International Journal of the Physical Sciences Vol. 6(34), pp. 7721 - 7728, 16 December, 2011
Available online at http://www.academicjournals.org/IJPS
ISSN 1992 - 1950

Full Length Research Paper

An approximate analytical solution of nonlinear fractional diffusion equation by homotopy analysis method

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Accepted 14 November, 2011

In the present paper, the approximate analytical solution of a nonlinear diffusion equation with fractional time derivative α ($0<\alpha<1$) and with the diffusion term as u^n ($n \neq 0$) are obtained with the help of analytical method of nonlinear problem called the Homotopy Analysis Method (HAM). By using initial value, the explicit solution of the equation for different particular cases have been derived which demonstrate the effectiveness, validity, potentiality and reliability of the method in reality. Numerical results of the fast and slow diffusion for different particular cases are presented graphically. The numerical solutions show that only a few iterations are needed to obtain accurate approximate solutions.

Key words: Partial differential equation, nonlinear fractional diffusion equation, Brownian motion, homotopy analysis method.

INTRODUCTION

Ovsiannikov (1959) investigated the solution of the nonlinear diffusion equation,

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(g(u)\frac{\partial u}{\partial x}\right) \tag{1}$$

by symmetry method, where $u(x,t)$ represents mass concentration. This type of equation appears in plasma physics, kinetic theory of gases, transport in porous medium, etc. In many cases $g(u)$ is approximated as $g(u)=u^n$. Then, Equation 1 is called fast diffusion equation for $-2<n<0$ and slow diffusion equation for $n>0$. In the first case, the spread of mass is faster than the linear case $n=0$ and in the second case it is slower.

AMS/MSC Classification codes: 02.30.Jr, 02.60.-x, 05.30.Pr, 05.40.Jc

Gandarias (2001) investigated the fast diffusion equation for $n=-1$. Later, Popovych et al. (2007) developed his idea and obtained new wider classes of potential non-classical symmetries of the fast diffusion equation.

Guo and Guo (2001) studied the large time behaviors of the global and non-global solutions of the Cauchy problem for a fast diffusion equation with source. Recently, Fa and Lenzi (2007) have used the Green function method to find the solution of the diffusion equation:

$$\frac{\partial \rho(x,t)}{\partial t} = \frac{\partial}{\partial x}\left(D(x,t)\frac{\partial \rho(x,t)}{\partial x}\right) \tag{2}$$

in a finite interval with diffusion coefficient $D(x,t)=D(t)|x|^{-\theta}$ and initial condition $\rho(x,0)=\rho_0(x)$ subject to absorbing boundaries. But to the best of the authors knowledge, the diffusion equation with fractional time derivative:

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial}{\partial x}\left(u^n \frac{\partial u}{\partial x}\right) \quad (3)$$

with the initial condition:

$$u(x,0) = f(x) \quad (4)$$

The equation 4 condition has not yet been solved by any researcher. The aforementioned type of anomalous diffusion is a ubiquitous phenomenon in nature, and it appears in different branches of science and engineering. Most of the nonlinear problems do not have a precise analytical solution; specially, it is very hard to get it for the fractional nonlinear equations. So, these types of equations should be solved by approximate analytical methods.

The approach used is based on homotopy analysis method (HAM) which is first proposed by Chinese Mathematician Shijun Liao (1992) in our Ph.D. dissertation. This extremely simple and highly powerful algorithm gives the numerical results compatible with those obtained by making use of Adomian's polynomials which are too complex and lengthy to evaluate. In 2003, Liao claimed that HAM is a general analytical approach to get series solutions of various types of linear and nonlinear problem. In reality, it is seen that HAM provides a simple way to ensure the convergence of the series solution and therefore, it is valid for strongly nonlinear problems. HAM, which is based on homotopy and a fundamental concept of topology, has a freedom in choosing initial approximations and auxiliary linear operators which often helps to transfer the complicated nonlinear problem to its simpler form. This method has been successfully applied (Liao, 1998; Hayat et al., 2004a, b, c; Liao, 2006; Wu and Liao, 2004) to solve the different nonlinear problems. The basic idea of the method and its applications in Science and Engineering for solving nonlinear problems and its comparison with the other analytical techniques can be found in the monograph of Liao (1992).

