear stress-velocity gradient relationship in Newtonian fluid theory. Several non-Newtonian fluid models were therefore proposed to explain the complex behavior. Usually, the stress constitutive relations of such models inherit complexities which lead to highly nonlinear equations of motion with many terms. To simplify the extremely complex equations, one alternative is to use boundary layer theory which is known to effectively reduce the complexity of Navier-Stokes equations and reduce drastically the computational time. Since there are many non-Newtonian models and new models are being proposed continuously, boundary layer theory for each proposed model also appear in the literature. It is beyond the scope of this work to review vast literature on the boundary layers of non-Newtonian fluids. A limited work on the topic can be referred as examples [1-23].

In this work, boundary layer equations are developed for the first time for the Powel-Eyring fluid. One of the interesting features of Powell-Eyring fluids is that, the stress constitutive relation of such fluids can be derived using the kinetic theory of liquids. This model correctly reduces to Newtonian behavior for low and high shear rates. Scaling symmetry is well known to exist for boundary layer type problems leading to useful solutions and for this reason, the specific form of the scaling symmetry which leaves the equations invariant is determined. Using the symmetry, the partial differential system is transformed into an ordinary differential system. See [13-15, 23-

25] for applications of scaling symmetries to various problems. Resulting ordinary differential system is numerically solved by a finite difference algorithm. Effect of non-Newtonian parameters on the velocity profiles are shown in the graphs.

Some of the recent work on Powel-Eyring fluids is as follows: Yürüsoy [26] studied a slider bearing lubricated with a Powel-Eyring fluid. Islam *et al.* [27] reconsidered the same problem and found solutions with homotopy perturbation method which verified the results of [26] obtained by perturbation method. Ai and Vafai [28] investigated Stokes second problem for eight different non-Newtonian fluid models including Powel-Eyring model. The boundary layer equations given in this work is new and do not exist in the previous literature.

## 2. BOUNDARY LAYER EQUATIONS

The Cauchy stress tensor for Powel-Eyring fluid is

$$\mathbf{T} = -p\mathbf{I} + \mathbf{S} \quad (1)$$

where

$$\mathbf{S} = \mu(\nabla\mathbf{V}) + \frac{1}{\beta} \sinh^{-1}\left(\frac{1}{c} \nabla\mathbf{V}\right) \quad (2)$$

where  $\mathbf{V}$  is the velocity vector. Steady-state, two dimensional, incompressible equations of motion including mass conservation can be written as

$$\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0 \quad (3)$$

$$\rho\left(u^* \frac{\partial u^*}{\partial x^*} + v^* \frac{\partial u^*}{\partial y^*}\right) = -\frac{\partial p^*}{\partial x^*} + \frac{\partial S_{xx}}{\partial x^*} + \frac{\partial S_{xy}}{\partial y^*} \quad (4)$$

$$\rho\left(u^* \frac{\partial v^*}{\partial x^*} + v^* \frac{\partial v^*}{\partial y^*}\right) = -\frac{\partial p^*}{\partial y^*} + \frac{\partial S_{yx}}{\partial x^*} + \frac{\partial S_{yy}}{\partial y^*} \quad (5)$$

where  $x^*$  is the spatial coordinate along the surface,  $y^*$  is vertical to it,  $u^*$  and  $v^*$  are the velocity components in the  $x^*$  and  $y^*$  coordinates.

To calculate the stress components in (2), the function is approximated first

$$\sinh^{-1}\left(\frac{1}{c} \nabla\mathbf{V}\right) \cong \frac{1}{c}(\nabla\mathbf{V}) - \frac{1}{6}\left(\frac{1}{c} \nabla\mathbf{V}\right)^3 \quad (6)$$

and then substituted into (2)

$$\mathbf{S} \cong \left(\mu + \frac{1}{\beta c}\right)(\nabla\mathbf{V}) - \frac{1}{6\beta c^3}(\nabla\mathbf{V})^3 \quad (7)$$

