Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

Boundedness for commutators of fractional p-adic Hardy operators

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Abstract
In this paper we prove that the commutators generated by the fractional p-adic Hardy operators and the central BMO function are bounded on weighted homogeneous Herz spaces.
MSC: 11E95; 11K70; 42B99

Keywords: fractional p-adic Hardy operator; central BMO function; weighted homogeneous Herz space

1 Introduction

In recent years, p-adic fields have been introduced into some aspects of mathematical physics. There are a lot of articles where different applications of the p-adic analysis in the string theory, quantum mechanics, stochastics, the theory of dynamical systems, cognitive sciences, and psychology are studied [1–9] (see also the references therein). As a consequence, new mathematical problems have emerged, among them, the study of harmonic analysis on a p-adic field has been drawing more and more concern (*cf.* [10–14] and the references therein).

For a prime number p, let $\mathbb{Q}_p^n$ be the field of p-adic numbers. It is defined as the completion of the field of rational numbers $\mathbb{Q}$ with respect to the non-Archimedean p-adic norm $|\cdot|_p$. This norm is defined as follows: $|0|_p = 0$. If any non-zero rational number x is represented as $x = p^\gamma \frac{m}{n}$, where m and n are integers which are not divisible by p, and γ is an integer, then $|x|_p = p^{-\gamma}$. It is not difficult to show that the norm satisfies the following properties:

$$|xy|_p = |x|_p|y|_p, \qquad |x+y|_p \le \max\{|x|_p, |y|_p\}.$$

It follows from the second property that when $|x|_p \ne |y|_p$, then $|x+y|_p = \max\{|x|_p, |y|_p\}$. From the standard p-adic analysis [7], we see that any non-zero p-adic number $x \in \mathbb{Q}_p$ can be uniquely represented in the canonical series

$$x = p^\gamma \sum_{j=0}^{\infty} a_j p^j, \quad \gamma = \gamma(x) \in \mathbb{Z}, \tag{1.1}$$

where a_j are integers, $0 \le a_j \le p-1$, $a_0 \ne 0$. The series (1.1) converges in the p-adic norm because $|a_j p^j|_p = p^{-j}$. Set $\mathbb{Q}_p^* = \mathbb{Q}_p \setminus \{0\}$.

The space $\mathbb{Q}_p^n$ consists of points $x = (x_1, x_2, \ldots, x_n)$, where $x_j \in \mathbb{Q}_p$, $j = 1, 2, \ldots, n$. The p-adic norm on $\mathbb{Q}_p^n$ is

$$|x|_p := \max_{1 \le j \le n} |x_j|_p, \quad x \in \mathbb{Q}_p^n. \tag{1.2}$$

Denote by

$$B_\gamma(a) = \{x \in \mathbb{Q}_p^n : |x - a|_p \le p^\gamma\},$$

the ball with center at $a \in \mathbb{Q}_p^n$ and radius p^γ, and

$$S_\gamma(a) = \{x \in \mathbb{Q}_p^n : |x - a|_p = p^\gamma\} = B_\gamma(a) \setminus B_{\gamma-1}(a).$$

Since $\mathbb{Q}_p^n$ is a locally compact commutative group under addition, it follows from the standard analysis that there exists a Haar measure dx on $\mathbb{Q}_p^n$, which is unique up to a positive constant multiple and is translation invariant. We normalize the measure dx by the equality

$$\int_{B_0(0)} dx = |B_0(0)|_H = 1,$$

where $|E|_H$ denotes the Haar measure of a measurable subset E of $\mathbb{Q}_p^n$. By simple calculation, we can obtain that

$$|B_\gamma(a)|_H = p^{\gamma n}, \qquad |S_\gamma(a)|_H = p^{\gamma n}(1 - p^{-n}),$$

for any $a \in \mathbb{Q}_p^n$. For a more complete introduction to the p-adic field, see [15] or [7].

The classical Hardy operators are defined by

$$Hf(x) := \frac{1}{x}\int_0^x f(t)\,dt, \qquad H^* f(x) := \int_x^\infty \frac{f(t)}{t}\,dt, \quad x > 0,$$

for a non-negative integrable function f on $\mathbb{R}^+$. Obviously, $\mathcal{H}$ and $\mathcal{H}^*$ satisfy

$$\int_{\mathbb{R}^n} g(x)\mathcal{H}f(x)\,dx = \int_{\mathbb{R}^n} f(x)\mathcal{H}^* g(x)\,dx.$$

The well-known Hardy integral inequality [16] tells us that for $1 < q < \infty$,

$$\|Hf\|_{L^q(\mathbb{R}^+)} \le \frac{q}{q-1}\|f\|_{L^q(\mathbb{R}^+)},$$

where the constant $\frac{q}{q-1}$ is the best possible. The generalized result [17] is that

$$\|H^* f\|_{L^{q'}(\mathbb{R}^+)} \le \frac{q}{q-1}\|f\|_{L^{q'}(\mathbb{R}^+)},$$

and

$$\|H^*\|_{L^{q'}(\mathbb{R}^+) \to L^{q'}(\mathbb{R}^+)} = \frac{q}{q-1},$$

where $\frac{1}{q} + \frac{1}{q'} = 1$.

The Hardy integral inequalities have received considerable attention due to their usefulness in analysis and their applications. There are numerous papers dealing with their various generalizations, variants and applications (*cf.* [18–20] and the references cited therein). We have obtained the Hardy integral inequalities for p-adic Hardy operators and their commutators [21]. The boundedness of commutators is an active topic in harmonic analysis because of its important applications; for example, it can be applied to characterizing some function spaces. There are a lot of works about the boundedness of commutators of various Hardy-type operators on Euclidean spaces (*cf.* [22, 23], *etc.*). In this paper, we will establish the Hardy integral inequalities for commutators generated by fractional p-adic Hardy operators and CMO functions.

