

Fractional Poisson Bracket

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Received 01.05.2008

Abstract

In the present paper fractional Hamilton-Jacobi equation has been derived for dynamical systems involving the Caputo derivative. Fractional Poisson-bracket is introduced. Further Hamilton's canonical equations are formulated and quantum wave equation corresponds to the fractional Hamilton-Jacobi equation is suggested. Illustrative examples have been worked out to explain the formalism.

Key Words: Hamilton-Jacobi quantized equation; Riemann-Liouville derivative; Caputo derivative; Canonical transformation; Generating function; fractional Poisson-bracket

1. Introduction

In seminal papers Riewe [1, 2] has formulated Lagrangian and Hamiltonian mechanics to include derivatives of fractional order [3, 4, 5, 6, 7, 8, 9]. It has been shown that Lagrangian involving fractional time derivatives leads to equations of motion with non conservative classical forces such as friction [1]. Motivated by this approach many researchers have explored this area giving new insight into this problem [10, 11, 12, 13, 14, 15, 16]. Agrawal [12] has developed fractional calculus of variations dealing with problems in which either the objective functional or the constraint equations, or both, contain at least one fractional derivative term. Agrawal [12] has dealt with Lagrangian involving Riemann-Liouville (R-L) fractional derivatives. R-L derivatives are nonlocal. R-L derivative of a constant is not zero, and in many applications it involves fractional initial conditions which are nonphysical. For these reasons the Caputo derivative [8, 9] has widely been used in recent literature. Agrawal [13, 14] in the recent papers has presented fractional Euler-Lagrange equations involving Caputo derivatives. In conclusion it is emphasized that both (the R-L and Caputo) fractional derivatives arise in the formulation, even when the fractional variational problem is defined only in terms of one type of derivative. Thus fractional boundary conditions may be necessary even when the problem is defined in terms of Caputo derivative. Further fractional Hamiltonian formulation has been developed in terms of Caputo derivatives by Baleanu and coworkers [15, 16]. As a pursuit of this in the present paper we investigate the fractional Hamiltonian involving Caputo derivative and derive the Hamilton-Jacobi equations. Poisson brackets constitute important part of Hamiltonian mechanics. Entire Hamiltonian mechanics can be restated in terms of Poisson-bracket. In view of this a generalization of Poisson-bracket (fractional version) is suggested. Hamilton's canonical equations (fractional case) have been expressed in terms of fractional Poisson bracket. Further fractional quantum wave equation is suggested. Illustrative examples are presented to explain the formalism.

2. Fractional Calculus

Fractional calculus deals with generalizations of integer-order derivatives and integrals to arbitrary order. In this section we present basic definitions and properties which will be used in the subsequent sections [3, 4, 5, 6, 7].

Definition 2.1 If $f(x) \in C[a, b]$ and $\alpha > 0$ then [4, 5, 6]

$${}_a I_x^\alpha f(x) := \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(t)}{(x-t)^{1-\alpha}} dt, \quad x > a, \quad (1)$$

$${}_x I_b^\alpha f(x) := \frac{1}{\Gamma(\alpha)} \int_x^b \frac{f(t)}{(x-t)^{1-\alpha}} dt, \quad x < b, \quad (2)$$

are called the left sided and the right sided Riemann-Liouville fractional integral of order α , respectively.

Definition 2.2 Let $n - 1 \leq \alpha < n$, then

$${}_a D_x^\alpha f(x) := \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dx}\right)^n \int_a^x \frac{f(t)}{(x-t)^{-n+\alpha+1}} dt, \quad (3)$$

$${}_x D_b^\alpha f(x) := \frac{1}{\Gamma(n-\alpha)} \left(-\frac{d}{dx}\right)^n \int_x^b \frac{f(t)}{(t-x)^{-n+\alpha+1}} dt, \quad (4)$$

are called the left sided and the right sided Riemann-Liouville fractional derivative of order α , respectively, whenever the RHS exists.