The shear stress components are inserted into the equations of motion and the usual boundary layer assumptions are made i.e.  $x^* \sim O(1)$ ,  $y^* \sim O(\delta)$ ,  $u^* \sim O(1)$ ,  $v^* \sim O(\delta)$ . The highest order terms are retained and the momentum equations become

$$\rho\left(u^* \frac{\partial u^*}{\partial x^*} + v^* \frac{\partial u^*}{\partial y^*}\right) = -\frac{\partial p^*}{\partial x^*} + \left(\mu + \frac{1}{\beta c}\right) \frac{\partial^2 u^*}{\partial y^{*2}} - \frac{1}{2\beta c^3} \left(\frac{\partial u^*}{\partial y^*}\right)^2 \frac{\partial^2 u^*}{\partial y^{*2}} \quad (8)$$

$$0 = \frac{\partial p^*}{\partial y^*} \quad (9)$$

from which dependence of pressure on  $y^*$  is eliminated. In deriving the equations  $\mu \sim O(\delta^2)$ ,  $1/\beta \sim O(\delta)$  and  $1/c \sim O(\delta)$  assumptions are made. Dimensionless variables and parameters are defined as follows

$$x = \frac{x^*}{L}, \quad y = \frac{y^*}{L}, \quad u = \frac{u^*}{V}, \quad v = \frac{v^*}{V}, \quad p = \frac{p^*}{\rho V^2}$$

$$\varepsilon_1 = \frac{\mu}{\rho V L}, \quad \varepsilon_2 = \frac{1}{\rho \beta V^2}, \quad \varepsilon_3 = \frac{V}{c L} \quad (10)$$

where  $L$  is a characteristic length and  $V$  is a reference velocity. Expressing the pressure in terms of outer velocity, the final dimensionless boundary layer equations become

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (11)$$

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{dp}{dx} + (\varepsilon_1 + \varepsilon_2 \varepsilon_3) \frac{\partial^2 u}{\partial y^2} - \frac{1}{2} \varepsilon_2 \varepsilon_3^2 \left( \frac{\partial u}{\partial y} \right)^2 \frac{\partial^2 u}{\partial y^2} \quad (12)$$

The classical boundary conditions for the problem are

$$u(x, 0) = 0, \quad v(x, 0) = 0, \quad u(x, \infty) = U(x) \quad (13)$$

For  $\varepsilon_2 = 0$  or  $\varepsilon_3 = 0$ , the equations reduce to those of Newtonian fluid.

### 3. SIMILARITY TRANSFORMATIONS

Scaling symmetry is one of the most common symmetries producing useful solutions in boundary layer type equations [13-15, 23]. All variables are rescaled as follows

$$\bar{x} = \lambda^a x, \quad \bar{y} = \lambda^b y, \quad \bar{u} = \lambda^c u, \quad \bar{v} = \lambda^d v, \quad \bar{U} = \lambda^e U \quad (14)$$

Substituting (14) into (11) and (12) and requiring that the equations be invariant under the transformation yields

$$b + c - a - d = 0, \quad 2c - 2e = 0, \quad 2b + c - a = 0, \quad 4b - c - a = 0 \quad (15)$$

All parameters are solved in terms of parameter  $b$

$$a = 3b, \quad c = b, \quad d = -b, \quad e = b \quad (16)$$

The associated equations for this transformation which define similarity variables are

$$\frac{dx}{3x} = \frac{dy}{y} = \frac{du}{u} = \frac{dv}{-v} = \frac{dU}{U} \quad (17)$$

The similarity variable and functions are

$$\xi = \frac{y}{x^{1/3}}, \quad u = x^{1/3} f(\xi), \quad v = \frac{g(\xi)}{x^{1/3}}, \quad U = x^{1/3} \quad (18)$$

Substituting all into the boundary layer equations yields the ordinary differential system

$$f - \xi f' + 3g' = 0 \quad (19)$$

$$f^2 - \xi f f' + 3g f' = 1 + 3(\varepsilon_1 + \varepsilon_2 \varepsilon_3) f'' - \frac{3}{2} \varepsilon_2 \varepsilon_3^2 (f')^2 f'' \quad (20)$$

The boundary conditions also transform

$$f(0) = 0, \quad g(0) = 0, \quad f(\infty) = 1 \quad (21)$$

#### 4. NUMERICAL RESULTS

Equations (19) and (20) are numerically integrated using a finite difference scheme subject to the boundary conditions (21). In Figure 1,  $f$  function and in Figure 2,  $g$  function related to the  $x$  and  $y$  components of the velocities are drawn for different  $\varepsilon_1$  parameters. Boundary layer becomes thicker for an increase in  $\varepsilon_1$ . There is an increase in  $y$  component of velocity which is negative over the whole domain for an increase in  $\varepsilon_1$  as can be seen from Figure 2. A similar trend is observed for parameter  $\varepsilon_2$  (See Figures 3 and 4). The last parameter  $\varepsilon_3$  has also an effect of thickening of the boundary layers as can be seen from Figure 5 in the parameter range considered. The increase in  $y$  component of velocity as  $\varepsilon_3$  increases is observed from Figure 6. Note that for higher values of  $\varepsilon_3$ , it becomes extremely hard to numerically integrate the equations. A bigger  $\varepsilon_3$  may also violate the validity of the Taylor series approximation for the stress tensor.

