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Generalized conditions for starlikeness and convexity of certain analytic functions

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Abstract

For analytic functions $f(z)$ in the open unit disk $\mathbb{U}$ with $f(0) = 0$ and $f'(0) = 1$, Nunokawa et al. (Turk J Math 34, 333-337, 2010)have shown some conditions for starlikeness and convexity of $f(z)$. The object of the present paper is to derive some generalized conditions for starlikeness and convexity of functions $f(z)$ with examples.

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1 Introduction

Let $\mathcal{A}$ denote the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C}: |z| < 1\}$. Let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of functions $f(z)$ which are univalent in $\mathbb{U}$. A function $f(z) \in \mathcal{S}$ is said to be starlike with respect to the origin in $\mathbb{U}$ if $f(\mathbb{U})$ is the starlike domain. We denote by $\mathcal{S}^*$ the class of all starlike functions $f(z)$ with respect to the origin in $\mathbb{U}$. Furthermore, if a function $f(z) \in \mathcal{S}$ satisfies $zf'(z) \in \mathcal{S}^*$, then $f(z)$ is said to be convex in $\mathbb{U}$. We also denote by $\mathcal{K}$ the class of all convex functions in $\mathbb{U}$. Note that $\mathcal{K} \subset \mathcal{S}^* \subset \mathcal{S} \subset \mathcal{A}$.

To discuss the univalency of $f(z) \in \mathcal{A}$, Nunokawa [1] has given

Lemma 1.1 *If* $f(z) \in \mathcal{A}$*satisfies* $|f''(z)| < 1$ $(z \in \mathbb{U})$, *then* $f(z) \in \mathcal{S}$. Also, Mocanu [2] has shown that

Lemma 1.2 *If* $f(z) \in \mathcal{A}$*satisfies*

$$|f'(z) - 1| < \frac{2}{\sqrt{5}} \quad (z \in \mathbb{U}),$$

then $f(z) \in \mathcal{S}^*$.

In view of Lemmas 1.1 and 1.2, Nunokawa et al. [3] have proved the following results.

Lemma 1.3 *If* $f(z) \in \mathcal{A}$*satisfies*

$$|f''(z)| \leqq \frac{2}{\sqrt{5}} = 0.8944\ldots \quad (z \in \mathbb{U}), \tag{1.2}$$

Then $f(z) \in \mathcal{S}^*$.

Lemma 1.4 *If* $f(z) \in \mathcal{A}$*satisfies*

$$|f''(z)| \leqq \frac{1}{\sqrt{5}} = 0.4472\ldots \quad |(z \in \mathbb{U}), \tag{1.3}$$

then $f(z) \in \mathcal{K}$.

The object of the present paper is to consider some generalized conditions for functions $f(z)$ to be in the classes $\mathcal{S}^*$ or $\mathcal{K}$.

2 Generalized conditions for starlikeness

We begin with the statement and the proof of generalized conditions for starlikeness.

Theorem 2.1 *If* $f(z) \in \mathcal{A}$*satisfies*

$$|f^{(j)}(z)| \leqq \frac{2}{\sqrt{5}} - M \quad (z \in \mathbb{U}), \tag{2.1}$$

for some $j (j = 2, 3, 4, \ldots)$, *then* $f(z) \in \mathcal{S}^*$, *where*

$$M = \begin{cases} 0 & (j = 2) \\ \sum_{n=2}^{j-1} |f^{(n)}(0)| & (j \geqq 3). \end{cases} \tag{2.2}$$

Proof For $j = 2$, the inequality (2.1) becomes (1.2) of Lemma 1.2. Thus, the theorem is hold true for $j = 2$. We need to prove the inequality for $j \geqq 3$. Note that

$$f''(z) = \int_0^z f'''(t)dt + f''(0). \tag{2.3}$$

We suppose that $|f'''(z)| \leqq N_3 (z \in \mathbb{U})$. Then, (2.3) gives us that

$$\begin{aligned} |f''(z)| &\leqq \int_0^{|z|} |f'''(\rho e^{i\theta})\mathrm{d}\rho| + |f''(0)| \\ &\leqq N_3|z| + |f''(0)| \\ &< N_3 + |f''(0)|. \end{aligned} \tag{2.4}$$

Therefore, if $f(z)$ satisfies

$$|f''(z)| < N_3 + |f''(0)| \leqq \frac{2}{\sqrt{5}} \qquad (z \in \mathbb{U}), \tag{2.5}$$

then $f(z) \in \mathcal{S}^*$ by Lemma 1.3. This means that if $f(z)$ satisfies

$$|f'''(z)| \leqq N_3 \leqq \frac{2}{\sqrt{5}} - |f''(0)| \quad (z \in \mathbb{U}), \tag{2.6}$$

then $f(z) \in \mathcal{S}^*$. Thus, the theorem is holds true for $j = 3$.