In this article, HAM is used to solve the nonlinear diffusion equation with fractional time derivative, where the domain of the space variable is unbounded. The approximate analytical solution of probability density function $u(x,t)$ for different fractional Brownian motions and also for the standard motions are derived successfully and presented graphically.

PRELIMINARIES AND NOTATIONS

Here, we give some definitions and properties of the fractional calculus and homotopy-derivative which are used further in this paper.

Fractional calculus

The following properties can found in Podlubny (1999):

Definition 1

A real function $f(t), t > 0$, is said to be in the space C_μ, $\mu \in \Re$, if there exists a real number $p > \mu$, such that $f(t) = t^p f_1(t)$, where $f_1(t) \in C(0,\infty)$, and it is said to be in the space $C_\mu{}^n$ if and only if $f^{(n)} \in C_\mu, n \in N$.

Definition 2

The Riemann-Liouville fractional integral operator (J^α) of order $\alpha \geq 0$, of a function $f \in C_\mu$, $\mu \geq -1$, is defined as:

$$J^\alpha f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\xi)^{\alpha-1} f(\xi)\, d\xi,\ \alpha > 0,\ t > 0, \quad (5)$$

$$J^0 f(t) = f(t).$$

where $\Gamma(\alpha)$ is the well-known gamma function. Some of the properties of the operator J^α, which we will need here, are as follows.

For $f \in C_\mu, \mu \geq -1, \alpha, \beta \geq 0$ and $\gamma \geq -1$:

1. $J^\alpha J^\beta f(t) = J^{\alpha+\beta} f(t),$
2. $J^\alpha J^\beta f(t) = J^\beta J^\alpha f(t),$
3. $J^\alpha t^\gamma = \dfrac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} t^{\alpha+\gamma},$

Definition 3

The fractional derivative (D^α) of $f(t)$, in the Caputo sense is defined as:

$$D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-\xi)^{n-\alpha-1} f^{(n)}(\xi)\, d\xi, \quad (6)$$

for $n-1 < \alpha < n,\ n \in N,\ t > 0,\ f \in C_{-1}^n.$

The following are two basic properties of the Caputo fractional derivative (Gorenflo and Mainardi, 1997):

1. Let $f \in C_{-1}^n$, $n \in N$, then $D^\alpha f, 0 \leq \alpha \leq n$ is well

defined and $D^{\alpha} f \in C_{-1}$.

2. Let $n-1 \le \alpha \le n, n \in N$ and $f \in C_{\mu}^{n}$, $\mu \ge -1$. Then:

$$(J^{\alpha} D^{\alpha}) f(t) = f(t) - \sum_{k=0}^{n-1} f^{(k)}(0^{+}) \frac{t^{k}}{k!}. \qquad (7)$$

Homotopy-derivative

The following properties can be found in Liao (2009).

Definition 4

Let ϕ be a function of the homotopy parameter q, then:

$$D_m(\phi) = \frac{1}{m!} \frac{\partial^m \phi}{\partial q^m}\bigg|_{q=0} \qquad (8)$$

is called the *m*th-order homotopy-derivative of ϕ, where $m \ge 0$ is an integer.

Properties for homotopy-series:

$$\phi_1 = \sum_{i=0}^{\infty} u_i q^i, \quad \phi_2 = \sum_{i=0}^{\infty} v_i q^i$$

it holds:

1. $D_m(\phi_1) = u_m$.
2. $D_m(q\phi_1) = D_{m-1}(\phi_1)$.
3. $D_m(\phi_1 \phi_2) = \sum_{i=0}^{m} \phi_{1,i}\, \phi_{2,m-i}$.
4. If L be a linear operator independent of the homotopy parameter q. For homotopy series, then $D_m(L\phi_1) = L\, D_m(\phi_1)$.
5. If f and g be functions independent of the homotopy parameter q, then $D_m(f\phi_1 \pm g\phi_2) = f\, D_m(\phi_1) \pm g\, D_m(\phi_2)$.