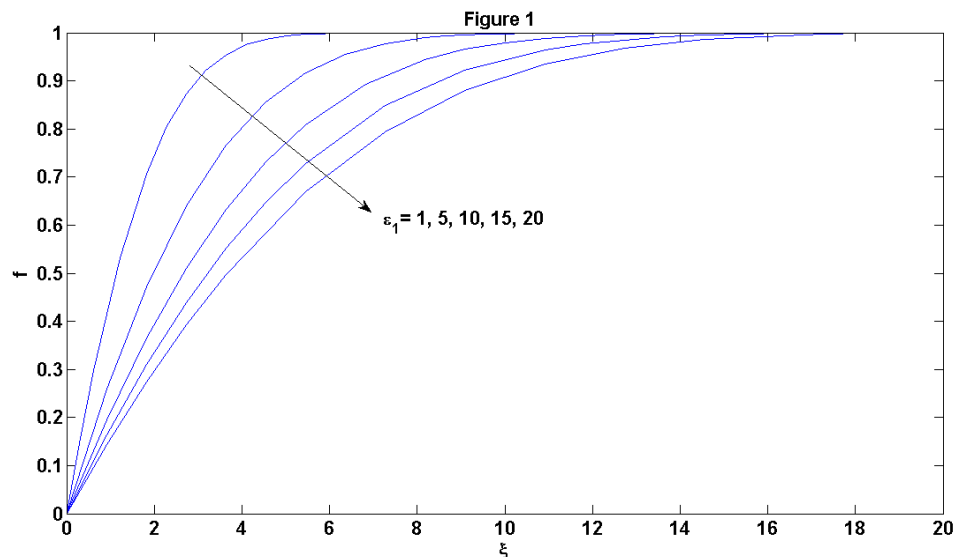


Figure 1. Effect of  $\varepsilon_1$  parameter on the similarity function  $f$  related to the  $x$  component of velocity ( $\varepsilon_2=1$ ,  $\varepsilon_3=1$ ).

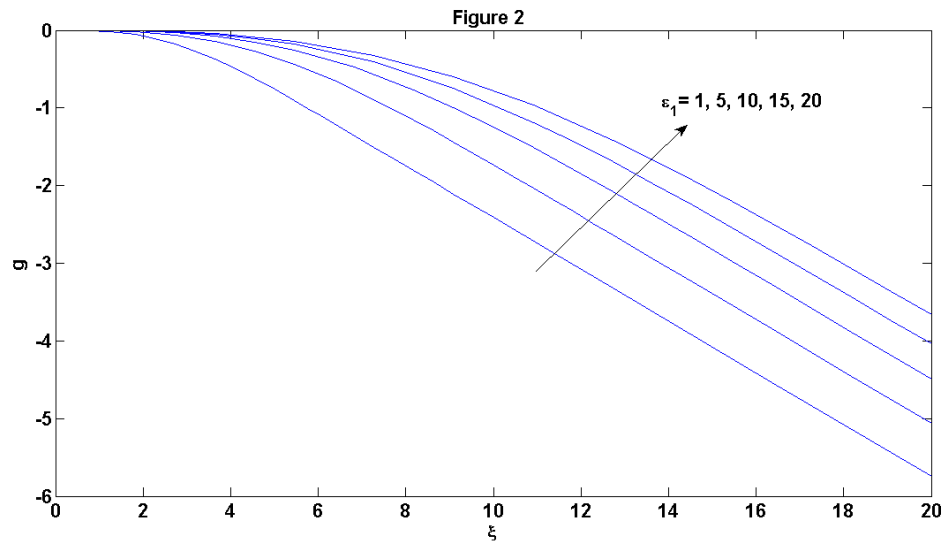


Figure 2. Effect of  $\varepsilon_1$  parameter on the similarity function  $g$  related to the y component of velocity ( $\varepsilon_2=1$ ,  $\varepsilon_3=1$ ).