Definition 1.1 For a function f on $\mathbb{Q}_p^n$, we define the *p-adic Hardy operators* as follows:

$$\begin{aligned} \mathcal{H}^p f(x) &= \frac{1}{|x|_p^n}\int_{B(0,|x|_p)} f(t)\,dt, \\ \mathcal{H}^{p,*} f(x) &= \int_{\mathbb{Q}_p^n\setminus B(0,|x|_p)} \frac{f(t)}{|t|_p^n}\,dt, \quad x\in\mathbb{Q}_p^n\setminus\{0\}, \end{aligned} \tag{1.3}$$

where $B(0,|x|_p)$ is a ball in $\mathbb{Q}_p^n$ with center at $0\in\mathbb{Q}_p^n$ and radius $|x|_p$.

Definition 1.2 Let $f\in L_{\mathrm{loc}}(\mathbb{Q}_p^n)$, $0\le\beta<n$. *The fractional p-adic Hardy operators* are defined by

$$\begin{aligned} \mathcal{H}_\beta^p f(x) &= \frac{1}{|x|_p^{n-\beta}}\int_{B(0,|x|_p)} f(t)\,dt, \\ \mathcal{H}_\beta^{p,*} f(x) &= \int_{\mathbb{Q}_p^n\setminus B(0,|x|_p)} \frac{f(t)}{|t|_p^{n-\beta}}\,dt, \quad x\in\mathbb{Q}_p^n\setminus\{0\}, \end{aligned} \tag{1.4}$$

where $B(0,|x|_p)$ is the ball as in Definition 1.1.

It is clear that when $\beta=0$, then $\mathcal{H}_\beta^p$ becomes $\mathcal{H}^p$.

Definition 1.3 Let $b\in L_{\mathrm{loc}}(\mathbb{Q}_p^n)$, $0\le\beta<n$. *The commutators of fractional p-adic Hardy operators* are defined by

$$\mathcal{H}_{\beta,b}^p f = b\mathcal{H}_\beta^p f - \mathcal{H}_\beta^p(bf), \qquad \mathcal{H}_{\beta,b}^{p,*} f = b\mathcal{H}_\beta^{p,*} f - \mathcal{H}_\beta^{p,*}(bf). \tag{1.5}$$

In [24–26], the CMO spaces (central BMO spaces) on $\mathbb{R}^n$ have been introduced and studied. CMO spaces bear a simple relationship with BMO: $g\in BMO$ precisely when g and all of its translates belong to BMO spaces uniformly a.e. Many precise analogies exist between CMO spaces and BMO spaces from the point of view of real Hardy spaces. Similarly, we define the CMO^q spaces on $\mathbb{Q}_p^n$.

Definition 1.4 Let $1\le q<\infty$, a function $f\in L_{\mathrm{loc}}^q(\mathbb{Q}_p^n)$ is said to be in $CMO^q(\mathbb{Q}_p^n)$ if

$$\|f\|_{CMO^q(\mathbb{Q}_p^n)} := \sup_{\gamma\in\mathbb{Z}}\left(\frac{1}{|B_\gamma(0)|_H}\int_{B_\gamma(0)}\left|f(x)-f_{B_\gamma(0)}\right|^q dx\right)^{\frac{1}{q}} < \infty,$$

where

$$f_{B_\gamma(0)} = \frac{1}{|B_\gamma(0)|_H} \int_{B_\gamma(0)} f(x)\,dx.$$

Remark 1.1 It is obvious that $L^\infty(\mathbb{Q}_p^n) \subset BMO(\mathbb{Q}_p^n) \subset CMO^q(\mathbb{Q}_p^n)$.

Let $B_k = B_k(0) = \{x \in \mathbb{Q}_p^n : |x|_p \le p^k\}$, $S_k = B_k \setminus B_{k-1}$ and χ_k be the characteristic function of the set S_k.

Definition 1.5 [27] Suppose that $\alpha \in \mathbb{R}$, $0 < q < \infty$ and $0 < r < \infty$. The homogeneous p-adic Herz space $K_r^{\alpha,q}(\mathbb{Q}_p^n)$ is defined by

$$K_r^{\alpha,q}(\mathbb{Q}_p^n) = \{f \in L^r_{\mathrm{loc}}(\mathbb{Q}_p^n) : \|f\|_{K_r^{\alpha,q}(\mathbb{Q}_p^n)} < \infty\},$$

where

$$\|f\|_{K_r^{\alpha,q}(\mathbb{Q}_p^n)} = \left(\sum_{k=-\infty}^{+\infty} p^{k\alpha q} \|f\chi_k\|^q_{L^r(\mathbb{Q}_p^n)}\right)^{\frac{1}{q}},$$

with the usual modifications made when $q = \infty$ or $r = \infty$.

Remark 1.2 $K_r^{\alpha,q}(\mathbb{Q}_p^n)$ is the generalization of $L^q(|x|_p^\alpha\,dx)$, and $K_q^{0,q}(\mathbb{Q}_p^n) = L^q(\mathbb{Q}_p^n)$, $K_q^{\frac{\alpha}{q},q}(\mathbb{Q}_p^n) = L^q(|x|_p^\alpha\,dx)$ for all $0 < q \le \infty$ and $\alpha \in \mathbb{R}$.

Motivated by [22], we get the following operator boundedness results. Throughout this paper, we use C to denote different positive constants which are independent of the essential variables.