SOLUTION OF THE PROBLEM BY HAM

In the present paper, the nonlinear fractional diffusion equation:

$$\frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}} = \frac{\partial}{\partial x}\left(u^n \frac{\partial u}{\partial x} \right), \qquad 0 < \alpha \le 1 \qquad (9)$$

is considered with the initial condition:

$$u(x,0) = f(x) \qquad (10)$$

To solve Equation 9 by HAM, we choose the initial approximation:

$$u_0(x,t) = f(x), \qquad (11)$$

and the linear operator,

$$L[\phi(x,t;q)] = \frac{\partial^{\alpha} \phi(x,t;q)}{\partial t^{\alpha}}, \quad 0 < \alpha \le 1 \qquad (12)$$

with the property:

$$L[c] = 0, \qquad (13)$$

where c is integral constant. Furthermore, Equation 9 suggests that we define an equation of nonlinear operator as:

$$N[\phi(x,t;q)] = \frac{\partial^{\alpha} \phi(x,t;q)}{\partial t^{\alpha}} - \frac{\partial}{\partial x}\left((\phi(x,t;q))^n \frac{\partial \phi(x,t;q)}{\partial x} \right) \qquad (14)$$

Now, we construct the zeroth-order deformation equation:

$$(1-q) L[\phi(x,t;q) - u_0(x,t)] = q \hbar N[\phi(x,t;q)], \qquad (15)$$

Obviously, when $q = 0$ and $q = 1$:

$$\phi(x,t;0) = u_0(x,t) \text{ and } \phi(x,t;1) = u(x,t). \qquad (16)$$

Therefore, as the embedding parameter q increases from zero to unity, $\phi(x,t;q)$ varies from the initial guess $u_0(x,t)$ to the solution $u(x,t)$. Expanding $\phi(x,t;q)$ in Taylor series with respect to q one has:

$$\phi(x,t;q) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t) q^m, \qquad (17)$$

where,

$$u_m(x,t) = \frac{1}{m!} \frac{\partial^m \phi(x,t;q)}{\partial q^m}\bigg|_{q=0}.$$

If the auxiliary linear operator, the initial guess and the auxiliary parameter $\hbar$ is properly chosen, the above series is convergent at $q=1$, then one has:

$$u(x,t)=u_0(x,t)+\sum_{m=1}^{\infty}u_m(x,t)\ , \tag{18}$$

which must be one of the solution of the original nonlinear equation, as proved by Liao (2003). Now we define the vector:

$$\vec{u}_l(x,t)=\{u_0(x,t),u_1(x,t),\ldots\ldots,u_l(x,t)\}, \tag{19}$$

So the mth-order deformation equation is:

$$L[u_m(x,t)-\chi_m\,u_{m-1}(x,t)]=\hbar\,R_m\left(\vec{u}_{m-1}(x,t)\right), \tag{20}$$

with the initial condition:

$$u_m(x,0)=0\,, \tag{21}$$

where,

$$R_m\left(\vec{u}_{m-1}(x,t)\right)=\frac{1}{(m-1)!}\left.\frac{\partial^{m-1}N[\phi(x,t;q)]}{\partial q^{m-1}}\right|_{q=0}$$

and

$$\chi_m=\begin{cases}0, & m\le 1,\\ 1, & m>1.\end{cases}$$

Now, the solution of the *m*th-order deformation Equation 20 for $m\ge 1$ becomes:

$$u_m(x,t)=\chi_m\,u_{m-1}(x,t)+\hbar\,J_t^{\alpha}\left[R_m\left(\vec{u}_{m-1}(x,t)\right)\right]+c\,, \tag{22}$$

where c is the integration constant, which is determined by the initial condition of Equation 21.

