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On a more accurate Hardy-Mulholland-type inequality

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Abstract

By using weight coefficients, technique of real analysis, and Hermite-Hadamard's inequality, we give a more accurate Hardy-Mulholland-type inequality with multiparameters and a best possible constant factor related to the beta function. The equivalent forms, the reverses, the operator expressions, and some particular cases are also considered.

MSC: 26D15; 47A07

Keywords: Hardy-Mulholland-type inequality; weight coefficient; equivalent form; reverse; operator

1 Introduction

Assuming that $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $a_m, b_n \geq 0$, $a = \{a_m\}_{m=1}^{\infty} \in l^p$, $b = \{b_n\}_{n=1}^{\infty} \in l^q$, $\|a\|_p = (\sum_{m=1}^{\infty} a_m^p)^{\frac{1}{p}} > 0$, and $\|b\|_q > 0$, we have the following Hardy-Hilbert inequality with the best possible constant factor $\frac{\pi}{\sin(\pi/p)}$ (see [1], Theorem 315):

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\pi/p)}\|a\|_p\|b\|_q. \tag{1}$$

The more accurate and extended inequality of (1) is given as follows (see [1], Theorem 323 and [2]):

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n-\alpha} < \frac{\pi}{\sin(\pi/p)}\|a\|_p\|b\|_q \quad (0 \leq \alpha \leq 1), \tag{2}$$

where the constant factor $\frac{\pi}{\sin(\pi/p)}$ is the best possible. Also, we have the following Mulholland inequality similar to (1) with the same best possible constant factor $\frac{\pi}{\sin(\pi/p)}$ (see [3] or [1], Theorem 343, replacing $\frac{a_m}{n}$, $\frac{b_n}{n}$ by a_m, b_n):

$$\sum_{m=2}^{\infty}\sum_{n=2}^{\infty}\frac{a_m b_n}{\ln mn} < \frac{\pi}{\sin(\pi/p)}\left(\sum_{m=2}^{\infty}\frac{a_m^p}{m^{1-p}}\right)^{\frac{1}{p}}\left(\sum_{n=2}^{\infty}\frac{b_n^q}{n^{1-q}}\right)^{\frac{1}{q}}. \tag{3}$$

Inequalities (1)-(3) are important in analysis and its applications (see [1, 2, 4–20]).

Suppose that $\mu_i, \upsilon_j > 0$ ($i, j \in \mathbf{N} = \{1, 2, \ldots\}$) and

$$U_m := \sum_{i=1}^{m} \mu_i, \qquad V_n := \sum_{j=1}^{n} \upsilon_j \quad (m, n \in \mathbf{N}). \tag{4}$$

Then we have the following Hardy-Hilbert-type inequality ([1], Theorem 321):

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{\mu_m^{1/q}\upsilon_n^{1/p} a_m b_n}{U_m + V_n} < \frac{\pi}{\sin(\pi/p)} \|a\|_p \|b\|_q. \tag{5}$$

For $\mu_i = \upsilon_j = 1$ ($i, j \in \mathbf{N}$), inequality (5) reduces to (1). Replacing $\mu_m^{1/q} a_m$ and $\upsilon_n^{1/p} b_n$ by a_m and b_n in (5), respectively, we obtain the equivalent form of (5) as follows:

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{a_m b_n}{U_m + V_n} < \frac{\pi}{\sin(\frac{\pi}{p})} \left(\sum_{m=1}^{\infty} \frac{a_m^p}{\mu_m^{p-1}}\right)^{\frac{1}{p}} \left(\sum_{n=1}^{\infty} \frac{b_n^q}{\upsilon_n^{q-1}}\right)^{\frac{1}{q}}. \tag{6}$$

In 2015, Yang [21] gave the following extension of (6). For $0 < \lambda_1, \lambda_2 \leq 1$, $\lambda_1 + \lambda_2 = \lambda$, decreasing sequences $\{\mu_m\}_{m=1}^{\infty}$ and $\{\upsilon_n\}_{n=1}^{\infty}$, and $U_\infty = V_\infty = \infty$, we have the following inequality with the best possible constant factor $B(\lambda_1, \lambda_2)$:

$$\begin{aligned} &\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{a_m b_n}{(U_m + V_n)^{\lambda}} \\ &\quad < B(\lambda_1, \lambda_2) \left[\sum_{m=1}^{\infty} \frac{U_m^{p(1-\lambda_1)-1} a_m^p}{\mu_m^{p-1}}\right]^{\frac{1}{p}} \left[\sum_{n=1}^{\infty} \frac{V_n^{q(1-\lambda_2-1)} b_n^q}{\upsilon_n^{q-1}}\right]^{\frac{1}{q}}, \end{aligned} \tag{7}$$

where $B(u, v)$ is the beta function (see [22]):

$$B(u, v) := \int_0^{\infty} \frac{t^{u-1}}{(1+t)^{u+v}}\, dt \quad (u, v > 0). \tag{8}$$

In this paper, by using weight coefficients, technique of real analysis, and the Hermite-Hadamard inequality, we give a Hardy-Mulholland-type inequality with a best possible constant factor $\frac{\pi}{\sin(\pi/p)}$ as follows.

For $\mu_1 = \upsilon_1 =1$, decreasing sequences $\{\mu_m\}_{m=1}^{\infty}$ and $\{\upsilon_n\}_{n=1}^{\infty}$, and $U_\infty = V_\infty = \infty$, we have

$$\begin{aligned} &\sum_{m=2}^{\infty}\sum_{n=2}^{\infty} \frac{a_m b_n}{\ln(U_m V_n)} \\ &\quad < \frac{\pi}{\sin(\pi/p)} \left[\sum_{m=2}^{\infty} \left(\frac{U_m}{\mu_{m+1}}\right)^{p-1} a_m^p\right]^{\frac{1}{p}} \left[\sum_{n=2}^{\infty} \left(\frac{V_n}{\upsilon_{n+1}}\right)^{q-1} b_n^q\right]^{\frac{1}{q}}, \end{aligned} \tag{9}$$

which is an extension of (3). So, we have obtained a more accurate and extended inequality of (9) with multiparameters and a best possible constant factor $B(\lambda_1, \lambda_2)$. We also consider the equivalent forms, the reverses, the operator expressions, and some particular cases.

2 Some lemmas and an example

In the following, we make appointment that $p \neq 0,1$, $\frac{1}{p}+\frac{1}{q}=1$, $0<\lambda_1,\lambda_2 \leq 1$, $\lambda_1+\lambda_2=\lambda$, $\mu_i,\upsilon_j>0$ $(i,j\in \mathbf{N})$, with $\mu_1=\upsilon_1=1$, U_m and V_n are defined by (4), $\frac{1}{1+\frac{\mu_2}{2}} \leq \alpha \leq 1$, $\frac{1}{1+\frac{\upsilon_2}{2}} \leq \beta \leq 1$, $a_m,b_n \geq 0$, $\|a\|_{p,\Phi_\lambda} := (\sum_{m=2}^{\infty}\Phi_\lambda(m)a_m^p)^{\frac{1}{p}}$, and $\|b\|_{q,\Psi_\lambda} := (\sum_{n=2}^{\infty}\Psi_\lambda(n)b_n^q)^{\frac{1}{q}}$, where

$$\begin{aligned} \Phi_\lambda(m) &:= \left(\frac{U_m}{\mu_{m+1}}\right)^{p-1}(\ln \alpha U_m)^{p(1-\lambda_1)-1} \quad \bigl(m\in \mathbf{N}\backslash\{1\}\bigr), \\ \Psi_\lambda(n) &:= \left(\frac{V_n}{\upsilon_{n+1}}\right)^{q-1}(\ln \beta V_n)^{q(1-\lambda_2)-1} \quad \bigl(n\in \mathbf{N}\backslash\{1\}\bigr). \end{aligned} \tag{10}$$

Lemma 1 *If $a\in \mathbf{R}$, $f(x)$ is continuous in $[a-\frac{1}{2},a+\frac{1}{2}]$, and $f'(x)$ is strictly increasing in the intervals $(a-\frac{1}{2},a)$ and $(a,a+\frac{1}{2})$ and satisfying*

$$\lim_{x\to a-} f'(x) = f'(a-0) \leq f'(a+0) = \lim_{x\to a+} f'(x),$$

then we have the following Hermite-Hadamard inequality (*cf.* [23]):

$$f(a) < \int_{a-\frac{1}{2}}^{a+\frac{1}{2}} f(x)\,dx. \tag{11}$$

Proof Since $f'(a-0)$ $(\leq f'(a+0))$ is finite, we define the linear function $g(x)$ as follows:

$$g(x) := f'(a-0)(x-a)+f(a), \quad x\in \left[a-\frac{1}{2},a+\frac{1}{2}\right].$$