Theorem 1.1 *Suppose that* $\beta \ge 0$, $0 < q_1 \le q_2 < \infty$, $\frac{1}{r_1} - \frac{1}{r_2} = \frac{\beta}{n}$, $1 < r_1 < \infty$, $\frac{1}{r_1} + \frac{1}{r_1'} = 1$, $b \in CMO^{\max\{r_1',r_2\}}(\mathbb{Q}_p^n)$. *Then*

(1) *If* $\alpha < \frac{n}{r_1'}$, *then*

$$\left\|\mathcal{H}^p_{\beta,b} f\right\|_{K_{r_2}^{\alpha,q_2}(\mathbb{Q}_p^n)} \le C\|b\|_{CMO^{\max\{r_1',r_2\}}(\mathbb{Q}_p^n)} \|f\|_{K_{r_1}^{\alpha,q_1}(\mathbb{Q}_p^n)}. \tag{1.6}$$

(2) *If* $\alpha > -\frac{n}{r_2}$, *then*

$$\left\|\mathcal{H}^{p,*}_{\beta,b} f\right\|_{K_{r_2}^{\alpha,q_2}(\mathbb{Q}_p^n)} \le C\|b\|_{CMO^{\max\{r_1',r_2\}}(\mathbb{Q}_p^n)} \|f\|_{K_{r_1}^{\alpha,q_1}(\mathbb{Q}_p^n)}. \tag{1.7}$$

When $\alpha = 0$, $q_j = r_j$, $j = 1, 2$, we can get the following result.

Corollary 1.1 *Suppose that* $\beta \ge 0$, $0 < q_1 \le q_2 < \infty$, $\frac{1}{q_1} - \frac{1}{q_2} = \frac{\beta}{n}$, $1 < q_1 < \infty$, $\frac{1}{q_1} + \frac{1}{q_1'} = 1$, $b \in CMO^{\max\{q_1',q_2\}}(\mathbb{Q}_p^n)$. *Then*

$$\left\|\mathcal{H}^p_{\beta,b} f\right\|_{L^{q_2}(\mathbb{Q}_p^n)} \le C\|b\|_{CMO^{\max\{q_1',q_2\}}(\mathbb{Q}_p^n)} \|f\|_{L^{q_1}(\mathbb{Q}_p^n)}, \tag{1.8}$$

and

$$\left\|\mathcal{H}_{\beta,b}^{p,*}f\right\|_{L^{q_2}(\mathbb{Q}_p^n)} \le C\|b\|_{CMO^{\max\{q_1',q_2\}}(\mathbb{Q}_p^n)}\|f\|_{L^{q_1}(\mathbb{Q}_p^n)}. \tag{1.9}$$

When $\beta = 0$, we can get the boundedness of a p-adic Hardy operator in [21].

Corollary 1.2 *Let* $0 < q_1 \le q_2 < \infty$, $1 < r < \infty$, $b \in CMO^{\max\{r,r'\}}(\mathbb{Q}_p^n)$. *Then*

(1) *If* $\alpha < \frac{n}{r'}$, *then*

$$\left\|\mathcal{H}_b^p f\right\|_{K_r^{\alpha,q_2}(\mathbb{Q}_p^n)} \le C\|b\|_{CMO^{\max\{r,r'\}}(\mathbb{Q}_p^n)}\|f\|_{K_r^{\alpha,q_1}(\mathbb{Q}_p^n)}. \tag{1.10}$$

(2) *If* $\alpha > -\frac{n}{r}$, *then*

$$\left\|\mathcal{H}_b^{p,*} f\right\|_{K_{r_2}^{\alpha,q_2}(\mathbb{Q}_p^n)} \le C\|b\|_{CMO^{\max\{r,r'\}}(\mathbb{Q}_p^n)}\|f\|_{K_{r_1}^{\alpha,q_1}(\mathbb{Q}_p^n)}. \tag{1.11}$$

By the similar proof of Theorem 1.1, we can obtain the following result.

Corollary 1.3 *Suppose that* $\beta \ge 0$, $0 < q_1 \le q_2 < \infty$, $\frac{1}{r_1} - \frac{1}{r_2} = \frac{\beta}{n}$, $1 < r_1 < \infty$, $\frac{1}{r_1} + \frac{1}{r_1'} = 1$. *Then*

(1) *If* $\alpha < \frac{n}{r_1'}$, *then*

$$\left\|\mathcal{H}_\beta^p f\right\|_{K_{r_2}^{\alpha,q_2}(\mathbb{Q}_p^n)} \le C\|f\|_{K_{r_1}^{\alpha,q_1}(\mathbb{Q}_p^n)}. \tag{1.12}$$

(2) *If* $\alpha > -\frac{n}{r_2}$, *then*

$$\left\|\mathcal{H}_\beta^{p,*} f\right\|_{K_{r_2}^{\alpha,q_2}(\mathbb{Q}_p^n)} \le C\|f\|_{K_{r_1}^{\alpha,q_1}(\mathbb{Q}_p^n)}. \tag{1.13}$$

2 Boundedness of commutators of fractional p-adic Hardy operator

In order to prove Theorem 1.1, we firstly give the following lemmas.

Lemma 2.1 *Suppose that* b *is a CMO function and* $1 \le q < r < \infty$, *then* $CMO^r(\mathbb{Q}_p^n) \subset CMO^q(\mathbb{Q}_p^n)$ *and* $\|b\|_{CMO^q} \le \|b\|_{CMO^r}$.