$$u_1(x,t)=-\hbar\left[\left(f(x)\right)^n f^{(2)}(x)+n\left(f(x)\right)^{n-1}\left(f'(x)\right)^2\right]\frac{t^{\alpha}}{\Gamma(\alpha+1)}, \tag{23}$$

$$\begin{aligned}u_2(x,t)=&-\hbar(\hbar+1)\left[\left(f(x)\right)^n f^{(2)}(x)+n\left(f(x)\right)^{n-1}\left(f'(x)\right)^2\right]\frac{t^{\alpha}}{\Gamma(\alpha+1)}+\hbar^2\left[\left(f(x)\right)^{2n} f^{(4)}(x)\right.\\&+6n\left(f(x)\right)^{2n-1}f'(x)f^{(3)}(x)+4n\left(f(x)\right)^{2n-1}\left(f^{(2)}(x)\right)^2+7n(2n-1)\left(f(x)\right)^{2n-2}\left(f'(x)\right)^2 f^{(2)}(x)\\&\left.+2n(n-1)(2n-1)\left(f(x)\right)^{2n-3}\left(f'(x)\right)^4\right]\frac{t^{2\alpha}}{\Gamma(2\alpha+1)},\end{aligned} \tag{24}$$

$$\begin{aligned}u_3(x,t)=&-\hbar(\hbar+1)^2\left[\left(f(x)\right)^n f^{(2)}(x)+n\left(f(x)\right)^{n-1}\left(f'(x)\right)^2\right]\frac{t^{\alpha}}{\Gamma(\alpha+1)}+2\hbar^2(\hbar+1)\left[\left(f(x)\right)^{2n} f^{(4)}(x)\right.\\&+6n\left(f(x)\right)^{2n-1}f'(x)f^{(3)}(x)+4n\left(f(x)\right)^{2n-1}\left(f^{(2)}(x)\right)^2+7n(2n-1)\left(f(x)\right)^{2n-2}\left(f'(x)\right)^2 f^{(2)}(x)\\&\left.+2n(n-1)(2n-1)\left(f(x)\right)^{2n-3}\left(f'(x)\right)^4\right]\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}-\hbar^3\left[6n(n-1)^2(2n-1)(3n-4)\left(f(x)\right)^{3n-5}\left(f'(x)\right)^6\right.\\&+3n(n-1)(78n^2-103n+32)\left(f(x)\right)^{3n-4}\left(f'(x)\right)^2\left(f^{(2)}(x)\right)^2+n(294n^2-353n+102)\\&\left(f(x)\right)^{3n-3}\left(f'(x)\right)^2\left(f^{(2)}(x)\right)^2+2n(79n^2-88n+24)\left(f(x)\right)^{3n-3}\left(f'(x)\right)^3 f^{(3)}(x)+2n(20n-9)\\&\left(f(x)\right)^{3n-2}\left(f^{(2)}(x)\right)^3+2n(93n-38)\left(f(x)\right)^{3n-2}f'(x)f^{(2)}(x)f^{(3)}(x)+n(59n-22)\\&\left(f(x)\right)^{3n-2}\left(f'(x)\right)^2 f^{(4)}(x)+14n\left(f(x)\right)^{3n-1}\left(f^{(3)}(x)\right)^2+23n\left(f(x)\right)^{3n-1}f^{(2)}(x)f^{(4)}(x)\\&+12n\left(f(x)\right)^{3n-1}f'(x)f^{(5)}(x)+n\left(f(x)\right)^{3n}f^{(6)}(x)+\frac{n\Gamma(2\alpha+1)}{\left(\Gamma(\alpha+1)\right)^2\Gamma(3\alpha+1)}\left\{\frac{3}{2}n^2(n-1)(3n-4)\right.\\&\left(f(x)\right)^{3n-5}\left(f'(x)\right)^6+\frac{3}{2}n(n-1)(15n-4)\left(f(x)\right)^{3n-4}\left(f'(x)\right)^4 f^{(2)}(x)+4n(2n-1)\left(f(x)\right)^{3n-3}\\&\left(f'(x)\right)^3 f^{(3)}(x)+\frac{1}{2}(51n^2-29n+2)\left(f(x)\right)^{3n-3}\left(f'(x)\right)^2\left(f^{(2)}(x)\right)^2+\frac{(7n-1)}{2}\left(f(x)\right)^{3n-2}\left(f^{(2)}(x)\right)^3\\&+2(6n-1)\left(f(x)\right)^{3n-2}f'(x)f^{(2)}(x)f^{(3)}(x)+n\left(f(x)\right)^{3n-2}\left(f'(x)\right)^2 f^{(4)}(x)\\&\left.\left.+\left(f(x)\right)^{3n-1}\left(f^{(3)}(x)\right)^2+\left(f(x)\right)^{3n-1}f^{(2)}(x)f^{(4)}(x)\right\}\right]\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}\end{aligned} \tag{25}$$

Proceeding in this manner, the components $u_k, k \geq 0$ of the HAM can be completely obtained and the series solutions are thus entirely obtained.