Definition 2.3 Let $f(x) \in C^n[a, b]$ and $n - 1 \leq \alpha < n$, then [4, 6]

$${}_a^C D_x^\alpha f(x) = {}_a I_x^{n-\alpha} D^n f(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x (x-t)^{n-\alpha-1} \left(\frac{d}{dt}\right)^n f(t) dt, \quad a < x < b \quad (5)$$

$${}_x^C D_b^\alpha f(x) = {}_x I_b^{n-\alpha} (-D)^n f(x) = \frac{1}{\Gamma(n-\alpha)} \int_x^b (t-x)^{n-\alpha-1} \left(-\frac{d}{dt}\right)^n f(t) dt, \quad a < x < b \quad (6)$$

are called as the left sided and the right sided Caputo fractional derivatives of order α , respectively, whenever the RHS exists.

The above have the following

Properties [14]:

(i)
$${}_a^C D_t^\alpha (f(t) + g(t)) = {}_a^C D_t^\alpha f(t) + {}_a^C D_t^\alpha g(t). \quad (7)$$

(ii)
$${}_a^C D_t^\alpha c = 0, \quad c \text{ is constant.} \quad (8)$$

(iii)
$$\int_a^b [{}_a^C D_t^\alpha f(t)] g(t) dt = \int_a^b f(t) [{}_t^C D_b^\alpha g(t)] dt. \quad (9)$$

(iv)
$${}_a D_t^\alpha (t-a)^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)} (t-a)^{\beta-\alpha} \quad (\beta > \alpha). \quad (10)$$

(v)
$${}_a I_t^\alpha (t-a)^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1+\alpha)} (t-a)^{\beta+\alpha}. \quad (11)$$

$${}_a I_t^\alpha {}_a D_t^\alpha x(t) = x(t) - \sum_{j=1}^n \frac{({}_a D_t^{\alpha-j} x)(a)}{\Gamma(\alpha + 1 - j)} (t - a)^{\alpha-j}. \tag{12}$$

$${}_t I_b^\alpha {}_t D_b^\alpha x(t) = x(t) - \sum_{j=1}^n \frac{({}_t D_b^{\alpha-j} x)(b)}{\Gamma(\alpha + 1 - j)} (b - t)^{\alpha-j}. \tag{13}$$

$${}_a I_t^\alpha {}^C D_t^\alpha x(t) = x(t) - \sum_{j=0}^{n-1} \frac{(D^j x)(a)}{\Gamma(j + 1)} (t - a)^j. \tag{14}$$

$${}_t I_b^\alpha {}^C D_b^\alpha x(t) = x(t) - \sum_{j=0}^{n-1} \frac{((-D)^j x)(b)}{\Gamma(j + 1)} (b - t)^j. \tag{15}$$

3. Fractional Mechanics

Agrawal and coworkers [13, 16] have presented Euler-Lagrange equations for fractional variational problems defined in terms of R-L and Caputo derivatives. In the following section we state a theorem regarding Lagrangian involving left and right Caputo derivatives, which will be used in further discussion.

Theorem 3.1 *Let $J[q]$ be a functional of the form*

$$J[q] = \int_a^b L(t, q, {}_a^C D_t^\alpha q, {}_t^C D_b^\beta q) dt, \tag{16}$$

where $0 < \alpha, \beta < 1$ and is defined on the set of functions $f(x)$ which have continuous left Caputo fractional derivative (LCFD) of order α and right Caputo fractional derivative (RCFD) of order β in $[a, b]$. A necessary condition for $J[q]$ to have an extremum for a given function $q(t)$ is that $q(t)$ satisfies the generalized Euler-Lagrange equation:

$$\frac{\partial L}{\partial q} + {}_t D_b^\alpha \frac{\partial L}{\partial {}_a^C D_t^\alpha q} + {}_a D_t^\beta \frac{\partial L}{\partial {}_t^C D_b^\beta q} = 0, \quad t \in [a, b], \tag{17}$$

and the transversality conditions:

$$\left[{}_t D_b^{\alpha-1} \frac{\partial L}{\partial {}_a^C D_t^\alpha q} - {}_a D_t^{\beta-1} \frac{\partial L}{\partial {}_t^C D_b^\beta q} \right] \eta(t)|_a^b = 0, \tag{18}$$

where ${}_t D_b^{\alpha-1}$ denotes the fractional integral of order $1 - \alpha$. See [13] for a proof.