Proof For any $b \in CMO^r(\mathbb{Q}_p^n)$, by Hölder's inequality, we have

$$\begin{aligned}
&\left(\frac{1}{|B_\gamma(0)|_H}\int_{B_\gamma(0)}\left|b(x) - b_{B_\gamma(0)}\right|^q dx\right)^{\frac{1}{q}} \\
&\quad\le \left\{\frac{1}{|B_\gamma(0)|_H}\left(\int_{B_\gamma(0)}\left|b(x) - b_{B_\gamma(0)}\right|^{q\cdot\frac{r}{q}} dx\right)^{\frac{q}{r}}\left(\int_{B_\gamma(0)} 1\,dx\right)^{1-\frac{q}{r}}\right\}^{\frac{1}{q}} \\
&\quad= \left\{\frac{1}{|B_\gamma(0)|_H}\left(\int_{B_\gamma(0)}\left|b(x) - b_{B_\gamma(0)}\right|^r dx\right)^{\frac{q}{r}}|B_\gamma(0)|_H^{1-\frac{q}{r}}\right\}^{\frac{1}{q}} \\
&\quad= \left(\frac{1}{|B_\gamma(0)|_H}\int_{B_\gamma(0)}\left|b(x) - b_{B_\gamma(0)}\right|^r dx\right)^{\frac{1}{r}} \\
&\quad\le \|b\|_{CMO^r}\left(\mathbb{Q}_p^n\right).
\end{aligned}$$

Therefore, $b \in CMO^q(\mathbb{Q}_p^n)$ and $\|b\|_{CMO^q} \le \|b\|_{CMO^r}$. This completes the proof. □

Lemma 2.2 *Suppose that b is a CMO function, $j, k \in \mathbb{Z}$, then*

$$\left|b(t) - b_{B_k}\right| \le \left|b(t) - b_{B_j}\right| + p^n |j-k| \|b\|_{CMO^1(\mathbb{Q}_p^n)}. \tag{2.1}$$

Proof For $i \in \mathbb{Z}$, recall that $b_{B_i} = \frac{1}{|B_i|_H}\int_{B_i} b(x)\,dx$, we have

$$\begin{aligned} |b_{B_i} - b_{B_{i+1}}| &\le \frac{1}{|B_i|_H}\int_{B_i}\left|b(t) - b_{B_{i+1}}\right| dt \\ &\le \frac{p^n}{|B_{i+1}|_H}\int_{B_{i+1}}\left|b(t) - b_{B_{i+1}}\right| dt \\ &\le p^n \|b\|_{CMO^1(\mathbb{Q}_p^n)}. \end{aligned} \tag{2.2}$$

For $j, k \in \mathbb{Z}$, without loss of generality, we can assume that $j \le k$, by (2.2), we get

$$\begin{aligned} \left|b(t) - b_{B_k}\right| &\le \left|b(t) - b_{B_j}\right| + \sum_{i=k}^{j-1} |b_{B_i} - b_{B_{i+1}}| \\ &\le \left|b(t) - b_{B_j}\right| + p^n |j-k| \|b\|_{CMO^1(\mathbb{Q}_p^n)}. \end{aligned} \tag{2.3}$$

The lemma is proved. □

Proof of Theorem 1.1 Denote $f(x)\chi_i(x) = f_i(x)$.

(1) By definition,

$$\begin{aligned} \left\|\left(\mathcal{H}^p_{\beta,b} f\right)\chi_k\right\|^{r_2}_{L^{r_2}(\mathbb{Q}_p^n)} &= \int_{S_k} |x|_p^{-r_2(n-\beta)}\left|\int_{B(0,|x|_p)} f(t)\left(b(x) - b(t)\right) dt\right|^{r_2} dx \\ &\le \int_{S_k} p^{-kr_2(n-\beta)}\left(\int_{B(0,p^k)}\left|f(t)\left(b(x) - b(t)\right)\right| dt\right)^{r_2} dx \\ &= p^{-kr_2(n-\beta)}\int_{S_k}\left(\sum_{j=-\infty}^{k}\int_{S_j}\left|f(t)\left(b(x) - b(t)\right)\right| dt\right)^{r_2} dx \\ &\le Cp^{-kr_2(n-\beta)}\int_{S_k}\left(\sum_{j=-\infty}^{k}\int_{S_j}\left|f(t)\left(b(x) - b_{B_k}\right)\right| dt\right)^{r_2} dx \\ &\quad + Cp^{-kr_2(n-\beta)}\int_{S_k}\left(\sum_{j=-\infty}^{k}\int_{S_j}\left|f(t)\left(b(t) - b_{B_k}\right)\right| dt\right)^{r_2} dx \\ &:= I + II. \end{aligned}$$

Now let us estimate I and II, respectively. For I, by Hölder's inequality ($\frac{1}{r_1} + \frac{1}{r_1'} = 1$), we have

$$\begin{aligned} I &= Cp^{-kr_2(n-\beta)}\left(\int_{S_k}\left|b(x) - b_{B_k}\right|^{r_2} dx\right)\left(\sum_{j=-\infty}^{k}\int_{S_j}\left|f(t)\right| dt\right)^{r_2} \\ &\le Cp^{\frac{-kr_2 n}{r_1'}}\left(\frac{1}{|B_k|_H}\int_{B_k}\left|b(x) - b_{B_k}\right|^{r_2} dx\right) \end{aligned}$$

$$\times \left\{ \sum_{j=-\infty}^{k} \left(\int_{S_j} \left|f(t)\right|^{r_1} dt \right)^{\frac{1}{r_1}} \left(\int_{S_j} dt \right)^{\frac{1}{r_1'}} \right\}^{r_2}$$

$$\leq C\|b\|^{r_2}_{CMO^{r_2}(\mathbb{Q}_p^n)} \left\{ \sum_{j=-\infty}^{k} p^{\frac{(j-k)n}{r_1'}} \|f_j\|_{L^{r_1}(\mathbb{Q}_p^n)} \right\}^{r_2}.$$