Finally, we approximate the analytical solution $u(x,t)$ by truncated series:

$$u(x,t) = \lim_{N \to \infty} \psi_N(x,t) \tag{26}$$

where, $\psi_N(x,t) = \sum_{k=0}^{N-1} u_k(x,t)\ ,\ N \geq 1.$

The above series solutions generally converge very rapidly. A classical approach of convergence of this type of series is already presented by Abbaoui and Cherruault (1996).

Particular cases

Case 1

For $n=1, \alpha=1$, Equation 3 reduces to the standard equation:

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial}{\partial x}\left(u(x,t)\frac{\partial u(x,t)}{\partial x}\right),$$

whose solution is:

$$u(x,t) = u_0 + u_1 + u_2 + u_3 + u_4 + \cdots\cdots$$

$$= x + t\ ; \tag{27}$$

Case 2

For $n=2, \alpha=1$, Equation 3 becomes:

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial}{\partial x}\left((u(x,t))^2 \frac{\partial u(x,t)}{\partial x}\right),$$

whose solution is:

$$u(x,t) = x + 2xt + 6xt^2 + 20xt^3 + 70xt^4 + \cdots\cdots$$

$$= \frac{x}{\sqrt{1-4t}}; \tag{28}$$

Case 3

For $n=-1, \alpha=1$ Equation 3 becomes:

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial}{\partial x}\left((u(x,t))^{-1}\frac{\partial u(x,t)}{\partial x}\right),$$

whose solution is:

$$u(x,t) = x\left[1 - \left(\frac{t}{x^3}\right) - 6\left(\frac{t}{x^3}\right)^2 - 91\left(\frac{t}{x^3}\right)^3 - 1905\left(\frac{t}{x^3}\right)^4 - \ldots\ldots\right],$$

$$= x - \left(\frac{t}{x^2}\right), \text{ if } \left|\frac{t}{x^3}\right| << 1 \tag{29}$$

NUMERICAL RESULTS AND DISCUSSION

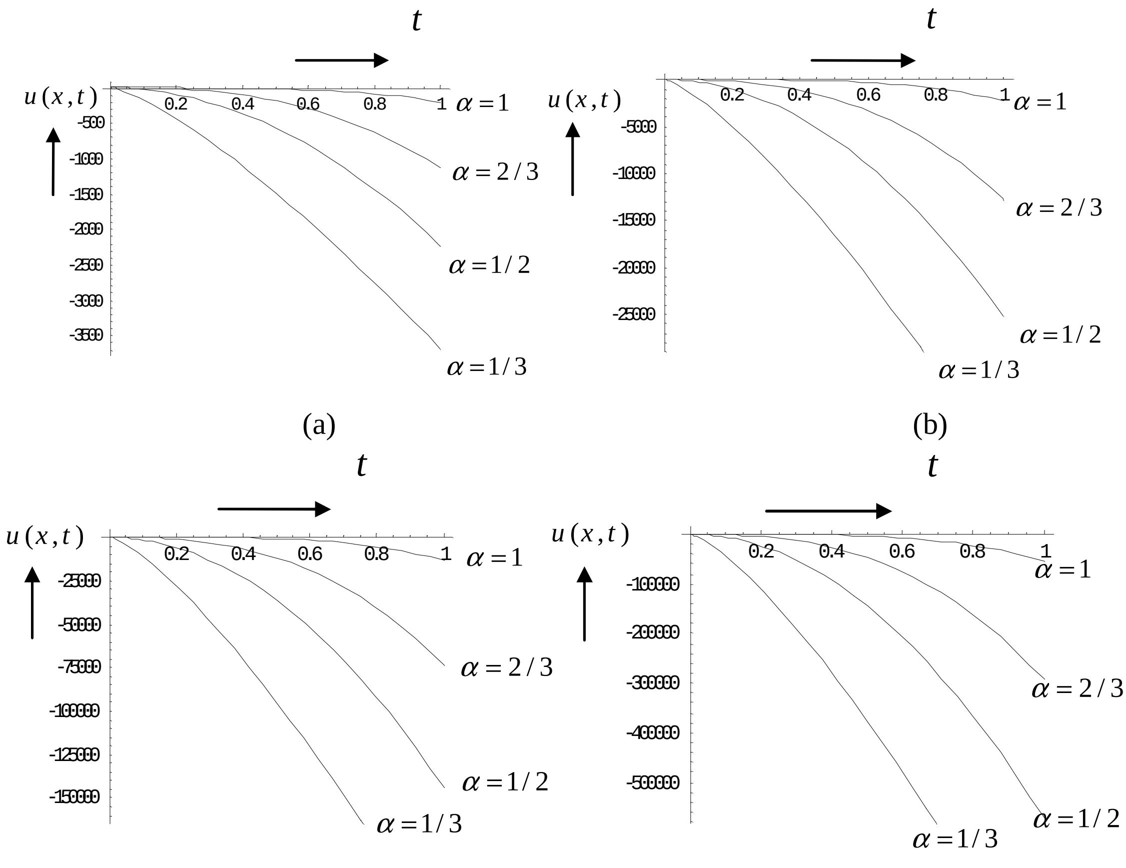
Here, the numerical results of $u(x,t)$ for different fractional Brownian motions $\alpha = \frac{1}{3}, \frac{1}{2}, \frac{2}{3}$ are also for the standard motion, $\alpha = 1$ are calculated for various values of time t with the various degree of the diffusion term at $x=1$ which are depicted through Figures 1 and 2 .