Since $f'(x)$ is strictly increasing in $(a-\frac{1}{2},a)$, we have that, for $x\in(a-\frac{1}{2},a)$,

$$\bigl(f(x)-g(x)\bigr)' = f'(x)-f'(a-0) < 0.$$

Since $f(a)-g(a)=0$, it follows that $f(x)-g(x)>0$, $x\in(a-\frac{1}{2},a)$. In the same way, we obtain $f(x)-g(x)>0$, $x\in(a,a+\frac{1}{2})$. Hence, we find

$$\int_{a-\frac{1}{2}}^{a+\frac{1}{2}} f(x)\,dx > \int_{a-\frac{1}{2}}^{a+\frac{1}{2}} g(x)\,dx = f(a),$$

that is, (11) follows. □

Example 1 If $\{\mu_m\}_{m=1}^{\infty}$ and $\{\upsilon_n\}_{n=1}^{\infty}$ are decreasing, then we define the functions $\mu(t):=\mu_m$, $t\in(m-1,m]$ $(m\in\mathbf{N})$; $\upsilon(t):=\upsilon_n$, $t\in(n-1,n]$ $(n\in\mathbf{N})$, and

$$U(x) := \int_0^x \mu(t)\,dt \quad (x\geq 0), \qquad V(y) := \int_0^y \upsilon(t)\,dt \quad (y\geq 0). \tag{12}$$

Then it follows that $U(m)=U_m$, $V(n)=V_n$, $U(\infty)=U_\infty$, $V(\infty)=V_\infty$, and

$$\begin{aligned} U'(x) &= \mu(x) = \mu_m, \quad x\in(m-1,m), \\ V'(y) &= \upsilon(y) = \upsilon_n, \quad y\in(n-1,n)\ (m,n\in\mathbf{N}). \end{aligned}$$

For fixed $m,n \in \mathbf{N}\backslash\{1\}$, we also define the function

$$f(x) := \frac{\ln^{\lambda_2-1}\beta V(x)}{V(x)(\ln\alpha U_m + \ln\beta V(x))^{\lambda}}, \quad x \in \left[n-\frac{1}{2}, n+\frac{1}{2}\right].$$

Then $f(x)$ is continuous in $[n-\frac{1}{2}, n+\frac{1}{2}]$. For $x \in (n-\frac{1}{2}, n)$ ($n \in \mathbf{N}\backslash\{1\}$), we find

$$\begin{aligned} f'(x) = {}& -\left[\frac{\ln^{\lambda_2-1}\beta V(x)}{V(x)} + \frac{\lambda\ln^{\lambda_2-1}\beta V(x)}{\ln\alpha U_m + \ln\beta V(x)} + \frac{1-\lambda_2}{V^{2-\lambda_2}(x)}\right] \\ & \times \frac{\upsilon_n}{V(x)(\ln\alpha U_m + \ln\beta V(x))^{\lambda}}. \end{aligned}$$

Since $1-\lambda_2 \geq 0$, it follows that $f'(x)$ (< 0) is strictly increasing in $(n-\frac{1}{2}, n)$ and

$$\begin{aligned} \lim_{x\to n-} f'(x) &= f'(n-0) \\ &= -\left[\frac{\ln^{\lambda_2-1}\beta V_n}{V_n} + \frac{\lambda\ln^{\lambda_2-1}\beta V_n}{\ln\alpha U_m + \ln\beta V_n} + \frac{1-\lambda_2}{V_n^{2-\lambda_2}}\right] \\ &\quad \times \frac{\upsilon_n}{V_n(\ln\alpha U_m + \ln\beta V_n)^{\lambda}}. \end{aligned}$$

In the same way, for $x \in (n, n+\frac{1}{2})$ ($n \in \mathbf{N}\backslash\{1\}$), we find

$$\begin{aligned} f'(x) = {}& -\left[\frac{\ln^{\lambda_2-1}\beta V(x)}{V(x)} + \frac{\lambda\ln^{\lambda_2-1}\beta V(x)}{\ln\alpha U_m + \ln\beta V(x)} + \frac{1-\lambda_2}{V^{2-\lambda_2}(x)}\right] \\ & \times \frac{\upsilon_{n+1}}{V(x)(\ln\alpha U_m + \ln\beta V(x))^{\lambda}}, \end{aligned}$$

so that $f'(x)$ (< 0) is strict increasing in $(n, n+\frac{1}{2})$. In view of $\upsilon_{n+1} \leq \upsilon_n$, it follows that

$$\lim_{x\to n+} f'(x) = f'(n+0) \geq f'(n-0).$$

Then by (11), for $m,n \in \mathbf{N}\backslash\{1\}$, we have

$$f(n) < \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} f(x)\,dx = \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} \frac{\ln^{\lambda_2-1}\beta V(x)}{V(x)(\ln\alpha U_m + \ln\beta V(x))^{\lambda}}\,dx. \tag{13}$$

Definition 1 Define the following weight coefficients:

$$\omega(\lambda_2, m) := \sum_{n=2}^{\infty} \frac{1}{\ln^{\lambda}(\alpha\beta U_m V_n)} \frac{\upsilon_{n+1}\ln^{\lambda_1}\alpha U_m}{V_n \ln^{1-\lambda_2}\beta V_n}, \quad m \in \mathbf{N}\backslash\{1\}, \tag{14}$$

$$\varpi(\lambda_1, n) := \sum_{m=2}^{\infty} \frac{1}{\ln^{\lambda}(\alpha\beta U_m V_n)} \frac{\mu_{m+1}\ln^{\lambda_2}\beta V_n}{U_m \ln^{1-\lambda_1}\alpha U_m}, \quad n \in \mathbf{N}\backslash\{1\}. \tag{15}$$

Lemma 2 *If* $\{\mu_m\}_{m=1}^{\infty}$ *and* $\{\upsilon_n\}_{n=1}^{\infty}$ *are decreasing and* $U_\infty = V_\infty = \infty$, *then for* $m,n \in \mathbf{N}\backslash\{1\}$, *we have the following inequalities*:

$$\omega(\lambda_2, m) < B(\lambda_1, \lambda_2) \quad (0 < \lambda_2 \leq 1, \lambda_1 > 0), \tag{16}$$

$$\varpi(\lambda_1, n) < B(\lambda_1, \lambda_2) \quad (0 < \lambda_1 \leq 1, \lambda_2 > 0). \tag{17}$$

Proof For $x \in (n - \frac{1}{2}, n + \frac{1}{2}) \backslash \{n\}$, $\upsilon_{n+1} \leq V'(x)$, by (13) we find

$$\begin{aligned}\omega(\lambda_2, m) &< \sum_{n=2}^{\infty} \upsilon_{n+1} \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} \frac{\ln^{\lambda_1} \alpha U_m \ln^{\lambda_2 - 1} \beta V(x)}{V(x)(\ln \alpha U_m + \ln \beta V(x))^{\lambda}} \, dx \\ &\leq \sum_{n=2}^{\infty} \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} \frac{\ln^{\lambda_1} \alpha U_m \ln^{\lambda_2 - 1} \beta V(x)}{(\ln \alpha U_m + \ln \beta V(x))^{\lambda}} \frac{V'(x)}{V(x)} \, dx \\ &= \int_{\frac{3}{2}}^{\infty} \frac{\ln^{\lambda_1} \alpha U_m \ln^{\lambda_2 - 1} \beta V(x)}{(\ln \alpha U_m + \ln \beta V(x))^{\lambda}} \frac{V'(x)}{V(x)} \, dx.\end{aligned}$$

Setting $t = \frac{\ln \beta V(x)}{\ln \alpha U_m}$, since $\beta V(\frac{3}{2}) = \beta(1 + \frac{\upsilon_2}{2}) \geq 1$ and $\frac{V'(x)}{V(x)} \, dx = (\ln \alpha U_m) \, dt$, we find

$$\omega(\lambda_2, m) < \int_0^{\infty} \frac{1}{(1+t)^{\lambda}} t^{\lambda_2 - 1} \, dt = B(\lambda_1, \lambda_2).$$

Hence, we obtain (16). In the same way, we obtain (17). □

Note For example, $\mu_n, \upsilon_n = \frac{1}{n^{\sigma}}$ $(0 \leq \sigma \leq 1)$ satisfy the conditions of Lemma 2.