For II, by Lemma 2.2, we get

$$\begin{aligned} II &= Cp^{-kr_2(n-\beta)} \int_{S_k} \left(\sum_{j=-\infty}^{k} \int_{S_j} \left|f(t)\big(b(t)-b_{B_k}\big)\right| dt \right)^{r_2} dx \\ &= Cp^{-kr_2(n-\beta)} p^{kn}\big(1-p^{-n}\big) \left(\sum_{j=-\infty}^{k} \int_{S_j} \left|f(t)\big(b(t)-b_{B_k}\big)\right| dt \right)^{r_2} \\ &\leq Cp^{\frac{-kr_2 n}{r_1'}} \left(\sum_{j=-\infty}^{k} \int_{S_j} \left|f(t)\big(b(t)-b_{B_j}\big)\right| dt \right)^{r_2} \\ &\quad + Cp^{\frac{-kr_2 n}{r_1'}} \|b\|^{r_2}_{CMO^1(\mathbb{Q}_p^n)} \left(\sum_{j=-\infty}^{k} (k-j) \int_{S_j} \left|f(t)\right| dt \right)^{r_2} = II_1 + II_2. \end{aligned}$$

For II_1 and II_2, by Hölder's inequality, we obtain

$$\begin{aligned} II_1 &\leq Cp^{\frac{-kr_2 n}{r_1'}} \left\{ \left(\sum_{j=-\infty}^{k} \int_{S_j} \left|f(t)\right|^{r_1} dt \right)^{\frac{1}{r_1}} \left(\int_{S_j} \left|b(t)-b_{B_j}\right|^{r_1'} dt \right)^{\frac{1}{r_1'}} \right\}^{r_2} \\ &\leq Cp^{\frac{-kr_2 n}{r_1'}} \left\{ \sum_{j=-\infty}^{k} \|f_j\|_{L^{r_1}(\mathbb{Q}_p^n)} p^{\frac{jn}{r_1'}} \left(\frac{1}{|B_j|_H} \int_{B_j} \left|b(t)-b_{B_j}\right|^{r_1'} dt \right)^{\frac{1}{r_1'}} \right\}^{r_2} \\ &\leq C\|b\|^{r_2}_{CMO^{r_1'}(\mathbb{Q}_p^n)} \left\{ \sum_{j=-\infty}^{k} p^{\frac{(j-k)n}{r_1'}} \|f_j\|_{L^{r_1}(\mathbb{Q}_p^n)} \right\}^{r_2} \end{aligned}$$

and

$$\begin{aligned} II_2 &\leq Cp^{\frac{-kr_2 n}{r_1'}} \|b\|^{r_2}_{CMO^1(\mathbb{Q}_p^n)} \left\{ \sum_{j=-\infty}^{k} (k-j) \left(\int_{S_j} \left|f(t)\right|^{r_1} dt \right)^{\frac{1}{r_1}} \left(\int_{S_j} dt \right)^{\frac{1}{r_1'}} \right\}^{r_2} \\ &\leq C\|b\|^{r_2}_{CMO^1(\mathbb{Q}_p^n)} \left\{ \sum_{j=-\infty}^{k} (k-j) p^{\frac{(j-k)n}{r_1'}} \|f_j\|_{L^{r_1}(\mathbb{Q}_p^n)} \right\}^{r_2}. \end{aligned}$$

Then the above inequalities together with Lemma 2.1 imply that

$$\begin{aligned} \left\|\mathcal{H}^p_{\beta,b} f\right\|_{K^{\alpha,q_2}_{r_2}(\mathbb{Q}_p^n)} &= \left(\sum_{k=-\infty}^{+\infty} p^{k\alpha q_2} \left\| \big(\mathcal{H}^p_{\beta,b} f\big)\chi_k \right\|^{q_2}_{L^{r_2}(\mathbb{Q}_p^n)} \right)^{\frac{1}{q_2}} \\ &\leq \left(\sum_{k=-\infty}^{+\infty} p^{k\alpha q_1} \left\| \big(\mathcal{H}^p_{\beta,b} f\big)\chi_k \right\|^{q_1}_{L^{r_2}(\mathbb{Q}_p^n)} \right)^{\frac{1}{q_1}} \end{aligned}$$

$$
\begin{aligned}
&\le C\left\{\sum_{k=-\infty}^{+\infty} p^{k\alpha q_1}\|b\|^{q_1}_{CMO^{r_2}(\mathbb{Q}_p^n)}\left(\sum_{j=-\infty}^{k} p^{\frac{(j-k)n}{r_1'}}\|f_j\|_{L^{r_1}(\mathbb{Q}_p^n)}\right)^{q_1}\right\}^{\frac{1}{q_1}} \\
&\quad + C\left\{\sum_{k=-\infty}^{+\infty} p^{k\alpha q_1}\|b\|^{q_1}_{CMO^{r_1'}(\mathbb{Q}_p^n)}\left(\sum_{j=-\infty}^{k} p^{\frac{(j-k)n}{r_1'}}\|f_j\|_{L^{r_1}(\mathbb{Q}_p^n)}\right)^{q_1}\right\}^{\frac{1}{q_1}} \\
&\quad + C\left\{\sum_{k=-\infty}^{+\infty} p^{k\alpha q_1}\|b\|^{q_1}_{CMO^{1}(\mathbb{Q}_p^n)}\left(\sum_{j=-\infty}^{k} (k-j)p^{\frac{(j-k)n}{r_1'}}\|f_j\|_{L^{r_1}(\mathbb{Q}_p^n)}\right)^{q_1}\right\}^{\frac{1}{q_1}} \\
&\le C\|b\|_{CMO^{\max\{r_1',r_2\}}(\mathbb{Q}_p^n)}\left\{\sum_{k=-\infty}^{+\infty} p^{k\alpha q_1}\left(\sum_{j=-\infty}^{k} (k-j)p^{\frac{(j-k)n}{r_1'}}\|f_j\|_{L^{r_1}(\mathbb{Q}_p^n)}\right)^{q_1}\right\}^{\frac{1}{q_1}} \\
&= J.
\end{aligned}
\tag{2.4}
$$