It is seen from the figures that the rate of increase of $u(x,t)$ with time decreases with the increase of α which conforms with the exponentially decay of regular Brownian motions. This result is in complete agreement with the results of Das (2009) and Giona and Roman (1992).

For positive values of the power of diffusivity coefficient, that is, for $0<n<2$ (Figure 3) anomalous diffusion has been observed where sub-diffusion occurs in the range of $0<n<0.6$ (Figure 4) and super-diffusion in the range $1.4<n<2$ (Figure 5). This phenomenon can be demonstrated from the normal length scale analysis. As revealed by Figure 3, a threshold is being observed to exist in the region of $0.6<n<1.4$, that is, after the occurrence of sub-diffusion and before that of super diffusion, but in the range of $-2<n<0$ (Figure 6), where no sub-diffusion or super-diffusion occurs, demarcation has been observed and $u(x,t)$ describes the asymptotic behavior with t.

Conclusion

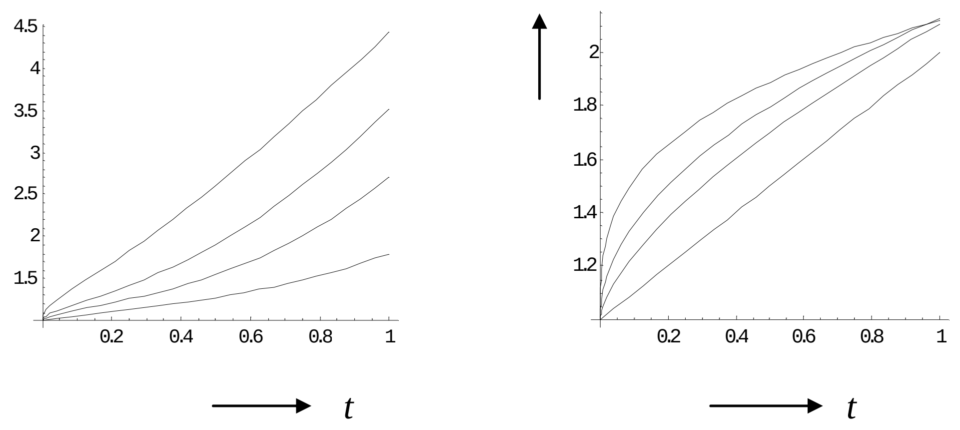
Homotopy analysis method is a powerful method of



(a) (b)

(c) (d)

Figure 1. Plot of $u(x,t)$ versus t at: (a) $n = -\frac{1}{2}$, (b) $n = -1$, (c) $n = -\frac{3}{2}$ and (d) $n = -2$ and $x = 1$ for different values of α.