Lemma 3 *With the assumptions of Lemma* 2, (i) *for* $m, n \in \mathbf{N} \backslash \{1\}$, *we have*

$$B(\lambda_1, \lambda_2)\bigl(1 - \theta(\lambda_2, m)\bigr) < \omega(\lambda_2, m) \quad (0 < \lambda_2 \leq 1, \lambda_1 > 0), \tag{18}$$

$$B(\lambda_1, \lambda_2)\bigl(1 - \vartheta(\lambda_1, n)\bigr) < \varpi(\lambda_1, n) \quad (0 < \lambda_1 \leq 1, \lambda_2 > 0), \tag{19}$$

where

$$\begin{aligned}\theta(\lambda_2, m) &= \frac{1}{B(\lambda_1, \lambda_2)} \frac{\ln^{\lambda_2} \beta(1 + \upsilon_2)}{\lambda_2 [1 + \frac{\ln \beta(1 + \theta(m)\upsilon_2)}{\ln \alpha U_m}]^{\lambda}} \frac{1}{\ln^{\lambda_2} \alpha U_m} \\ &= O\left(\frac{1}{\ln^{\lambda_2} \alpha U_m}\right) \\ &\in (0,1) \quad \left(\theta(m) \in \left(\frac{1 - \beta}{\beta \upsilon_2}, 1\right)\right),\end{aligned} \tag{20}$$

$$\begin{aligned}\vartheta(\lambda_1, n) &= \frac{1}{B(\lambda_1, \lambda_2)} \frac{\ln^{\lambda_1} \alpha(1 + \mu_2)}{\lambda_1 [1 + \frac{\ln \alpha(1 + \vartheta(n)\mu_2)}{\ln \beta V_n}]^{\lambda}} \frac{1}{\ln^{\lambda_1} \beta V_n} \\ &= O\left(\frac{1}{\ln^{\lambda_1} \beta V_n}\right) \\ &\in (0,1) \quad \left(\vartheta(n) \in \left(\frac{1 - \alpha}{\alpha \mu_2}, 1\right)\right);\end{aligned} \tag{21}$$

(ii) *for any* $c > 0$, *we have*

$$\sum_{m=2}^{\infty} \frac{\mu_{m+1}}{U_m \ln^{1+c} \alpha U_m} = \frac{1}{c} \left(\frac{1}{\ln^c \alpha(1 + \mu_2)} + cO(1)\right), \tag{22}$$

$$\sum_{n=2}^{\infty} \frac{\upsilon_{n+1}}{V_n \ln^{1+c} \beta V_n} = \frac{1}{c} \left(\frac{1}{\ln^c \beta(1 + \upsilon_2)} + c\widetilde{O}(1)\right). \tag{23}$$

Proof In view of $\beta \le 1$ and $\beta \ge \frac{1}{1+\upsilon_2/2} > \frac{1}{1+\upsilon_2}$, it follows that $1 \le \frac{1-\beta}{\beta\upsilon_2} + 1 < 2$. Since, by Example 1, $f(x)$ is strictly decreasing in $[n, n+1]$, for $m \in \mathbf{N}\backslash\{1\}$, we find

$$\begin{aligned}
\omega(\lambda_2, m) &> \sum_{n=2}^{\infty} \int_n^{n+1} \upsilon_{n+1} \frac{\ln^{\lambda_1} \alpha U_m \ln^{\lambda_2 - 1} \beta V(x)}{V(x)(\ln \alpha U_m + \ln \beta V(x))^{\lambda}} \, dx \\
&= \int_2^{\infty} \frac{\ln^{\lambda_1} \alpha U_m \ln^{\lambda_2 - 1} \beta V(x)}{(\ln \alpha U_m + \ln \beta V(x))^{\lambda}} \frac{V'(x)}{V(x)} \, dx \\
&= \int_{\frac{1-\beta}{\beta\upsilon_2}+1}^{\infty} \frac{\ln^{\lambda_1} \alpha U_m \ln^{\lambda_2 - 1} \beta V(x)}{(\ln \alpha U_m + \ln \beta V(x))^{\lambda}} \frac{V'(x)}{V(x)} \, dx \\
&\quad - \int_{\frac{1-\beta}{\beta\upsilon_2}+1}^{2} \frac{\ln^{\lambda_1} \alpha U_m \ln^{\lambda_2 - 1} \beta V(x)}{(\ln \alpha U_m + \ln \beta V(x))^{\lambda}} \frac{V'(x)}{V(x)} \, dx.
\end{aligned}$$

Setting $t = \frac{\ln \beta V(x)}{\ln \alpha U_m}$, we have $\ln \beta V(\frac{1-\beta}{\beta\upsilon_2} + 1) = \ln \beta(1 + \frac{1-\beta}{\beta\upsilon_2}\upsilon_2) = 0$ and

$$\begin{aligned}
\omega(\lambda_2, m) &> \int_0^{\infty} \frac{1}{(1+t)^{\lambda}} t^{\lambda_2 - 1} \, dt \\
&\quad - \int_{\frac{1-\beta}{\beta\upsilon_2}+1}^{2} \frac{\ln^{\lambda_1} \alpha U_m \ln^{\lambda_2 - 1} \beta V(x)}{(\ln \alpha U_m + \ln \beta V(x))^{\lambda}} \frac{V'(x)}{V(x)} \, dx \\
&= B(\lambda_1, \lambda_2)\bigl(1 - \theta(\lambda_2, m)\bigr),
\end{aligned}$$

where

$$\begin{aligned}
\theta(\lambda_2, m) &:= \frac{1}{B(\lambda_1, \lambda_2)} \int_{\frac{1-\beta}{\beta\upsilon_2}+1}^{2} \frac{\ln^{\lambda_1} \alpha U_m \ln^{\lambda_2 - 1} \beta V(x)}{(\ln \alpha U_m + \ln \beta V(x))^{\lambda}} \frac{V'(x)}{V(x)} \, dx \\
&\in (0, 1).
\end{aligned}$$

There exists $\theta(m) \in (\frac{1-\beta}{\beta\upsilon_2}, 1)$ such that

$$\begin{aligned}
\theta(\lambda_2, m) &= \frac{1}{B(\lambda_1, \lambda_2)} \frac{\ln^{\lambda_1} \alpha U_m}{(\ln \alpha U_m + \ln \beta V(1 + \theta(m)))^{\lambda}} \\
&\quad \times \int_{\frac{1-\beta}{\beta\upsilon_2}+1}^{2} \ln^{\lambda_2 - 1} \beta V(x) \frac{V'(x)}{V(x)} \, dx \\
&= \frac{1}{B(\lambda_1, \lambda_2)} \frac{\ln^{\lambda_1} \alpha U_m \ln^{\lambda_2} \beta(1 + \upsilon_2)}{\lambda_2(\ln \alpha U_m + \ln \beta V(1 + \theta(m)))^{\lambda}} \\
&= \frac{1}{B(\lambda_1, \lambda_2)} \frac{\ln^{\lambda_2} \beta(1 + \upsilon_2)}{\lambda_2[1 + \frac{\ln \beta(1 + \theta(m)\upsilon_2)}{\ln \alpha U_m}]^{\lambda}} \frac{1}{\ln^{\lambda_2} \alpha U_m}.
\end{aligned}$$

Since we find

$$0 < \theta(\lambda_2, m) \le \frac{\ln^{\lambda_2} \beta(1 + \upsilon_2)}{\lambda_2 B(\lambda_1, \lambda_2)} \frac{1}{\ln^{\lambda_2} \alpha U_m},$$

namely, $\theta(\lambda_2, m) = O(\frac{1}{\ln^{\lambda_2} \alpha U_m})$, we obtain (18) and (20). In the same way, we obtain (19) and (21).