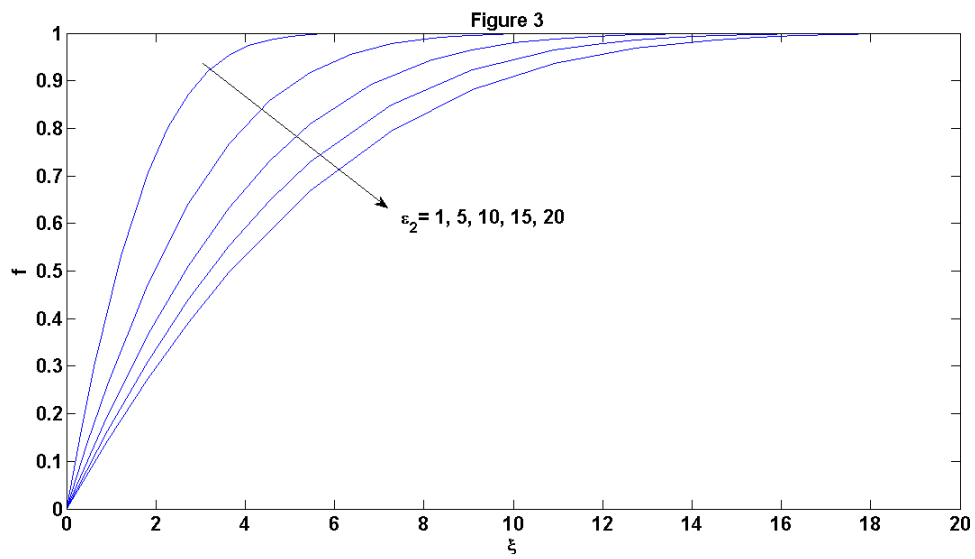


Figure 3. Effect of  $\varepsilon_2$  parameter on the similarity function  $f$  related to the x component of velocity ( $\varepsilon_1=1$ ,  $\varepsilon_3=1$ ).

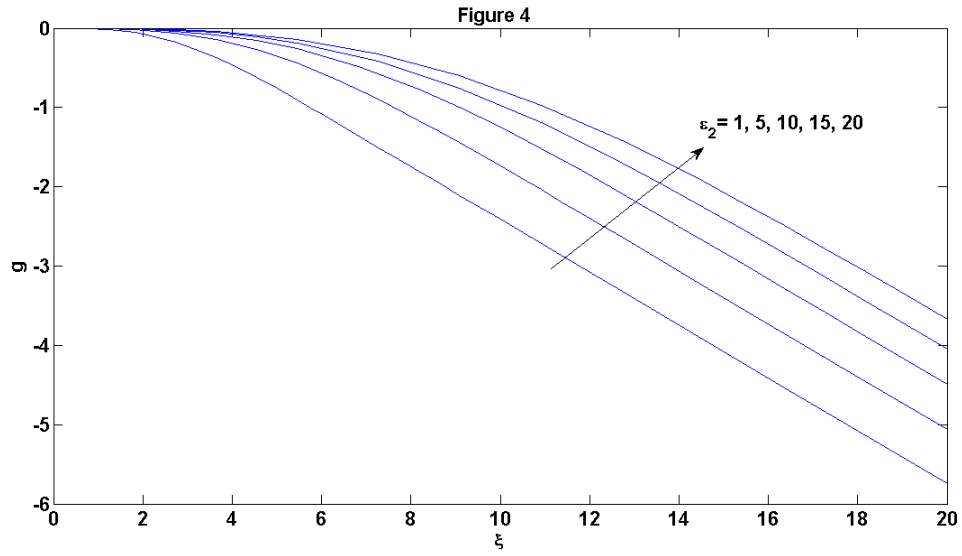


Figure 4. Effect of  $\varepsilon_2$  parameter on the similarity function  $g$  related to the  $y$  component of velocity ( $\varepsilon_1=1$ ,  $\varepsilon_3=1$ ).