For the case $0 < q_1 \le 1$, since $\alpha < \frac{n}{r_1'}$, we have

$$
\begin{aligned}
J^{q_1} &= C\|b\|^{q_1}_{CMO^{\max\{r_1',r_2\}}(\mathbb{Q}_p^n)}\sum_{k=-\infty}^{+\infty} p^{k\alpha q_1}\left(\sum_{j=-\infty}^{k} (k-j)p^{\frac{(j-k)n}{r_1'}}\|f_j\|_{L^{r_1}(\mathbb{Q}_p^n)}\right)^{q_1} \\
&= C\|b\|^{q_1}_{CMO^{\max\{r_1',r_2\}}(\mathbb{Q}_p^n)}\sum_{k=-\infty}^{+\infty}\left(\sum_{j=-\infty}^{k} p^{j\alpha}\|f_j\|_{L^{r_1}(\mathbb{Q}_p^n)}(k-j)p^{(j-k)(\frac{n}{r_1'}-\alpha)}\right)^{q_1} \\
&\le C\|b\|^{q_1}_{CMO^{\max\{r_1',r_2\}}(\mathbb{Q}_p^n)}\sum_{k=-\infty}^{+\infty}\sum_{j=-\infty}^{k} p^{j\alpha q_1}\|f_j\|^{q_1}_{L^{r_1}(\mathbb{Q}_p^n)}(k-j)^{q_1}p^{(j-k)(\frac{n}{r_1'}-\alpha)q_1} \\
&= C\|b\|^{q_1}_{CMO^{\max\{r_1',r_2\}}(\mathbb{Q}_p^n)}\sum_{j=-\infty}^{+\infty} p^{j\alpha q_1}\|f_j\|^{q_1}_{L^{r_1}(\mathbb{Q}_p^n)}\sum_{k=j}^{+\infty}(k-j)^{q_1}p^{(j-k)(\frac{n}{r_1'}-\alpha)q_1} \\
&= C\|b\|^{q_1}_{CMO^{\max\{r_1',r_2\}}(\mathbb{Q}_p^n)}\|f\|^{q_1}_{K_{r_1}^{\alpha,r_1}(\mathbb{Q}_p^n)}.
\end{aligned}
\tag{2.5}
$$

For the case $q_1 > 1$, by Hölder's inequality, we have

$$
\begin{aligned}
J^{q_1} &= C\|b\|^{q_1}_{CMO^{\max\{r_1',r_2\}}(\mathbb{Q}_p^n)}\sum_{k=-\infty}^{+\infty}\left(\sum_{j=-\infty}^{k} p^{j\alpha}\|f_j\|_{L^{r_1}(\mathbb{Q}_p^n)}(k-j)p^{(j-k)(\frac{n}{r_1'}-\alpha)}\right)^{q_1} \\
&\le C\|b\|^{q_1}_{CMO^{\max\{r_1',r_2\}}(\mathbb{Q}_p^n)}\sum_{k=-\infty}^{+\infty}\left(\sum_{j=-\infty}^{k} p^{j\alpha q_1}\|f_j\|^{q_1}_{L^{r_1}(\mathbb{Q}_p^n)}p^{\frac{(j-k)}{2}(\frac{n}{r_1'}-\alpha)q_1}\right) \\
&\quad \times\left(\sum_{j=-\infty}^{k}(k-j)^{q_1'}p^{\frac{(j-k)}{2}(\frac{n}{r_1'}-\alpha)q_1'}\right)^{\frac{q_1}{q_1'}} \\
&= C\|b\|^{q_1}_{CMO^{\max\{r_1',r_2\}}(\mathbb{Q}_p^n)}\sum_{j=-\infty}^{+\infty} p^{j\alpha q_1}\|f_j\|^{q_1}_{L^{r_1}(\mathbb{Q}_p^n)}\sum_{k=j}^{+\infty}p^{\frac{(j-k)}{2}(\frac{n}{r_1'}-\alpha)q_1} \\
&= C\|b\|^{q_1}_{CMO^{\max\{r_1',r_2\}}(\mathbb{Q}_p^n)}\|f\|^{q_1}_{K_{r_1}^{\alpha,r_1}(\mathbb{Q}_p^n)}.
\end{aligned}
\tag{2.6}
$$

Then (1.6) follows from (2.4)-(2.6).

(2) By definition,

$$\begin{aligned}
\|(\mathcal{H}^{p,*}_{\beta,b}f)\chi_k\|^{r_2}_{L^{r_2}(\mathbb{Q}^n_p)} &= \int_{S_k}\left|\int_{\mathbb{Q}^n_p\setminus B(0,|x|_p)}|t|^{\beta-n}_p f(t)\big(b(x)-b(t)\big)\,dt\right|^{r_2}dx \\
&\le \int_{S_k}\left(\int_{\mathbb{Q}^n_p\setminus B(0,p^k)}|t|^{\beta-n}_p\big|f(t)\big(b(x)-b(t)\big)\big|\,dt\right)^{r_2}dx \\
&= \int_{S_k}\left(\sum_{j=k}^{\infty}\int_{S_j}p^{j(\beta-n)}\big|f(t)\big(b(x)-b(t)\big)\big|\,dt\right)^{r_2}dx \\
&\le C\int_{S_k}\left(\sum_{j=k}^{\infty}\int_{S_j}p^{j(\beta-n)}\big|f(t)\big(b(x)-b_{B_k}\big)\big|\,dt\right)^{r_2}dx \\
&\quad + C\int_{S_k}\left(\sum_{j=k}^{\infty}\int_{S_j}p^{j(\beta-n)}\big|f(t)\big(b(t)-b_{B_k}\big)\big|\,dt\right)^{r_2}dx \\
&:= K+L.
\end{aligned}$$