For any $c > 0$, we find

$$\begin{aligned}
\sum_{m=2}^{\infty} \frac{\mu_{m+1}}{U_m \ln^{1+c} \alpha U_m} &\le \sum_{m=2}^{\infty} \frac{\mu_m}{U_m \ln^{1+c} \alpha U_m} \\
&= \frac{\mu_2}{U_2 \ln^{1+c} \alpha U_2} + \sum_{m=3}^{\infty} \frac{\mu_m}{U_m \ln^{1+c} \alpha U_m} \\
&= \frac{\mu_2}{U_2 \ln^{1+c} \alpha U_2} + \sum_{m=3}^{\infty} \int_{m-1}^{m} \frac{U'(x)}{U_m \ln^{1+c} \alpha U_m}\, dx \\
&< \frac{\mu_2}{U_2 \ln^{1+c} \alpha U_2} + \sum_{m=3}^{\infty} \int_{m-1}^{m} \frac{U'(x)}{U(x) \ln^{1+c} \alpha U(x)}\, dx \\
&= \frac{\mu_2}{U_2 \ln^{1+c} \alpha U_2} + \int_{2}^{\infty} \frac{U'(x)}{U(x) \ln^{1+c} \alpha U(x)}\, dx \\
&= \frac{\mu_2}{U_2 \ln^{1+c} \alpha U_2} + \frac{1}{c \ln^{c} \alpha (1+\mu_2)} \\
&= \frac{1}{c}\left[\frac{1}{\ln^{c} \alpha (1+\mu_2)} + c\frac{\mu_2}{U_2 \ln^{1+c} \alpha U_2}\right], \\
\sum_{m=2}^{\infty} \frac{\mu_{m+1}}{U_m \ln^{1+c} \alpha U_m} &= \sum_{m=2}^{\infty} \int_{m}^{m+1} \frac{U'(x)\, dx}{U_m \ln^{1+c} \alpha U_m} \\
&> \sum_{m=2}^{\infty} \int_{m}^{m+1} \frac{U'(x)}{U(x) \ln^{1+c} \alpha U(x)}\, dx \\
&= \int_{2}^{\infty} \frac{U'(x)\, dx}{U(x) \ln^{1+c} \alpha U(x)} = \frac{1}{c} \frac{1}{\ln^{c} \alpha (1+\mu_2)}.
\end{aligned}$$

Hence, we obtain (22). In the same way, we obtain (23). □

3 Main results and operator expressions

In the following, we also set

$$\begin{aligned}
\widetilde{\Phi}_{\lambda}(m) &:= \omega(\lambda_2, m)\left(\frac{U_m}{\mu_{m+1}}\right)^{p-1} (\ln \alpha U_m)^{p(1-\lambda_1)-1}, \\
\widetilde{\Psi}_{\lambda}(n) &:= \varpi(\lambda_1, n)\left(\frac{V_n}{\upsilon_{n+1}}\right)^{q-1} (\ln \beta V_n)^{q(1-\lambda_2)-1} \quad \bigl(m, n \in \mathbf{N}\backslash\{1\}\bigr).
\end{aligned} \tag{24}$$

Theorem 1 (i) *For $p > 1$, we have the following equivalent inequalities*:

$$I := \sum_{n=2}^{\infty} \sum_{m=2}^{\infty} \frac{a_m b_n}{\ln^{\lambda}(\alpha\beta U_m V_n)} \le \|a\|_{p,\widetilde{\Phi}_{\lambda}} \|b\|_{q,\widetilde{\Psi}_{\lambda}}, \tag{25}$$

$$J := \left\{\sum_{n=2}^{\infty} \frac{\upsilon_{n+1} \ln^{p\lambda_2 - 1} \beta V_n}{(\varpi(\lambda_1, n))^{p-1} V_n} \left[\sum_{m=2}^{\infty} \frac{a_m}{\ln^{\lambda}(\alpha\beta U_m V_n)}\right]^{p}\right\}^{\frac{1}{p}} \le \|a\|_{p,\widetilde{\Phi}_{\lambda}}; \tag{26}$$

(ii) *for $0 < p < 1$ (or $p < 0$), we have the equivalent reverses of* (25) *and* (26).

Proof (i) By Hölder's inequality with weight (see [23]) and (15) we have

$$\begin{aligned}
&\left[\sum_{m=2}^{\infty}\frac{a_m}{\ln^{\lambda}(\alpha\beta U_m V_n)}\right]^p \\
&= \left[\sum_{m=2}^{\infty}\frac{1}{\ln^{\lambda}(\alpha\beta U_m V_n)}\left(\frac{U_m^{1/q}(\ln\alpha U_m)^{(1-\lambda_1)/q}\upsilon_{n+1}^{1/p}}{(\ln\beta V_n)^{(1-\lambda_2)/p}\mu_{m+1}^{1/q}}a_m\right)\left(\frac{(\ln\beta V_n)^{(1-\lambda_2)/p}\mu_{m+1}^{1/q}}{U_m^{1/q}(\ln\alpha U_m)^{(1-\lambda_1)/q}\upsilon_{n+1}^{1/p}}\right)\right]^p \\
&\leq \sum_{m=2}^{\infty}\frac{1}{\ln^{\lambda}(\alpha\beta U_m V_n)}\frac{U_m^{p-1}(\ln\alpha U_m)^{(1-\lambda_1)p/q}\upsilon_{n+1}}{(\ln\beta V_n)^{1-\lambda_2}\mu_{m+1}^{p/q}}a_m^p \\
&\quad\times\left[\sum_{m=2}^{\infty}\frac{1}{\ln^{\lambda}(\alpha\beta U_m V_n)}\frac{(\ln\beta V_n)^{(1-\lambda_2)(q-1)}\mu_m}{U_m(\ln\alpha U_m)^{1-\lambda_1}\upsilon_{n+1}^{q-1}}\right]^{p-1} \\
&= \frac{(\varpi(\lambda_1,n))^{p-1}V_n}{(\ln\beta V_n)^{p\lambda_2-1}\upsilon_{n+1}}\sum_{m=2}^{\infty}\frac{\upsilon_{n+1}U_m^{p-1}(\ln\alpha U_m)^{(1-\lambda_1)(p-1)}a_m^p}{V_n\ln^{\lambda}(\alpha\beta U_m V_n)(\ln\beta V_n)^{1-\lambda_2}\mu_{m+1}^{p-1}}. \qquad (27)
\end{aligned}$$

Then by (14) we find

$$\begin{aligned}
J &\leq \left[\sum_{n=2}^{\infty}\sum_{m=2}^{\infty}\frac{\upsilon_{n+1}}{\ln^{\lambda}(\alpha\beta U_m V_n)}\frac{U_m^{p-1}(\ln\alpha U_m)^{(1-\lambda_1)(p-1)}}{V_n(\ln\beta V_n)^{1-\lambda_2}\mu_{m+1}^{p-1}}a_m^p\right]^{\frac{1}{p}} \\
&= \left[\sum_{m=2}^{\infty}\sum_{n=2}^{\infty}\frac{\upsilon_{n+1}(\ln\alpha U_m)^{\lambda_1}}{\ln^{\lambda}(\alpha\beta U_m V_n)}\frac{U_m^{p-1}(\ln\alpha U_m)^{p(1-\lambda_1)-1}}{V_n(\ln\beta V_n)^{1-\lambda_2}\mu_{m+1}^{p-1}}a_m^p\right]^{\frac{1}{p}} \\
&= \left[\sum_{m=2}^{\infty}\omega(\lambda_2,m)\left(\frac{U_m}{\mu_{m+1}}\right)^{p-1}(\ln\alpha U_m)^{p(1-\lambda_1)-1}a_m^p\right]^{\frac{1}{p}}, \qquad (28)
\end{aligned}$$

and then (26) follows.

By Hölder's inequality we have

$$\begin{aligned}
I &= \sum_{n=2}^{\infty}\left[\frac{(\ln\beta V_n)^{\lambda_2-\frac{1}{p}}\upsilon_{n+1}^{1/p}}{(\varpi(\lambda_1,n))^{\frac{1}{q}}V_n^{1/p}}\sum_{m=1}^{\infty}\frac{a_m}{\ln^{\lambda}(\alpha\beta U_m V_n)}\right] \\
&\quad\times\left[\left(\varpi(\lambda_1,n)\right)^{\frac{1}{q}}\frac{(\ln\beta V_n)^{\frac{1}{p}-\lambda_2}}{V_n^{-1/p}\upsilon_{n+1}^{1/p}}b_n\right] \leq J\|b\|_{q,\widetilde{\Psi}_\lambda}. \qquad (29)
\end{aligned}$$

Then by (26) we have (25).