(a) (b)

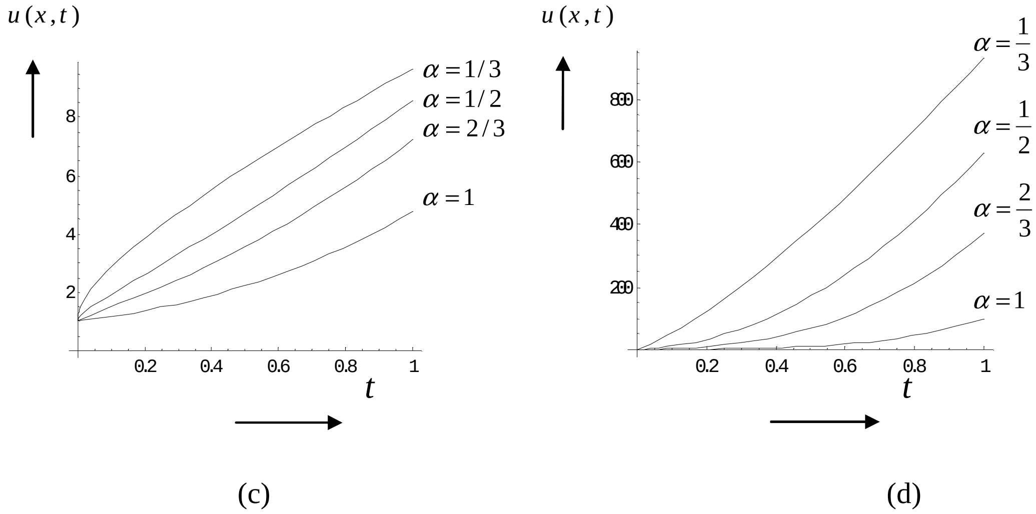


Figure 2. Plot of $u(x,t)$ versus t at: (a) $n=\frac{1}{2}$, (b) $n=1$, (c) $n=\frac{3}{2}$ and (d) $n=2$ and $x=1$ for different values of α.

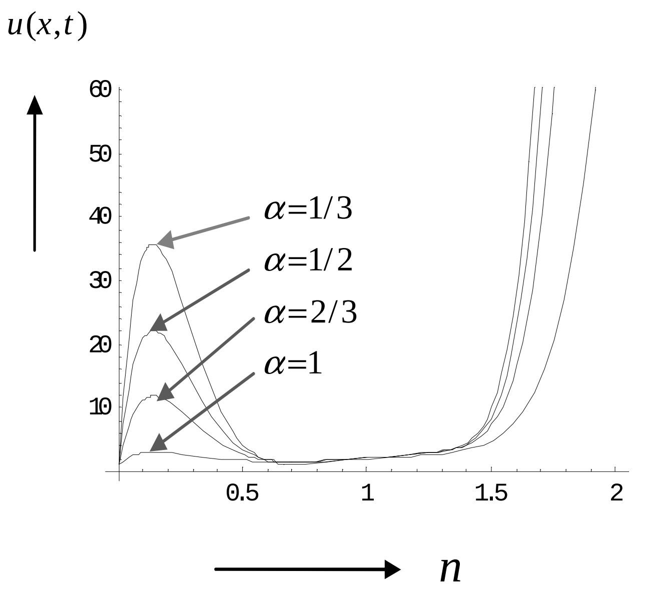


Figure 3. Plot of $u(x,t)$ versus positive values of n for various values of α at $x=1$ and $t=1$.