4. Fractional Canonical Transformations and Generating Functions

In this section we present the Hamiltonian formulation involving Caputo fractional derivatives. Consider the fractional Lagrangian given in equation (16). Then the canonical momenta p_α and p_β are

$$p_\alpha = \frac{\partial L}{\partial {}_a^C D_t^\alpha q}, \quad p_\beta = \frac{\partial L}{\partial {}_t^C D_b^\beta q}, \tag{19}$$

where p_α and p_β are independent. The fractional canonical Hamiltonian is

$$H = p_\alpha {}_a^C D_t^\alpha q + p_\beta {}_t^C D_b^\beta q - L. \tag{20}$$

Taking total differential of (20) and using (19), we obtain

$$dH = dp_\alpha {}^C D_t^\alpha q + dp_\beta {}^C D_b^\beta q - \frac{\partial L}{\partial q} dq - \frac{\partial L}{\partial t} dt. \quad (21)$$

Taking into account the fractional Euler-Lagrange equations (17), we get

$$dH = dp_\alpha {}^C D_t^\alpha q + dp_\beta {}^C D_b^\beta q + ({}_t D_b^\alpha p_\alpha + {}_a D_t^\beta p_\beta) dq - \frac{\partial L}{\partial t} dt. \quad (22)$$

Equation (22) shows that H is a function of p_α, p_β, q and t . Comparing total differential of H equation (22) we have [16].

$$\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}, \quad \frac{\partial H}{\partial p_\alpha} = {}^C D_t^\alpha q, \quad \frac{\partial H}{\partial p_\beta} = {}^C D_b^\beta q, \quad \frac{\partial H}{\partial q} = {}_a D_t^\beta p_\beta + {}_t D_b^\alpha p_\alpha. \quad (23)$$

Transformation of q, p_α, p_β into new variables $Q(q, p_\alpha, p_\beta, t), P_\alpha(q, p_\alpha, p_\beta, t), P_\beta(q, p_\alpha, p_\beta, t)$ is canonical if there exists a new Hamiltonian $\mathcal{H}(Q, P_\alpha, P_\beta, t)$ which satisfies modified Hamilton principle:

$$\delta \int_a^b (P_\alpha {}^C D_t^\alpha Q + P_\beta {}^C D_b^\beta Q - H) dt = 0. \quad (24)$$

As q, p_α , and p_β are canonically conjugate, we have

$$\delta \int_a^b (p_\alpha {}^C D_t^\alpha q + p_\beta {}^C D_b^\beta q - \mathcal{H}) dt = 0. \quad (25)$$

For these equations to hold, the integrands must differ by a total time derivative of an arbitrary function G , hence

$$(P_\alpha {}^C D_t^\alpha Q + P_\beta {}^C D_b^\beta Q - H) dt - (p_\alpha {}^C D_t^\alpha q + p_\beta {}^C D_b^\beta q - \mathcal{H}) dt = dG. \quad (26)$$

Since G is not varied at the end points, we get

$$\delta \int_a^b \frac{dG}{dt} dt = \delta[G(b) - G(a)] = 0. \quad (27)$$

The function G , which completely determines the transformation is called as a generating function. For mechanics involving fractional derivatives, we introduce variables $\bar{q}_\alpha, \bar{q}_\beta, \bar{Q}_\alpha, \bar{Q}_\beta$ satisfying

$$\frac{d\bar{q}_\alpha}{dt} = {}^C D_t^\alpha q, \quad \frac{d\bar{q}_\beta}{dt} = {}^C D_b^\beta q, \quad \frac{d\bar{Q}_\alpha}{dt} = {}^C D_t^\alpha Q, \quad \frac{d\bar{Q}_\beta}{dt} = {}^C D_b^\beta Q. \quad (28)$$

For integer-order derivatives, these new coordinates are the same as the usual canonical coordinates. However, while dealing with fractional derivatives, the coordinates $\bar{q}_\alpha, \bar{q}_\beta, \bar{Q}_\alpha, \bar{Q}_\beta$ will not be canonical, so all canonical expressions must be written in terms of the original coordinates ${}^C D_t^\alpha q, {}^C D_b^\beta q, {}^C D_t^\alpha Q, {}^C D_b^\beta Q$.