On the other hand, assuming that (25) is valid, we set

$$b_n := \frac{(\ln\beta V_n)^{p\lambda_2-1}\upsilon_{n+1}}{(\varpi(\lambda_1,n))^{p-1}V_n}\left[\sum_{m=2}^{\infty}\frac{a_m}{\ln^{\lambda}(\alpha\beta U_m V_n)}\right]^{p-1}, \quad n\in\mathbf{N}\backslash\{1\}. \qquad (30)$$

Then we find $J^p = \|b\|_{q,\widetilde{\Psi}_\lambda}^q$. If $J = 0$, then (26) is trivially valid; if $J = \infty$, then by (28), (26) takes the form of equality. Suppose that $0 < J < \infty$. By (25) it follows that

$$\|b\|_{q,\widetilde{\Psi}_\lambda}^q = J^p = I \leq \|a\|_{p,\widetilde{\Phi}_\lambda}\|b\|_{q,\widetilde{\Psi}_\lambda}, \qquad (31)$$

$$\|b\|_{q,\widetilde{\Psi}_\lambda}^{q-1} = J \leq \|a\|_{p,\widetilde{\Phi}_\lambda}, \qquad (32)$$

and then (26) follows, which is equivalent to (25).

(ii) For $0 < p < 1$ (or $p < 0$), by the reverse Hölder inequality with weight and (15), we obtain the reverse of (27) (or (27)), then we have the reverse of (28), and then the reverse of (26) follows. By Hölder's inequality we have the reverse of (29), and then by the reverse of (26) the reverse of (25) follows.

On the other hand, assuming that the reverse of (25) is valid, we set b_n as in (30). Then we find $J^p = \|b\|^q_{q,\widetilde{\Psi}_\lambda}$. If $J = \infty$, then the reverse of (26) is trivially valid; if $J = 0$, then by the reverse of (28), (26) takes the form of equality (= 0). Suppose that $0 < J < \infty$. By the reverse of (25) it follows that the reverses of (31) and (32) are valid, and then the reverse of (26) follows, which is equivalent to the reverse of (25). □

Theorem 2 *If $p > 1$, $\{\mu_m\}_{m=1}^\infty$ and $\{\upsilon_n\}_{n=1}^\infty$ are decreasing, $U_\infty = V_\infty = \infty$, $\|a\|_{p,\Phi_\lambda} \in \mathbf{R}_+$, and $\|b\|_{q,\Psi_\lambda} \in \mathbf{R}_+$, then we have the following equivalent inequalities*:

$$\sum_{n=2}^\infty \sum_{m=2}^\infty \frac{a_m b_n}{\ln^\lambda(\alpha\beta U_m V_n)} < B(\lambda_1,\lambda_2)\|a\|_{p,\Phi_\lambda}\|b\|_{q,\Psi_\lambda}, \tag{33}$$

$$J_1 := \left\{\sum_{n=2}^\infty \frac{\upsilon_{n+1}}{V_n}\ln^{p\lambda_2-1}\beta V_n\left[\sum_{m=2}^\infty \frac{a_m}{\ln^\lambda(\alpha\beta U_m V_n)}\right]^p\right\}^{\frac{1}{p}} < B(\lambda_1,\lambda_2)\|a\|_{p,\Phi_\lambda}, \tag{34}$$

where the constant factor $B(\lambda_1,\lambda_2)$ is the best possible.

Proof Using (16) and (17) in (25) and (26), we obtain equivalent inequalities (33) and (34).

For $\varepsilon \in (0, p\lambda_1)$, we set $\tilde{\lambda}_1 = \lambda_1 - \frac{\varepsilon}{p}$ $(\in (0,1))$, $\tilde{\lambda}_2 = \lambda_2 + \frac{\varepsilon}{p}$ (> 0), and

$$\begin{aligned} \tilde{a}_m &:= \frac{\mu_{m+1}}{U_m}\ln^{\tilde{\lambda}_1-1}\alpha U_m = \frac{\mu_{m+1}}{U_m}\ln^{\lambda_1-\frac{\varepsilon}{p}-1}\alpha U_m, \\ \tilde{b}_n &= \frac{\upsilon_{n+1}}{V_n}\ln^{\tilde{\lambda}_2-\varepsilon-1}\beta V_n = \frac{\upsilon_{n+1}}{V_n}\ln^{\lambda_2-\frac{\varepsilon}{q}-1}\beta V_n. \end{aligned} \tag{35}$$

Then by (22), (23), and (19) we have

$$\begin{aligned} \|\tilde{a}\|_{p,\Phi_\lambda}\|\tilde{b}\|_{q,\Psi_\lambda} &= \left(\sum_{m=2}^\infty \frac{\mu_{m+1}}{U_m \ln^{1+\varepsilon}\alpha U_m}\right)^{\frac{1}{p}}\left(\sum_{n=2}^\infty \frac{\upsilon_{n+1}}{V_n\ln^{1+\varepsilon}\beta V_n}\right)^{\frac{1}{q}} \\ &= \frac{1}{\varepsilon}\left[\frac{1}{\ln^\varepsilon \alpha(1+\mu_2)} + \varepsilon O(1)\right]^{\frac{1}{p}}\left[\frac{1}{\ln^\varepsilon \beta(1+\upsilon_2)} + \varepsilon\widetilde{O}(1)\right]^{\frac{1}{q}}, \end{aligned}$$

$$\begin{aligned} \tilde{I} &:= \sum_{n=2}^\infty\sum_{m=2}^\infty \frac{\tilde{a}_m\tilde{b}_n}{\ln^\lambda(\alpha\beta U_m V_n)} = \sum_{n=2}^\infty\left[\sum_{m=2}^\infty \frac{1}{\ln^\lambda(\alpha\beta U_m V_n)}\frac{\mu_{m+1}\ln^{\tilde{\lambda}_2}\beta V_n}{U_m\ln^{1-\tilde{\lambda}_1}\alpha U_m}\right]\frac{\upsilon_{n+1}}{V_n\ln^{\varepsilon+1}\beta V_n} \\ &= \sum_{n=2}^\infty \varpi(\tilde{\lambda}_1,n)\frac{\upsilon_{n+1}}{V_n\ln^{\varepsilon+1}\beta V_n} \ge B(\tilde{\lambda}_1,\tilde{\lambda}_2)\sum_{n=2}^\infty\left(1 - O\left(\frac{1}{\ln^{\tilde{\lambda}_1}\beta V_n}\right)\right)\frac{\upsilon_{n+1}}{V_n\ln^{\varepsilon+1}\beta V_n} \\ &= B(\tilde{\lambda}_1,\tilde{\lambda}_2)\left[\sum_{n=2}^\infty \frac{\upsilon_{n+1}}{V_n\ln^{\varepsilon+1}\beta V_n} - \sum_{n=2}^\infty O\left(\frac{\upsilon_{n+1}}{V_n(\ln\beta V_n)^{(\frac{\varepsilon}{q}+\lambda_1)+1}}\right)\right] \\ &= \frac{1}{\varepsilon}B(\tilde{\lambda}_1,\tilde{\lambda}_2)\left[\frac{1}{\ln^\varepsilon\beta(1+\upsilon_2)} + \varepsilon\left(\widetilde{O}(1) - O(1)\right)\right]. \end{aligned}$$

If there exists a positive constant $K \leq B(\lambda_1, \lambda_2)$ such that (33) is valid when replacing $B(\lambda_1, \lambda_2)$ by K, then, in particular, we have $\varepsilon\tilde{I} < \varepsilon K\|\tilde{a}\|_{p,\Phi_\lambda}\|\tilde{b}\|_{q,\Psi_\lambda}$, namely

$$\begin{aligned} &B\left(\lambda_1 - \frac{\varepsilon}{p}, \lambda_2 + \frac{\varepsilon}{p}\right)\left[\frac{1}{\ln^\varepsilon \beta(1+\upsilon_2)} + \varepsilon\left(\widetilde{O}(1) - O(1)\right)\right] \\ &\quad < K\left[\frac{1}{\ln^\varepsilon \alpha(1+\mu_2)} + \varepsilon O(1)\right]^{\frac{1}{p}}\left[\frac{1}{\ln^\varepsilon \beta(1+\upsilon_2)} + \varepsilon\widetilde{O}(1)\right]^{\frac{1}{q}}. \end{aligned}$$

It follows that $B(\lambda_1, \lambda_2) \leq K$ ($\varepsilon \to 0^+$). Hence, $K = B(\lambda_1, \lambda_2)$ is the best possible constant factor of (33).