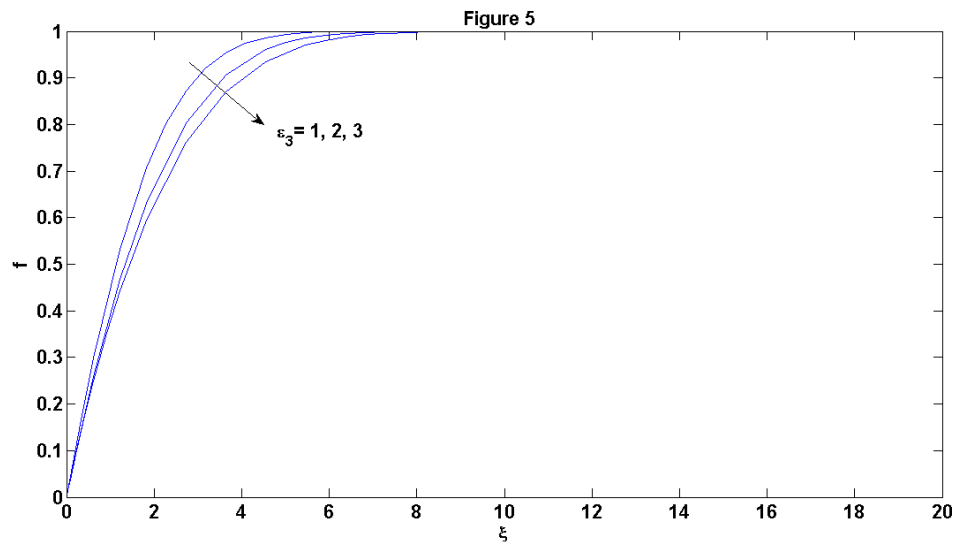


Figure 5. Effect of  $\varepsilon_3$  parameter on the similarity function  $f$  related to the  $x$  component of velocity ( $\varepsilon_1=1$ ,  $\varepsilon_2=1$ ).

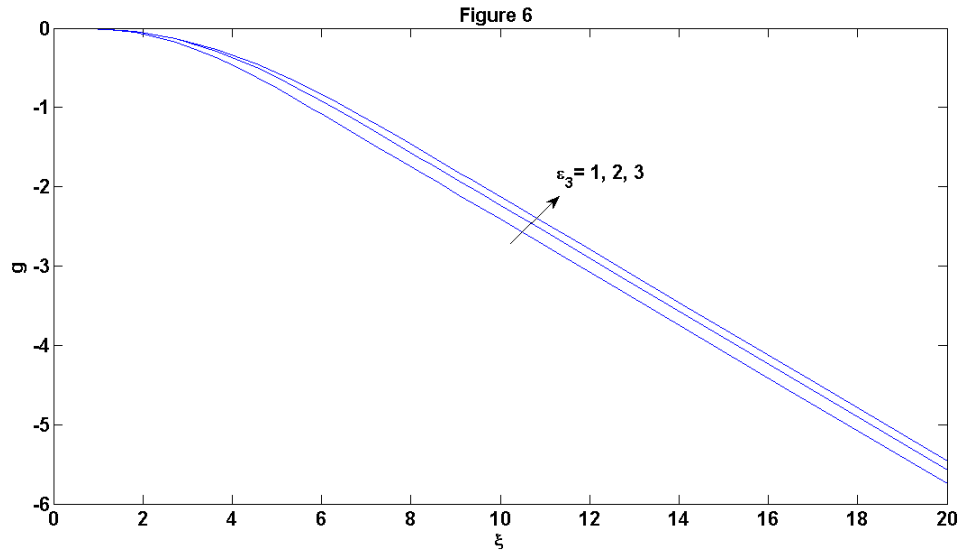


Figure 6. Effect of  $\varepsilon_3$  parameter on the similarity function  $g$  related to the  $y$  component of velocity ( $\varepsilon_1=1$ ,  $\varepsilon_2=1$ ).

Note the definitions of dimensionless parameters given in equation (10). An increase in the viscosity  $\mu$  would yield an increase in parameter  $\varepsilon_1$ . It is therefore natural for the boundary layer to become wider for a higher viscosity value. More energy will be dissipated near the boundary which in turn causes the outer velocity to resume higher from the boundary. The rheological parameter  $\beta$  has an inverse relation with dimensionless parameter  $\varepsilon_2$ . An increase in  $\varepsilon_2$  means a decrease in  $\beta$ . The same applies to the relationship between  $\varepsilon_3$  and the rheological parameter  $c$  also.

## 5. CONCLUDING REMARKS

Boundary layer equations of Powel-Eyring fluids are derived for the first time. Using a scaling transformation, the partial differential system is converted into an ordinary differential system which is solved numerically using a finite difference scheme. Effects of rheological parameters on the boundary layers are discussed in detail. It is found that an increase in all the dimensionless parameters  $\varepsilon_1$ ,  $\varepsilon_2$  and  $\varepsilon_3$  causes the boundary layers to thicken.

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