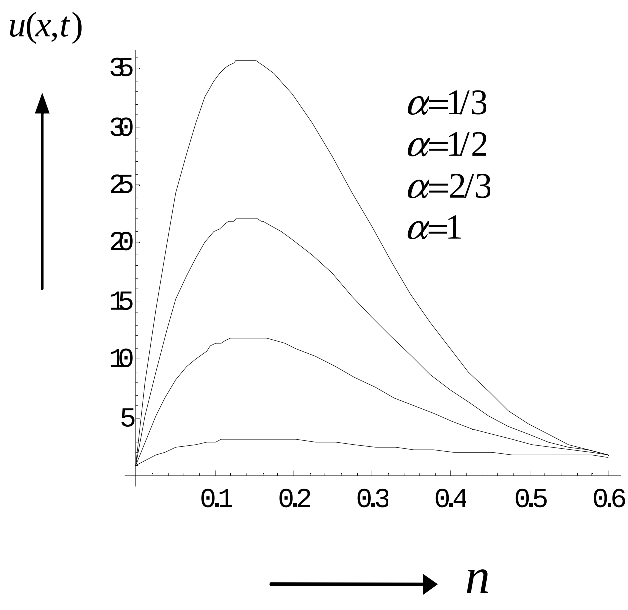


Figure 4. Plot of $u(x,t)$ versus positive values of $0<n<0.6$ for various values of α at $x=1$ and $t=1$.

solving nonlinear equations since it ensures exact solution as an infinite series of the functions. Since this series is quickly convergent and truncated series can be calculated, as shown in the article, so it is easy to find the approximate analytical solution of the nonlinear problem with a finite number of terms of the series solution. Here, we show that, even if a problem has a unique solution, there may be present unlimited solutions whose

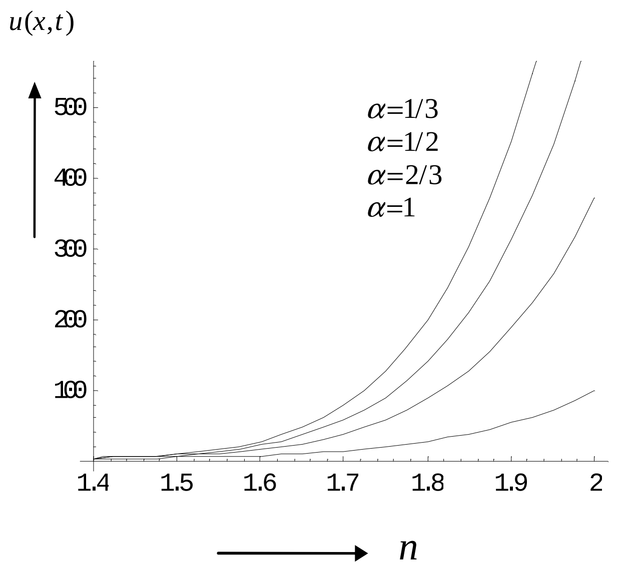


Figure 5. Plot of $u(x,t)$ versus positive values of $1.4 < n < 2$ for various values of α at $x = 1$ and $t = 1$.

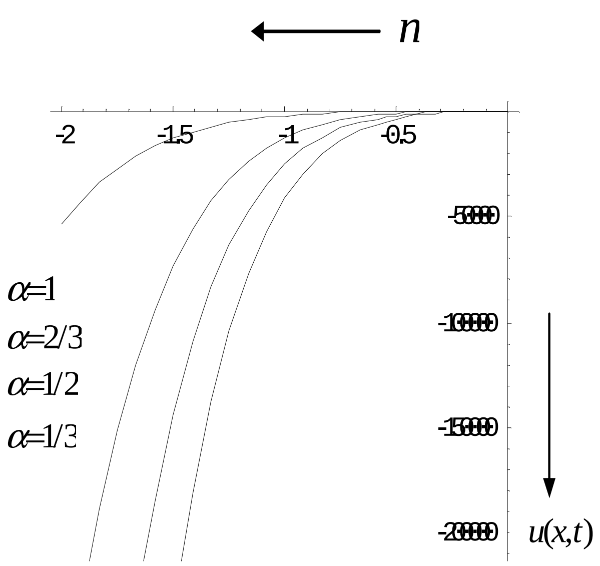


Figure 6. Plot of $u(x,t)$ versus negative values of n for various values of α at $x = 1$ and $t = 1$.

convergence region and rate are dependent on an auxiliary parameter. Unlike all the previous analytic techniques, this method provides us with an easy way to manage and adjust the convergence region and rate of solution series of nonlinear problems. Thus, this method is very useful for nonlinear problems with strong nonlinearity.

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