Similarly to (29), we still can find the following inequality:

$$I \leq J_1\|b\|_{q,\Psi_\lambda}. \tag{36}$$

Hence, we can prove that the constant factor $B(\lambda_1, \lambda_2)$ in (34) is the best possible. Otherwise, we would reach a contradiction by (36) that the constant factor in (33) is not the best possible. □

Remark 1 (i) For $\alpha = \beta = 1$ in (33) and (34), setting

$$\begin{aligned} \varphi_\lambda(m) &:= \left(\frac{U_m}{\mu_{m+1}}\right)^{p-1}(\ln U_m)^{p(1-\lambda_1)-1}, \\ \psi_\lambda(n) &:= \left(\frac{V_n}{\upsilon_{n+1}}\right)^{q-1}(\ln V_n)^{q(1-\lambda_2)-1} \quad \left(m, n \in \mathbf{N}\backslash\{1\}\right), \end{aligned}$$

we have the following equivalent Mulholland-type inequalities:

$$\sum_{n=2}^\infty\sum_{m=2}^\infty \frac{a_m b_n}{\ln^\lambda(U_m V_n)} < B(\lambda_1, \lambda_2)\|a\|_{p,\varphi_\lambda}\|b\|_{q,\psi_\lambda}, \tag{37}$$

$$\left\{\sum_{n=2}^\infty \frac{\upsilon_{n+1}}{V_n}\ln^{p\lambda_2-1} V_n\left[\sum_{m=2}^\infty \frac{a_m}{\ln^\lambda(U_m V_n)}\right]^p\right\}^{\frac{1}{p}} < B(\lambda_1, \lambda_2)\|a\|_{p,\varphi_\lambda}, \tag{38}$$

which are extensions of (9), and the following inequality:

$$\left\{\sum_{n=2}^\infty \frac{\upsilon_{n+1}}{V_n}\left[\sum_{m=2}^\infty \frac{a_m}{\ln(U_m V_n)}\right]^p\right\}^{\frac{1}{p}} < \frac{\pi}{\sin(\frac{\pi}{p})}\left[\sum_{m=2}^\infty\left(\frac{U_m}{\mu_{m+1}}\right)^{p-1} a_m^p\right]^{\frac{1}{p}}. \tag{39}$$

(ii) For $\mu_i = \upsilon_j = 1$ ($i, j \in \mathbf{N}$), $\lambda = 1$, $\lambda_1 = \frac{1}{q}$, $\lambda_2 = \frac{1}{p}$, (33) reduces to the following more accurate and extended Mulholland's inequality:

$$\sum_{m=2}^\infty\sum_{n=2}^\infty \frac{a_m b_n}{\ln(\alpha\beta mn)} < \frac{\pi}{\sin(\pi/p)}\left(\sum_{m=2}^\infty \frac{a_m^p}{m^{1-p}}\right)^{\frac{1}{p}}\left(\sum_{n=2}^\infty \frac{b_n^q}{n^{1-q}}\right)^{\frac{1}{q}}, \tag{40}$$

where $\frac{2}{3} \leq \alpha, \beta \leq 1$.

For $p > 1$, $\Psi_\lambda^{1-p}(n) = \frac{\upsilon_{n+1}}{V_n}(\ln \beta V_n)^{p\lambda_2 - 1}$, we define the following normed spaces:

$$l_{p,\Phi_\lambda} := \{a = \{a_m\}_{m=2}^\infty; \|a\|_{p,\Phi_\lambda} < \infty\},$$
$$l_{q,\Psi_\lambda} := \{b = \{b_n\}_{n=2}^\infty; \|b\|_{q,\Psi_\lambda} < \infty\},$$
$$l_{p,\Psi_\lambda^{1-p}} := \{c = \{c_n\}_{n=2}^\infty; \|c\|_{p,\Psi_\lambda^{1-p}} < \infty\}.$$

Assuming that $a = \{a_m\}_{m=2}^\infty \in l_{p,\Phi_\lambda}$ and setting

$$c = \{c_n\}_{n=2}^\infty, \qquad c_n := \sum_{m=2}^\infty \frac{a_m}{\ln^\lambda(\alpha\beta U_m V_n)}, \quad n \in \mathbf{N}\backslash\{1\},$$

we can rewrite (34) as follows:

$$\|c\|_{p,\Psi_\lambda^{1-p}} < B(\lambda_1, \lambda_2)\|a\|_{p,\Phi_\lambda} < \infty,$$

that is, $c \in l_{p,\Psi_\lambda^{1-p}}$.

Definition 2 Define the Mulholland-type operator $T : l_{p,\Phi_\lambda} \to l_{p,\Psi_\lambda^{1-p}}$ as follows: For any $a = \{a_m\}_{m=2}^\infty \in l_{p,\Phi_\lambda}$, there exists a unique representation $Ta = c \in l_{p,\Psi_\lambda^{1-p}}$. Define the formal inner product of Ta and $b = \{b_n\}_{n=2}^\infty \in l_{q,\Psi_\lambda}$ as follows:

$$(Ta, b) := \sum_{n=2}^\infty \left[\sum_{m=2}^\infty \frac{a_m}{\ln^\lambda(\alpha\beta U_m V_n)}\right] b_n. \tag{41}$$

Then we can rewrite (33) and (34) as follows:

$$(Ta, b) < B(\lambda_1, \lambda_2)\|a\|_{p,\Phi_\lambda}\|b\|_{q,\Psi_\lambda}, \tag{42}$$

$$\|Ta\|_{p,\Psi_\lambda^{1-p}} < B(\lambda_1, \lambda_2)\|a\|_{p,\Phi_\lambda}. \tag{43}$$

Define the norm of the operator T as follows:

$$\|T\| := \sup_{a(\neq\theta)\in l_{p,\Phi_\lambda}} \frac{\|Ta\|_{p,\Psi_\lambda^{1-p}}}{\|a\|_{p,\Phi_\lambda}}.$$

Then by (43) we find $\|T\| \leq B(\lambda_1, \lambda_2)$. Since the constant factor in (43) is the best possible, we have

$$\|T\| = B(\lambda_1, \lambda_2). \tag{44}$$

4 Some reverses

In the following, we also set

$$\begin{aligned} \widetilde{\Omega}_\lambda(m) &:= \bigl(1 - \theta(\lambda_2, m)\bigr)\left(\frac{U_m}{\mu_{m+1}}\right)^{p-1}(\ln \alpha U_m)^{p(1-\lambda_1)-1}, \\ \widetilde{F}_\lambda(n) &:= \bigl(1 - \vartheta(\lambda_1, n)\bigr)\left(\frac{V_n}{\upsilon_{n+1}}\right)^{q-1}(\ln \beta V_n)^{q(1-\lambda_2)-1} \quad \bigl(m, n \in \mathbf{N}\backslash\{1\}\bigr). \end{aligned} \tag{45}$$

For $0 < p < 1$ or $p < 0$, we still use the formal symbols $\|a\|_{p,\Phi_\lambda}$, $\|b\|_{q,\Psi_\lambda}$, $\|a\|_{p,\widetilde{\Omega}_\lambda}$, and $\|b\|_{q,\widetilde{F}_\lambda}$, and so on.

Theorem 3 *If* $0 < p < 1$, $\{\mu_m\}_{m=1}^\infty$ *and* $\{\upsilon_n\}_{n=1}^\infty$ *are decreasing,* $U_\infty = V_\infty = \infty$, $\|a\|_{p,\Phi_\lambda} \in \mathbf{R}_+$, *and* $\|b\|_{q,\Psi_\lambda} \in \mathbf{R}_+$, *then we have the following equivalent inequalities with the best possible constant factor* $B(\lambda_1,\lambda_2)$:

$$\sum_{n=2}^\infty \sum_{m=2}^\infty \frac{a_m b_n}{\ln^\lambda(\alpha\beta U_m V_n)} > B(\lambda_1,\lambda_2)\|a\|_{p,\widetilde{\Omega}_\lambda}\|b\|_{q,\Psi_\lambda}, \tag{46}$$

$$\left\{\sum_{n=2}^\infty \frac{\upsilon_{n+1}}{V_n}\ln^{p\lambda_2-1}\beta V_n\left[\sum_{m=2}^\infty \frac{a_m}{\ln^\lambda(\alpha\beta U_m V_n)}\right]^p\right\}^{\frac{1}{p}} > B(\lambda_1,\lambda_2)\|a\|_{p,\widetilde{\Omega}_\lambda}. \tag{47}$$

Proof Using (18) and (17) in the reverses of (25) and (26), since

$$\big(\omega(\lambda_2,m)\big)^{\frac{1}{p}} > \big(B(\lambda_1,\lambda_2)\big)^{\frac{1}{p}}\big(1-\theta(\lambda_2,m)\big)^{\frac{1}{p}} \quad (0 < p < 1),$$
$$\big(\varpi(\lambda_1,n)\big)^{\frac{1}{q}} > \big(B(\lambda_1,\lambda_2)\big)^{\frac{1}{q}} \quad (q < 0),$$

and

$$\frac{1}{(B(\lambda_1,\lambda_2))^{p-1}} > \frac{1}{(\varpi(\lambda_1,n))^{p-1}} \quad (0 < p < 1),$$

we obtain equivalent inequalities (46) and (47).