By Hölder's inequality, we get

$$\begin{aligned}
K &= C\left(\int_{S_k}\big|b(x)-b_{B_k}\big|^{r_2}\,dx\right)\left(\sum_{j=k}^{\infty}\int_{S_j}p^{j(\beta-n)}\big|f(t)\big|\,dt\right)^{r_2} \\
&\le Cp^{kn}\left(\frac{1}{|B_k|_H}\int_{B_k}\big|b(x)-b_{B_k}\big|^{r_2}\,dx\right) \\
&\quad \times\left\{\sum_{j=k}^{\infty}\left(\int_{S_j}\big|f(t)\big|^{r_1}\,dt\right)^{\frac{1}{r_1}}p^{j(\beta-\frac{n}{r_1})}\right\}^{r_2} \\
&\le C\|b\|^{r_2}_{CMO^{r_2}(\mathbb{Q}^n_p)}\left\{\sum_{j=k}^{\infty}p^{\frac{(k-j)n}{r_2}}\|f_j\|_{L^{r_1}(\mathbb{Q}^n_p)}\right\}^{r_2}.
\end{aligned} \tag{2.7}$$

By Lemma 2.2, we have

$$\begin{aligned}
L &= C\int_{S_k}\left(\sum_{j=k}^{\infty}\int_{S_j}p^{j(\beta-n)}\big|f(t)\big(b(t)-b_{B_k}\big)\big|\,dt\right)^{r_2}dx \\
&= Cp^{kn}\big(1-p^{-n}\big)\left(\sum_{j=k}^{\infty}\int_{S_j}p^{j(\beta-n)}\big|f(t)\big(b(t)-b_{B_k}\big)\big|\,dt\right)^{r_2} \\
&\le Cp^{kn}\left(\sum_{j=k}^{\infty}\int_{S_j}p^{j(\beta-n)}\big|f(t)\big(b(t)-b_{B_j}\big)\big|\,dt\right)^{r_2} \\
&\quad + Cp^{kn}\|b\|^{r_2}_{CMO^1(\mathbb{Q}^n_p)}\left(\sum_{j=k}^{\infty}(j-k)p^{j(\beta-n)}\int_{S_j}\big|f(t)\big|\,dt\right)^{r_2} \\
&= L_1+L_2.
\end{aligned}$$

For L_1 and L_2, by Hölder's inequality, we obtain

$$\begin{aligned}
L_1 &\le Cp^{kn}\left\{\sum_{j=k}^{\infty} p^{j(\beta-n)}\left(\int_{S_j}\left|b(t)-b_{B_j}\right|^{r_1'}dt\right)^{\frac{1}{r_1'}}\left(\int_{S_j}\left|f(t)\right|^{r_1}dt\right)^{\frac{1}{r_1}}\right\}^{r_2}\\
&\le Cp^{kn}\left\{\sum_{j=k}^{\infty} p^{\frac{jn}{r_1'}+j(\beta-n)}\|f_j\|_{L^{r_1}(\mathbb{Q}_p^n)}\left(\frac{1}{|B_j|_H}\int_{B_j}\left|b(t)-b_{B_j}\right|^{r_1'}dt\right)^{\frac{1}{r_1'}}\right\}^{r_2}\\
&\le C\|b\|^{r_2}_{CMO^{r_1'}(\mathbb{Q}_p^n)}\left\{\sum_{j=k}^{\infty} p^{\frac{(k-j)n}{r_2}}\|f_j\|_{L^{r_1}(\mathbb{Q}_p^n)}\right\}^{r_2}
\end{aligned}$$

and

$$\begin{aligned}
L_2 &\le C\|b\|^{r_2}_{CMO^1(\mathbb{Q}_p^n)}p^{kn}\left\{\sum_{j=k}^{\infty}(j-k)p^{j(\beta-n)}\left(\int_{S_j}\left|f(t)\right|^{r_1}dt\right)^{\frac{1}{r_1}}\left(\int_{S_j}dt\right)^{\frac{1}{r_1'}}\right\}^{r_2}\\
&\le C\|b\|^{r_2}_{CMO^1(\mathbb{Q}_p^n)}\left\{\sum_{j=k}^{\infty}(j-k)p^{\frac{(k-j)n}{r_2}}\|f_j\|_{L^{r_1}(\mathbb{Q}_p^n)}\right\}^{r_2}. \qquad (2.8)
\end{aligned}$$

Then (2.7)-(2.8) together with Lemma 2.1 imply that

$$\begin{aligned}
\left\|\mathcal{H}^{p,*}_{\beta,b}f\right\|_{K^{\alpha,q_2}_{r_2}(\mathbb{Q}_p^n)} &= \left(\sum_{k=-\infty}^{+\infty}p^{k\alpha q_2}\left\|\left(\mathcal{H}^{p,*}_{\beta,b}f\right)\chi_k\right\|^{q_2}_{L^{r_2}(\mathbb{Q}_p^n)}\right)^{\frac{1}{q_2}}\\
&\le \left(\sum_{k=-\infty}^{+\infty}p^{k\alpha q_1}\left\|\left(\mathcal{H}^{p,*}_{\beta,b}f\right)\chi_k\right\|^{q_1}_{L^{r_2}(\mathbb{Q}_p^n)}\right)^{\frac{1}{q_1}}\\
&\le C\left(\sum_{k=-\infty}^{+\infty}p^{k\alpha q_1}\|b\|^{q_1}_{CMO^{r_2}(\mathbb{Q}_p^n)}\left(\sum_{j=k}^{\infty}p^{\frac{(k-j)n}{r_2}}\|f_j\|_{L^{r_1}(\mathbb{Q}_p^n)}\right)^{q_1}\right)^{\frac{1}{q_1}}\\
&\quad + C\left(\sum_{k=-\infty}^{+\infty}p^{k\alpha q_1}\|b\|^{q_1}_{CMO^{r_1'}(\mathbb{Q}_p^n)}\left(\sum_{j=k}^{\infty}p^{\frac{(k-j)n}{r_2}}\|f_j\|_{L^{r_1}(\mathbb{Q}_p^n)}\right)^{q_1}\right)^{\frac{1}{q_1}}\\
&\quad + C\left(\sum_{k=-\infty}^{+\infty}p^{k\alpha q_1}\|b\|^{q_1}_{CMO^{1}(\mathbb{Q}_p^n)}\left(\sum_{j=k}^{\infty}(k-j)p^{\frac{(j-k)n}{r_2}}\|f_j\|_{L^{r_1}(\mathbb{Q}_p^n)}\right)^{q_1}\right)^{\frac{1}{q_1}}\\
&\le C\|b\|_{CMO^{\max\{r_1',r_2\}}(\mathbb{Q}_p^n)}\left(\sum_{k=-\infty}^{+\infty}p^{k\alpha q_1}\left(\sum_{j=k}^{\infty}(j-k)p^{\frac{(k-j)n}{r_2}}\|f_j\|_{L^{r_1}(\mathbb{Q}_p^n)}\right)^{q_1}\right)^{\frac{1}{q_1}}\\
&= S.
\end{aligned}$$