5. Canonical Transformation of The First Kind

For a generating function $G(\bar{q}_\alpha, \bar{q}_\beta, \bar{Q}_\alpha, \bar{Q}_\beta, t)$, the transformation is

$$(p_\alpha {}^C D_t^\alpha q + p_\beta {}^C D_b^\beta q - H) dt - (P_\alpha {}^C D_t^\alpha Q + P_\beta {}^C D_b^\beta Q - \mathcal{H}) dt = dG(\bar{q}_\alpha, \bar{q}_\beta, \bar{Q}_\alpha, \bar{Q}_\beta, t). \quad (29)$$

We have

$$dG = \frac{\partial G}{\partial \bar{q}_\alpha} d\bar{q}_\alpha + \frac{\partial G}{\partial \bar{q}_\beta} d\bar{q}_\beta + \frac{\partial G}{\partial \bar{Q}_\alpha} d\bar{Q}_\alpha + \frac{\partial G}{\partial \bar{Q}_\beta} d\bar{Q}_\beta + \frac{\partial G}{\partial t} dt. \quad (30)$$

Equations (27–29) yield

$$\frac{\partial G}{\partial \bar{q}_\alpha} = p_\alpha, \quad \frac{\partial G}{\partial \bar{q}_\beta} = p_\beta, \quad \frac{\partial G}{\partial \bar{Q}_\alpha} = -P_\alpha, \quad \frac{\partial G}{\partial \bar{Q}_\beta} = -P_\beta, \quad \frac{\partial G}{\partial t} = \mathcal{H} - H. \quad (31)$$

6. Canonical Transformation of The Second Kind

Let \mathcal{S} be a generating function dependent on $\bar{q}_\alpha, \bar{q}_\beta, P_\alpha, P_\beta, t$. From equations (28), (29) we have

$$dG = p_\alpha d\bar{q}_\alpha - P_\alpha d\bar{Q}_\alpha + p_\beta d\bar{q}_\beta - P_\beta d\bar{Q}_\beta + (\mathcal{H} - H)dt \quad (32)$$

$$= p_\alpha d\bar{q}_\alpha - d(P_\alpha \bar{Q}_\alpha) + \bar{Q}_\alpha dP_\alpha + p_\beta d\bar{q}_\beta - d(P_\beta \bar{Q}_\beta) + \bar{Q}_\beta dP_\beta + (\mathcal{H} - H)dt. \quad (33)$$

It is easy to observe that

$$d(G + P_\alpha \bar{Q}_\alpha + P_\beta \bar{Q}_\beta) = p_\alpha d\bar{q}_\alpha + \bar{Q}_\alpha dP_\alpha + p_\beta d\bar{q}_\beta + \bar{Q}_\beta dP_\beta + (\mathcal{H} - H)dt. \quad (34)$$

Let $\mathcal{S} = G + P_\alpha \bar{Q}_\alpha + P_\beta \bar{Q}_\beta$ then

$$d\mathcal{S} = p_\alpha d\bar{q}_\alpha + \bar{Q}_\alpha dP_\alpha + p_\beta d\bar{q}_\beta + \bar{Q}_\beta dP_\beta + (\mathcal{H} - H)dt. \quad (35)$$

Since \mathcal{S} is a function of $\bar{q}_\alpha, \bar{q}_\beta, P_\alpha, P_\beta, t$, we can write

$$\frac{\partial \mathcal{S}}{\partial \bar{q}_\alpha} = p_\alpha, \frac{\partial \mathcal{S}}{\partial \bar{q}_\beta} = p_\beta, \frac{\partial \mathcal{S}}{\partial P_\alpha} = \bar{Q}_\alpha, \frac{\partial \mathcal{S}}{\partial P_\beta} = \bar{Q}_\beta, \frac{\partial \mathcal{S}}{\partial t} = \mathcal{H} - H. \quad (36)$$

7. Fractional Poisson Bracket

Hamiltonian mechanics can be written in terms of Poisson brackets. In the present section a generalization of Poisson bracket has been introduced, which is useful for generalizing fractional mechanics involving Caputo derivatives.

Definition 7.1 *If functions $F(t, q, p_\alpha, p_\beta)$ and $G(t, q, p_\alpha, p_\beta)$ depend on the position coordinate, fractional momenta and time, fractional Poisson (FP) bracket of F and G , denoted as $[F, G]_{FP}$, is defined to be:*

$$[F, G]_{FP} = \frac{\partial F}{\partial q} \left(\frac{\partial G}{\partial p_\alpha} + \frac{\partial G}{\partial p_\beta} \right) - \frac{\partial G}{\partial q} \left(\frac{\partial F}{\partial p_\alpha} + \frac{\partial F}{\partial p_\beta} \right). \quad (37)$$