For $\varepsilon \in (0, p\lambda_1)$, we set $\tilde{\lambda}_1$, $\tilde{\lambda}_2$, $\tilde{a}_m$, and $\tilde{b}_n$ as in (35). Then by (22), (23), and (17) we find

$$\begin{aligned}
\|a\|_{p,\widetilde{\Omega}_\lambda}\|b\|_{q,\Psi_\lambda} &= \left[\sum_{m=2}^\infty \frac{(1-\theta(\lambda_2,m))\mu_{m+1}}{U_m \ln^{1+\varepsilon}\alpha U_m}\right]^{\frac{1}{p}}\left(\sum_{n=2}^\infty \frac{\upsilon_{n+1}}{V_n \ln^{1+\varepsilon}\beta V_n}\right)^{\frac{1}{q}} \\
&= \left(\sum_{m=2}^\infty \frac{\mu_{m+1}}{U_m \ln^{1+\varepsilon}\alpha U_m} - \sum_{m=2}^\infty O\left(\frac{\mu_{m+1}}{U_m \ln^{1+\lambda_2+\varepsilon}\alpha U_m}\right)\right)^{\frac{1}{p}} \\
&\quad \times \left(\sum_{n=2}^\infty \frac{\upsilon_{n+1}}{V_n \ln^{1+\varepsilon}\beta V_n}\right)^{\frac{1}{q}} \\
&= \frac{1}{\varepsilon}\left[\frac{1}{\ln^\varepsilon \alpha(1+\mu_2)} + \varepsilon\big(O(1) - O_1(1)\big)\right]^{\frac{1}{p}}\left[\frac{1}{\ln^\varepsilon \beta(1+\upsilon_2)} + \varepsilon\widetilde{O}(1)\right]^{\frac{1}{q}},
\end{aligned}$$

$$\begin{aligned}
\tilde{I} &:= \sum_{n=2}^\infty \sum_{m=2}^\infty \frac{\tilde{a}_m \tilde{b}_n}{\ln^\lambda(\alpha\beta U_m V_n)} \\
&= \sum_{n=2}^\infty \left[\sum_{m=2}^\infty \frac{1}{\ln^\lambda(\alpha\beta U_m V_n)} \frac{\mu_{m+1}\ln^{\tilde{\lambda}_2}\beta V_n}{U_m \ln^{1-\tilde{\lambda}_1}\alpha U_m}\right]\frac{\upsilon_{n+1}}{V_n \ln^{\varepsilon+1}\beta V_n} \\
&= \sum_{n=2}^\infty \varpi(\tilde{\lambda}_1,n)\frac{\upsilon_{n+1}}{V_n \ln^{\varepsilon+1}\beta V_n} \le B(\tilde{\lambda}_1,\tilde{\lambda}_2)\sum_{n=2}^\infty \frac{\upsilon_{n+1}}{V_n \ln^{\varepsilon+1}\beta V_n} \\
&= \frac{1}{\varepsilon}B(\tilde{\lambda}_1,\tilde{\lambda}_2)\left[\frac{1}{\ln^\varepsilon \beta(1+\upsilon_2)} + \varepsilon\widetilde{O}(1)\right].
\end{aligned}$$

If there exists a positive constant $K \geq B(\lambda_1, \lambda_2)$ such that (46) is valid when replacing $B(\lambda_1, \lambda_2)$ by K, then, in particular, we have $\varepsilon\tilde{I} > \varepsilon K\|\tilde{a}\|_{p,\tilde{\Omega}_\lambda}\|\tilde{b}\|_{q,\Psi_\lambda}$, namely

$$\begin{aligned} &B\left(\lambda_1 - \frac{\varepsilon}{p}, \lambda_2 + \frac{\varepsilon}{p}\right)\left[\frac{1}{\ln^\varepsilon \beta(1+\upsilon_2)} + \varepsilon\widetilde{O}(1)\right] \\ &\quad > K\left[\frac{1}{\ln^\varepsilon \alpha(1+\mu_2)} + \varepsilon\bigl(O(1) - O_1(1)\bigr)\right]^{\frac{1}{p}} \\ &\qquad \times \left[\frac{1}{\ln^\varepsilon \beta(1+\upsilon_2)} + \varepsilon\widetilde{O}(1)\right]^{\frac{1}{q}}. \end{aligned}$$

It follows that $B(\lambda_1, \lambda_2) \geq K$ ($\varepsilon \to 0^+$). Hence, $K = B(\lambda_1, \lambda_2)$ is the best possible constant factor of (46).

The constant factor $B(\lambda_1, \lambda_2)$ in (47) is still the best possible. Otherwise, we would reach a contradiction by the reverse of (36) that the constant factor in (46) is not the best possible. □

Remark 2 For $\alpha = \beta = 1$, setting

$$\begin{aligned} \tilde{\theta}(\lambda_2, m) &= \frac{1}{B(\lambda_1, \lambda_2)} \frac{\ln^{\lambda_2}(1+\upsilon_2)}{\lambda_2[1 + \frac{\ln(1+\theta(m)\upsilon_2)}{\ln U_m}]^\lambda} \frac{1}{\ln^{\lambda_2} U_m} \\ &= O\left(\frac{1}{\ln^{\lambda_2} U_m}\right) \in (0,1) \quad \bigl(\theta(m) \in (0,1)\bigr), \\ \tilde{\varphi}_\lambda(m) &:= \bigl(1 - \tilde{\theta}(\lambda_2, m)\bigr)\left(\frac{U_m}{\mu_{m+1}}\right)^{p-1}(\ln U_m)^{p(1-\lambda_1)-1}, \end{aligned}$$

it is evident that (46) and (47) are extensions of the following equivalent inequalities:

$$\sum_{n=2}^\infty \sum_{m=2}^\infty \frac{a_m b_n}{\ln^\lambda(U_m V_n)} > B(\lambda_1, \lambda_2)\|a\|_{p,\tilde{\varphi}_\lambda}\|b\|_{q,\psi_\lambda}, \tag{48}$$

$$\left\{\sum_{n=2}^\infty \frac{\upsilon_{n+1}}{V_n} \ln^{p\lambda_2 - 1} V_n \left[\sum_{m=2}^\infty \frac{a_m}{\ln^\lambda(U_m V_n)}\right]^p\right\}^{\frac{1}{p}} > B(\lambda_1, \lambda_2)\|a\|_{p,\tilde{\varphi}_\lambda}, \tag{49}$$

where the constant factor $B(\lambda_1, \lambda_2)$ is still the best possible.

Theorem 4 *If $p < 0$, $\{\mu_m\}_{m=1}^\infty$ and $\{\upsilon_n\}_{n=1}^\infty$ are decreasing, $U_\infty = V_\infty = \infty$, $\|a\|_{p,\Phi_\lambda} \in \mathbf{R}_+$, and $\|b\|_{q,\Psi_\lambda} \in \mathbf{R}_+$, then we have the following equivalent inequalities with the best possible constant factor $B(\lambda_1, \lambda_2)$:*

$$\sum_{n=1}^\infty \sum_{m=1}^\infty \frac{a_m b_n}{\ln^\lambda(\alpha\beta U_m V_n)} > B(\lambda_1, \lambda_2)\|a\|_{p,\Phi_\lambda}\|b\|_{q,\widetilde{F}_\lambda}, \tag{50}$$

$$\begin{aligned} J_2 &:= \left\{\sum_{n=1}^\infty \frac{\upsilon_{n+1} \ln^{p\lambda_2 - 1} \beta V_n}{(1 - \vartheta(\lambda_1, n))^{p-1} V_n}\left[\sum_{m=1}^\infty \frac{a_m}{\ln^\lambda(\alpha\beta U_m V_n)}\right]^p\right\}^{\frac{1}{p}} \\ &\quad > B(\lambda_1, \lambda_2)\|a\|_{p,\Phi_\lambda}. \end{aligned} \tag{51}$$

Proof Using (16) and (19) in the reverses of (25) and (26), since

$$\begin{aligned}
&\big(\omega(\lambda_2,m)\big)^{\frac{1}{p}} > \big(B(\lambda_1,\lambda_2)\big)^{\frac{1}{p}} \quad (p<0),\\
&\big(\varpi(\lambda_1,n)\big)^{\frac{1}{q}} > \big(B(\lambda_1,\lambda_2)\big)^{\frac{1}{q}}\big(1-\vartheta(\lambda_1,n)\big)^{\frac{1}{q}} \quad (0<q<1),
\end{aligned}$$

and

$$\left[\frac{1}{(B(\lambda_1,\lambda_2))^{p-1}(1-\vartheta(\lambda_1,n))^{p-1}}\right]^{\frac{1}{p}} > \left[\frac{1}{(\varpi(\lambda_1,n))^{p-1}}\right]^{\frac{1}{p}} \quad (p<0),$$

we obtain equivalent inequalities (50) and (51).