For the case $0 < q_1 \le 1$, since $\alpha > -\frac{n}{r_2}$, we have

$$\begin{aligned}
S^{q_1} &= C\|b\|^{q_1}_{CMO^{\max\{r_1',r_2\}}(\mathbb{Q}_p^n)}\sum_{k=-\infty}^{+\infty}p^{k\alpha q_1}\left(\sum_{j=k}^{\infty}(j-k)p^{\frac{(k-j)n}{r_2}}\|f_j\|_{L^{r_1}(\mathbb{Q}_p^n)}\right)^{q_1}\\
&= C\|b\|^{q_1}_{CMO^{\max\{r_1',r_2\}}(\mathbb{Q}_p^n)}\sum_{k=-\infty}^{+\infty}\left(\sum_{j=k}^{\infty}p^{j\alpha}\|f_j\|_{L^{r_1}(\mathbb{Q}_p^n)}(j-k)p^{(k-j)(\frac{n}{r_2}+\alpha)}\right)^{q_1}
\end{aligned}$$

$$
\begin{aligned}
&\le C\|b\|^{q_1}_{CMO^{\max\{r_1',r_2\}}(\mathbb{Q}_p^n)} \sum_{k=-\infty}^{+\infty} \sum_{j=k}^{\infty} p^{j\alpha q_1} \|f_j\|^{q_1}_{L^{r_1}(\mathbb{Q}_p^n)} (j-k)^{q_1} p^{(k-j)(\frac{n}{r_2}+\alpha)q_1} \\
&= C\|b\|^{q_1}_{CMO^{\max\{r_1',r_2\}}(\mathbb{Q}_p^n)} \sum_{j=-\infty}^{+\infty} p^{j\alpha q_1} \|f_j\|^{q_1}_{L^{r_1}(\mathbb{Q}_p^n)} \sum_{k=-\infty}^{j} (j-k)^{q_1} p^{(k-j)(\frac{n}{r_2}+\alpha)q_1} \\
&= C\|b\|^{q_1}_{CMO^{\max\{r_1',r_2\}}(\mathbb{Q}_p^n)} \|f\|^{q_1}_{K_{r_1}^{\alpha,q_1}(\mathbb{Q}_p^n)}.
\end{aligned}
$$

For the case $q_1 > 1$, by Hölder's inequality, we have

$$
\begin{aligned}
S^{q_1} &= C\|b\|^{q_1}_{CMO^{\max\{r_1',r_2\}}(\mathbb{Q}_p^n)} \sum_{k=-\infty}^{+\infty} p^{k\alpha q_1} \left(\sum_{j=k}^{\infty} (j-k) p^{\frac{(k-j)n}{r_2}} \|f_j\|_{L^{r_1}(\mathbb{Q}_p^n)} \right)^{q_1} \\
&\le C\|b\|^{q_1}_{CMO^{\max\{r_1',r_2\}}(\mathbb{Q}_p^n)} \sum_{k=-\infty}^{+\infty} \left(\sum_{j=k}^{\infty} p^{j\alpha q_1} \|f_j\|^{q_1}_{L^{r_1}(\mathbb{Q}_p^n)} p^{\frac{(k-j)}{2}(\frac{n}{r_2}+\alpha)q_1} \right) \\
&\quad \times \left(\sum_{j=k}^{\infty} (j-k)^{q_1'} p^{\frac{(k-j)}{2}(\frac{n}{r_2}+\alpha)q_1'} \right)^{\frac{q_1}{q_1'}} \\
&= C\|b\|^{q_1}_{CMO^{\max\{r_1',r_2\}}(\mathbb{Q}_p^n)} \sum_{j=-\infty}^{+\infty} p^{j\alpha q_1} \|f_j\|^{q_1}_{L^{r_1}(\mathbb{Q}_p^n)} \sum_{k=-\infty}^{j} p^{\frac{(k-j)}{2}(\frac{n}{r_2}+\alpha)q_1} \\
&= C\|b\|^{q_1}_{CMO^{\max\{r_1',r_2\}}(\mathbb{Q}_p^n)} \|f\|^{q_1}_{K_{r_1}^{\alpha,q_1}(\mathbb{Q}_p^n)}.
\end{aligned}
$$

Then the above inequalities imply (1.7). Theorem 1.1 is proved. □

Competing interests
The author declares that she has no competing interests.

Acknowledgements
The author sincerely thanks Professor Zunwei Fu for his useful discussions. This work was supported by NSF of China (Grant No. 11126203), NSF of Shandong Province (Grant Nos. ZR2010AL006).

Received: 20 July 2012 Accepted: 23 November 2012 Published: 11 December 2012

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doi:10.1186/1029-242X-2012-293
Cite this article as: Wu: Boundedness for commutators of fractional p-adic Hardy operators. *Journal of Inequalities and Applications* 2012 **2012**:293.