The following properties can be observed:

(a)

$$[F, G]_{FP} = -[G, F]_{FP},$$

(b)

$$[F_1 + F_2, G]_{FP} = [F_1, G]_{FP} + [F_2, G]_{FP},$$

(c)

$$[F_1, [F_2, F_3]_{FP}]_{FP} + [F_2, [F_3, F_1]_{FP}]_{FP} + [F_3, [F_1, F_2]_{FP}]_{FP} = 0 \quad (\text{Jacobi's identity}),$$

(d)

$$[F, q]_{FP} = -\left(\frac{\partial F}{\partial p_\alpha} + \frac{\partial F}{\partial p_\beta} \right),$$

(f)

$$[F, p_\alpha]_{FP} = [F, p_\beta]_{FP} = \frac{\partial F}{\partial q},$$

(g)

$$[q, q]_{FP} = [p_\alpha, p_\alpha]_{FP} = [p_\alpha, p_\beta]_{FP} = 0,$$

$$[p_\alpha, q]_{FP} = [p_\beta, q]_{FP} = -1.$$

8. Hamilton's Canonical Equation In terms of Poisson Bracket

Hamilton's canonical equations in terms of Poisson brackets can be expressed as

$$[q, H]_{FP} = \left(\frac{\partial H}{\partial p_\alpha} + \frac{\partial H}{\partial p_\beta} \right), \quad (38)$$

$$[p_\alpha, H]_{FP} = [p_\beta, H]_{FP} = -({}_a D_t^\beta p_\beta + {}_t D_b^\alpha p_\alpha) = -\frac{\partial H}{\partial q}. \quad (39)$$

9. Fractional Quantum Wave Equation

As in conventional mechanics, the Hamilton-Jacobi (H-J) equation results from a canonical transformation for which the new variables are constant. For integer-order derivatives, such a transformation will follow automatically if the new Hamiltonian \mathcal{H} is identically zero, since from the equations of motion we then have

$$\dot{Q} = \frac{\partial \mathcal{H}}{\partial P} = 0, \quad \dot{P} = -\frac{\partial \mathcal{H}}{\partial Q}. \quad (40)$$

For fractional derivatives, we can derive a similar relationship by putting

$$\frac{\partial \mathcal{S}}{\partial t} + H(q, p_\alpha, p_\beta, t) = 0. \quad (41)$$

In view of (36), (41) yields fractional version of H-J equation, i.e.

$$\frac{\partial \mathcal{S}}{\partial t} + H\left(q, \frac{\partial \mathcal{S}}{\partial \bar{q}_\alpha}, \frac{\partial \mathcal{S}}{\partial \bar{q}_\beta}, t\right) = 0. \quad (42)$$

Hence the quantum wave equation corresponding to Hamilton-Jacobi involving fractional Caputo derivative is suggested to be [1]

$$\left[H\left(q, -i\hbar \frac{\partial}{\partial \bar{q}_\alpha}, -i\hbar \frac{\partial}{\partial \bar{q}_\beta}, t\right) \psi = i\hbar \frac{\partial \psi}{\partial t}, \quad (43)$$

where ψ is a wave function.

10. Examples

Example 1. The total energy of the fractional oscillator is given as [17]

$$H = \frac{1}{2} kx^2 + \frac{1}{2} m_\alpha ({}_a^C D_t^\alpha x)^2, \quad 0 < \alpha < 1, a < t < b. \quad (44)$$

The generalized energy of fractional oscillator in uniform electric field E will be

$$H_{FO} = qEx + \frac{1}{2} kx^2 + \frac{1}{2} m_\alpha ({}_a^C D_t^\alpha x)^2, \quad 0 < \alpha < 1, a < t < b, \quad (45)$$

where q is the charge of oscillator, $m_\alpha = \gamma m$ and dimension of γ is $T^{\alpha-1}$. In view of equation (20), Lagrangian for FO is

$$L_{FO} = \frac{1}{2} m_\alpha ({}_a^C D_t^\alpha x)^2 - \frac{1}{2} kx^2 - qEx. \quad (46)$$