For $\varepsilon \in (0,q\lambda_2)$, we set $\tilde{\lambda}_1 = \lambda_1 + \frac{\varepsilon}{q}$ (>0), $\tilde{\lambda}_2 = \lambda_2 - \frac{\varepsilon}{q}$ $(\in (0,1))$, and

$$\begin{aligned}
&\tilde{a}_m := \frac{\mu_{m+1}}{U_m}\ln^{\tilde{\lambda}_1-\varepsilon-1}\alpha U_m = \frac{\mu_{m+1}}{U_m}\ln^{\lambda_1-\frac{\varepsilon}{p}-1}\alpha U_m,\\
&\tilde{b}_n = \frac{\upsilon_{n+1}}{V_n}\ln^{\tilde{\lambda}_2-1}\beta V_n = \frac{\upsilon_{n+1}}{V_n}\ln^{\lambda_2-\frac{\varepsilon}{q}-1}\beta V_n.
\end{aligned}$$

Then by (22), (23), and (16) we have

$$\begin{aligned}
\|\tilde{a}\|_{p,\Phi_\lambda}\|\tilde{b}\|_{q,\widetilde{F}_\lambda} &= \left(\sum_{m=2}^{\infty}\frac{\mu_{m+1}}{U_m\ln^{\varepsilon+1}\alpha U_m}\right)^{\frac{1}{p}}\left[\sum_{n=2}^{\infty}\frac{(1-\vartheta(\lambda_1,n))\upsilon_{n+1}}{V_n\ln^{\varepsilon+1}\beta V_n}\right]^{\frac{1}{q}}\\
&= \left(\sum_{m=2}^{\infty}\frac{\mu_{m+1}}{U_m\ln^{\varepsilon+1}\alpha U_m}\right)^{\frac{1}{p}}\\
&\quad\times\left[\sum_{n=2}^{\infty}\frac{\upsilon_{n+1}}{V_n\ln^{\varepsilon+1}\beta V_n} - \sum_{n=2}^{\infty}O\left(\frac{\upsilon_{n+1}}{V_n\ln^{1+(\lambda_1+\varepsilon)}\beta V_n}\right)\right]^{\frac{1}{q}}\\
&= \frac{1}{\varepsilon}\left[\frac{1}{\ln^{\varepsilon}\alpha(1+\mu_2)}+\varepsilon O(1)\right]^{\frac{1}{p}}\\
&\quad\times\left[\frac{1}{\ln^{\varepsilon}\beta(1+\upsilon_2)}+\varepsilon\big(\widetilde{O}(1)-O_1(1)\big)\right]^{\frac{1}{q}},
\end{aligned}$$

$$\begin{aligned}
\tilde{I} &= \sum_{m=2}^{\infty}\sum_{n=2}^{\infty}\frac{\tilde{a}_m\tilde{b}_n}{\ln^{\lambda}(\alpha\beta U_m V_n)}\\
&= \sum_{m=2}^{\infty}\left[\sum_{n=2}^{\infty}\frac{\ln^{\tilde{\lambda}_1}\alpha U_m}{\ln^{\lambda}(\alpha\beta U_m V_n)}\frac{\upsilon_{n+1}}{V_n}\ln^{\tilde{\lambda}_2-1}\beta V_n\right]\frac{\mu_{m+1}}{U_m\ln^{\varepsilon+1}\alpha U_m}\\
&= \sum_{m=2}^{\infty}\omega(\tilde{\lambda}_2,m)\frac{\mu_{m+1}}{U_m\ln^{\varepsilon+1}\alpha U_m}\\
&\le B(\tilde{\lambda}_1,\tilde{\lambda}_2)\sum_{n=2}^{\infty}\frac{\mu_{m+1}}{U_m\ln^{\varepsilon+1}\alpha U_m}\\
&= \frac{1}{\varepsilon}B(\tilde{\lambda}_1,\tilde{\lambda}_2)\left[\frac{1}{\ln^{\varepsilon}\alpha(1+\mu_2)}+\varepsilon O(1)\right].
\end{aligned}$$

If there exists a positive constant $K \geq B(\lambda_1, \lambda_2)$ such that (50) is valid when replacing $B(\lambda_1, \lambda_2)$ by K, then, in particular, we have $\varepsilon\tilde{I} > \varepsilon K\|\tilde{a}\|_{p,\Phi_\lambda}\|\tilde{b}\|_{q,\widetilde{F}_\lambda}$, namely

$$\begin{aligned} &B\left(\lambda_1 + \frac{\varepsilon}{q}, \lambda_2 - \frac{\varepsilon}{q}\right)\left[\frac{1}{\ln^\varepsilon \alpha(1+\mu_2)} + \varepsilon O(1)\right] \\ &\quad > K\left[\frac{1}{\ln^\varepsilon \alpha(1+\mu_2)} + \varepsilon O(1)\right]^{\frac{1}{p}} \\ &\qquad \times \left[\frac{1}{\ln^\varepsilon \beta(1+\upsilon_2)} + \varepsilon\big(\widetilde{O}(1) - O_1(1)\big)\right]^{\frac{1}{q}}. \end{aligned}$$

It follows that $B(\lambda_1, \lambda_2) \geq K$ ($\varepsilon \to 0^+$). Hence, $K = B(\lambda_1, \lambda_2)$ is the best possible constant factor of (50).

Similarly to the reverse of (29), we still can find that

$$I \geq J_2\|b\|_{q,\widetilde{F}_\lambda}. \tag{52}$$

Hence, the constant factor $B(\lambda_1, \lambda_2)$ in (51) is still the best possible. Otherwise, we would reach a contradiction by (52) that the constant factor in (50) is not the best possible. □

Remark 3 For $\alpha = \beta = 1$, setting

$$\begin{aligned} \tilde{\vartheta}(\lambda_1, n) &= \frac{1}{B(\lambda_1, \lambda_2)} \frac{\ln^{\lambda_1}(1+\mu_2)}{\lambda_1[1 + \frac{\ln(1+\vartheta(n)\mu_2)}{\ln V_n}]^{\lambda}} \frac{1}{\ln^{\lambda_1} V_n} \\ &= O\left(\frac{1}{\ln^{\lambda_1} V_n}\right) \in (0,1) \quad \big(\vartheta(n) \in (0,1)\big), \\ \tilde{\psi}_\lambda(n) &:= \big(1 - \tilde{\vartheta}(\lambda_1, n)\big)\left(\frac{V_n}{\upsilon_{n+1}}\right)^{q-1} (\ln V_n)^{q(1-\lambda_2)-1}, \end{aligned}$$

it is evident that (50) and (51) are extensions of the following equivalent inequalities:

$$\sum_{n=2}^{\infty}\sum_{m=2}^{\infty} \frac{a_m b_n}{\ln^\lambda(U_m V_n)} > B(\lambda_1, \lambda_2)\|a\|_{p,\varphi_\lambda}\|b\|_{q,\tilde{\psi}_\lambda}, \tag{53}$$

$$\left\{\sum_{n=2}^{\infty} \frac{\upsilon_{n+1}\ln^{p\lambda_2-1} V_n}{(1-\tilde{\vartheta}(\lambda_1, n))^{p-1} V_n}\left[\sum_{m=2}^{\infty} \frac{a_m}{\ln^\lambda(U_m V_n)}\right]^p\right\}^{\frac{1}{p}} > B(\lambda_1, \lambda_2)\|a\|_{p,\varphi_\lambda}, \tag{54}$$

where the constant factor $B(\lambda_1, \lambda_2)$ is still the best possible.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
BY carried out the mathematical studies, participated in the sequence alignment, and drafted the manuscript. QC participated in the design of the study and performed the numerical analysis. Both authors read and approved the final manuscript.

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