The generalized Euler-Lagrange equation equation (17) and the transversality condition equation (18) yield:

$$(a) \quad -qE - kx + m_\alpha {}_t D_b^\alpha {}_a^C D_t^\alpha x = 0, \quad (b) \quad {}_t D_b^{\alpha-1} {}_a^C D_t^\alpha x|_{t=b} = e_1, \quad (47)$$

respectively. Equation (47) can be solved to get explicit expression for $x(t)$, as follows:
 Let $\omega_\alpha^2 = \frac{k}{m_\alpha}$ and $\frac{qE}{m_\alpha} = \gamma$. The Euler-Lagrange equation then takes the form

$${}_t D_b^\alpha {}^C D_t^\alpha x = \omega_\alpha^2 x + \gamma. \quad (48)$$

To find the solution of equation (48) with initial condition $x(0) = e_0$, we apply ${}_t I_b^\alpha$ to both sides of equation (48). Further, using (13) and (47)(b), we get

$${}_a^C D_t^\alpha x(t) = \omega_\alpha^2 {}_t I_b^\alpha x(t) + \frac{\gamma}{\Gamma(\alpha+1)}(b-t)^\alpha + \frac{e_1}{\Gamma(\alpha)}(b-t)^{\alpha-1}. \quad (49)$$

Similarly, applying ${}_a I_t^\alpha$ to both sides of equation (49) and using equation (14), we obtain

$$x(t) = \omega_\alpha^2 {}_a I_t^\alpha {}_t I_b^\alpha x + {}_a I_t^\alpha \left[\frac{e_1}{\Gamma(\alpha)}(b-t)^{\alpha-1} + \frac{\gamma}{\Gamma(\alpha+1)}(b-t)^\alpha \right] + e_0. \quad (50)$$

as ($x(0) = e_0$). Equation (50) can be thought of as Volterra-type integral that has composite integral operators and can be solved to obtain

$$x(t) = (1 - \omega_\alpha^2 {}_a I_t^\alpha {}_t I_b^\alpha)^{-1} ({}_a I_t^\alpha \left[\frac{e_1}{\Gamma(\alpha)}(b-t)^{\alpha-1} + \frac{\gamma}{\Gamma(\alpha+1)}(b-t)^\alpha \right] + e_0) \quad (51)$$

$$= \sum_{j=0}^{\infty} (\omega_\alpha^2 {}_a I_t^\alpha {}_t I_b^\alpha)^j ({}_a I_t^\alpha \left[\frac{e_1}{\Gamma(\alpha)}(b-t)^{\alpha-1} + \frac{\gamma}{\Gamma(\alpha+1)}(b-t)^\alpha \right] + e_0). \quad (52)$$

The generalized momentum for FO is

$$p_\alpha = \frac{\partial L_{FO}}{\partial {}^C D_t^\alpha q} = m_\alpha {}^C D_t^\alpha x, \quad (53)$$

and generalized Hamiltonian in terms of p_α is

$$H_{FO} = \frac{1}{2m_\alpha} p_\alpha^2 + \frac{1}{2} kx^2 + qEx. \quad (54)$$

Hence the Hamilton's equations (equation (23)) become

$$\frac{\partial H_{FO}}{\partial p_\alpha} = \frac{p_\alpha}{m_\alpha} = {}^C D_t^\alpha q, \quad \frac{\partial H_{FO}}{\partial q} = kx + qE = {}_t D_b^\alpha p_\alpha, \quad (55)$$

which are equivalent to generalized Euler-Lagrange equations given in equation (47)(a).

Further, we present Hamilton-Jacobi equation for this system. equation (42) yields the Hamilton-Jacobi Equation.

$$\frac{\partial \mathcal{S}_{FO}}{\partial t} + \frac{1}{2m_\alpha} \left(\frac{\partial \mathcal{S}_{FO}}{\partial \bar{x}_\alpha} \right)^2 + \frac{1}{2} kx^2 + qEx = 0, \quad (56)$$

where $\frac{d\bar{x}_\alpha}{dt} = {}^C D_t^\alpha x(t)$. Assume that a solution to equation (56) is the form of $\mathcal{S}_{FO} = S_1(t) + S_2(\bar{x}_\alpha)$. Then

$$\frac{1}{2m_\alpha} \left(\frac{dS_2}{d\bar{x}_\alpha} \right)^2 + \frac{1}{2} kx^2 + qEx = -\frac{dS_1}{dt} = \beta, \quad (57)$$

where β is a constant. Therefore

$$\frac{dS_2}{d\bar{x}_\alpha} = \sqrt{2m_\alpha(\beta - \frac{1}{2}kx^2 - qEx)}, S_1 = -\beta t, \quad (58)$$

so that

$$\mathcal{S}_{FO} = \bar{x}_\alpha \sqrt{2m_\alpha(\beta - \frac{1}{2}kx^2 - qEx)} - \beta t. \quad (59)$$

Further, we identify β with the new momentum coordinate P_α . Hence the new position variable Q_α will be

$$Q_\alpha = \frac{\partial \mathcal{S}_{FO}}{\partial \beta} = \frac{\partial}{\partial \beta} [\bar{x}_\alpha \sqrt{2m_\alpha(\beta - \frac{1}{2}kx^2 - qEx)} - \beta t] \quad (60)$$

$$= \frac{m_\alpha}{\sqrt{2m_\alpha(H - \frac{1}{2}kx^2 - qEx)}} \bar{x}_\alpha - t = \gamma, \quad (61)$$

where γ is a constant. Solving equation (61) for \bar{x}_α we, get

$$m_\alpha \bar{x}_\alpha = (\gamma + t) \sqrt{2m_\alpha(\beta - \frac{1}{2}kx^2 - qEx)} \quad (62)$$

Further differentiating (62) with respect to t , we get

$$m_\alpha \frac{d\bar{x}_\alpha}{dt} = \sqrt{2m_\alpha(\beta - \frac{1}{2}kx^2 - qEx)} = p_\alpha. \quad (63)$$

In view of equation (42), we can identify β with the Hamiltonian H . Then in view of (54), $m \frac{d\bar{x}_\alpha}{dt} = \sqrt{2m(\beta - \frac{1}{2}kx^2 - qEx)} = p_\alpha$. But

$$m_\alpha {}^C D_t^\alpha x = p_\alpha. \quad (64)$$

Applying ${}_t D_b^\alpha$ to both sides of equation (64), we get

$$m_\alpha {}_t D_b^\alpha {}^C D_t^\alpha x = {}_t D_b^\alpha p_\alpha. \quad (65)$$

Using equation (55) we obtain

$$m_\alpha {}_t D_b^\alpha {}^C D_t^\alpha x = kx + qE, \quad (66)$$

which is the same equation that we derive from Euler-Lagrange and Hamilton equations. This result can also be obtained using fractional Poisson brackets.

$$[x, H_{FO}]_{FP} = [x, \frac{1}{2}kx^2 + qEx + \frac{1}{2}m_\alpha({}^C D_t^\alpha x)^2]_{FP}.$$

Since the generalized momentum for FO is

$$p_\alpha = \frac{\partial L_{FO}}{\partial {}^C D_t^\alpha x} = m_\alpha {}^C D_t^\alpha x, \quad (67)$$

we can write,

$$[x, H_{FO}]_{FP} = [x, \frac{1}{2}kx^2 + qEx + \frac{p_\alpha^2}{2m_\alpha}]_{FP} = \frac{p_\alpha}{m_\alpha} = {}^C D_t^\alpha x, \quad (68)$$

and

$$[p_\alpha, H_{FO}]_{FP} = [p_\alpha, \frac{1}{2}kx^2 + qEx + \frac{p_\alpha^2}{2m_\alpha}]_{FP} = -kx - qE = -{}_t D_b^\alpha p_\alpha. \quad (69)$$

These equations are the same as the Hamilton's equations equation (55). Therefore, both the methods yield the following equation, for fractional oscillator in uniform electric field

$$-kx - qE + m_\alpha {}_t D_b^\alpha {}^C D_t^\alpha x = 0. \quad (70)$$

Equation (43) yields the following wave equation corresponding to fractional oscillator in the uniform electric field E

$$\left(\frac{1}{2m_\alpha}(-i\hbar \frac{\partial}{\partial \bar{x}_\alpha})\right)^2 + \frac{1}{2}kx^2 + qEx \psi = i\hbar \frac{\partial \psi}{\partial t}. \quad (71)$$

The wave function for the quantized FO is of the form

$$\psi(x, \bar{x}_\alpha, t) = A(x, \bar{x}_\alpha, t) e^{\frac{i}{\hbar} \mathcal{S}_{FO}(x, \bar{x}_\alpha, t)}, \quad (72)$$

where $A(x, \bar{x}_\alpha, t)$ and $\mathcal{S}_{FO}(x, \bar{x}_\alpha, t)$ are amplitude and phase, respectively. Substituting equation (72) in equation (71) we get

$$-\frac{\hbar^2}{2m_\alpha} \left(\frac{\partial A}{\partial \bar{x}_\alpha} \right)^2 + \frac{1}{2m_\alpha} A \left(\frac{\partial \mathcal{S}_{FO}}{\partial \bar{x}_\alpha} \right)^2 + \left(\frac{1}{2} kx^2 + qEx \right) A = i\hbar \frac{\partial A}{\partial t} - A \frac{\partial \mathcal{S}_{FO}}{\partial t}. \quad (73)$$

Separating this expression into real and imaginary parts, we get the two equations:

$$\frac{1}{2m_\alpha} \left(\frac{\partial \mathcal{S}_{FO}}{\partial \bar{x}_\alpha} \right)^2 + \frac{\partial \mathcal{S}_{FO}}{\partial t} + \frac{1}{2} kx^2 + qEx = \frac{\hbar^2}{2m_\alpha} \left(\frac{\partial A}{\partial \bar{x}_\alpha} \right)^2 \quad \text{and} \quad \frac{\partial A}{\partial t} = 0. \quad (74)$$

Equation (74) reduces to fractional Hamilton-Jacobi, when $\hbar = 0$.

Example 2. Consider FO with dissipative force

$$F(x) = -\gamma {}^C D_b^\beta x, \quad (75)$$

where dimension of γ is $MLT^{-(1+\beta)}$. Using potential corresponding to this force, we have the Lagrangian as [1]:

$$L = \frac{1}{2} m_\alpha ({}^C D_t^\alpha x)^2 - \frac{1}{2} kx^2 - i \frac{\gamma}{2(-1)^{\frac{\beta}{2}}} ({}^C D_b^{\frac{\beta}{2}} x)^2. \quad (76)$$

In view of equation (17), we obtain generalized Euler-Lagrange equation as

$$-kx + m_\alpha {}_t D_b^\alpha {}^C D_t^\alpha x - i \frac{\gamma}{(-1)^{\frac{\beta}{2}}} {}_a D_t^{\frac{\beta}{2}} {}^C D_b^{\frac{\beta}{2}} x = 0. \quad (77)$$

The generalized momenta are

$$p_\alpha = m_\alpha {}^C D_t^\alpha x \quad \text{and} \quad p_{\frac{\beta}{2}} = -i \frac{\gamma}{(-1)^{\frac{\beta}{2}}} {}^C D_b^{\frac{\beta}{2}} x. \quad (78)$$

The corresponding Hamiltonian is

$$H = \frac{1}{2} m_\alpha ({}^C D_t^\alpha x)^2 + \frac{1}{2} kx^2 + i \frac{\gamma}{2(-1)^{\frac{\beta}{2}}} ({}^C D_b^{\frac{\beta}{2}} x)^2. \quad (79)$$

Using equations (78), the Hamiltonian for this system takes the form

$$H = \frac{p_\alpha^2}{2m_\alpha} + \frac{1}{2} kx^2 + \frac{p_{\frac{\beta}{2}}^2}{2i\gamma(-1)^{\frac{\beta}{2}}}. \quad (80)$$

Using Hamilton's canonical Poisson bracket, we have

$$[p_\alpha, H] = [p_\alpha, \frac{p_\alpha^2}{2m_\alpha} + \frac{1}{2} kx^2 + \frac{p_{\frac{\beta}{2}}^2}{2i\gamma(-1)^{\frac{\beta}{2}}}] = -kx = [p_{\frac{\beta}{2}}, H] = -({}_a D_t^{\frac{\beta}{2}} p_{\frac{\beta}{2}} + {}_t D_b^\alpha p_\alpha). \quad (81)$$

Substituting equations (78) in equation (81) we arrive at equation (77).

11. Results and Conclusion

Fractional mechanics describes both conservative and non-conservative systems. With this motivation a generalization of Poisson bracket is introduced. Further generalized Hamilton's canonical equations have been derived in the case of Lagrangian involving Caputo derivatives. Subsequently Hamilton-Jacobi equation has been derived and fractional quantum wave is suggested.

Acknowledgements

The author is grateful to Professors V. Daftardar-Gejji. and A. D. Gangal for useful discussions.

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