Next, we suppose that the theorem is true for $j = 2, 3, 4, \ldots, (k - 1)$. Then, letting $|f^{(k)}(z)| \leqq N_k \quad (z \in \mathbb{U})$, we have that

$$\begin{aligned} |f^{(k-1)}(z)| &= \left| \int_0^z f^{(k)}(t)\mathrm{d}t + f^{(k-1)}(0) \right| \\ &\leqq N_k|z| + |f^{(k-1)}(0)| \\ &< N_k + |f^{(k-1)}(0)|. \end{aligned} \tag{2.7}$$

Thus, if $f(z)$ satisfies

$$\begin{aligned} |f^{(k-1)}(z)| &< N_k + |f^{(k-1)}(0)| \\ &\leqq \frac{2}{\sqrt{5}} - \sum_{n=2}^{k-2} |f^{(n)}(0)|, \end{aligned} \tag{2.8}$$

then $f(z) \in \mathcal{S}^*$. This is equivalent to

$$|f^{(k)}(z)| \leqq N_k \leqq \frac{2}{\sqrt{5}} - \sum_{n=2}^{k-1} |f^{(n)}(0)|. \tag{2.9}$$

Therefore, the theorem holds true for $j = k$. Thus, applying the mathematical induction, we complete the proof of the theorem.

Example 2.1 Let us consider a function

$$f(z) = z + a_2z^2 + a_3z^3 + a_4z^4. \tag{2.10}$$

Since

$$|f'''(z)| = 24|a_4|,$$

if $f(z)$ satisfies

$$24|a_4| \leqq \frac{2}{\sqrt{5}} - 2|a_2| - 6|a_3|,$$

then $f(z) \in \mathcal{S}^*$. This is equivalent to

$$\sqrt{5}|a_2| + 3\sqrt{5}|a_3| + 12\sqrt{5}|a_4| \leqq 1.$$

Therefore, we put

$$a_2 = \frac{\mathrm{e}^{i\theta_1}}{2\sqrt{5}}, \quad a_3 = \frac{\mathrm{e}^{i\theta_2}}{9\sqrt{5}}, \quad a_4 = \frac{\mathrm{e}^{i\theta_3}}{72\sqrt{5}}.$$

Consequently, we see that the function

$$f(z) = z + \frac{\mathrm{e}^{i\theta_1}}{2\sqrt{5}}z^2 + \frac{\mathrm{e}^{i\theta_2}}{9\sqrt{5}}z^3 + \frac{\mathrm{e}^{i\theta_3}}{72\sqrt{5}}z^4$$

is in the class $\mathcal{S}^*$.

3 Generalized conditions for convexity

For the convexity of $f(z)$, we derive

Theorem 3.1 *If* $f(z) \in \mathcal{A}$*satisfies*

$$|f^{(j)}(z)| \leqq \frac{1}{j!}\left(\frac{4}{\sqrt{5}} - P\right) \quad (z \in \mathbb{U}). \tag{3.1}$$

for some $j (j = 3, 4, 5, \ldots)$, *then* $f(z) \in \mathcal{K}$, *where*

$$P = \sum_{n=2}^{j-1} n \cdot n! |f^{(n)}(0)|. \tag{3.2}$$

Proof We have to prove for $j \geqq 3$. Note that

$$(zf'(z))'' = 2f''(z) + zf'''(z) = 2\left(\int_0^z f'''(t)\mathrm{d}t + f''(0)\right) + zf'''(z). \tag{3.3}$$

If $|f'''(z)| \leqq N_3 \quad (z \in \mathbb{U})$, then we have that

$$\begin{aligned} |(zf'(z))''| &\leqq 2\left|\int_0^z f'''(t)\mathrm{d}t + f''(0)\right| + |zf'''(z)| \\ &\leqq 2\int_0^{|z|} |f'''(\rho e^{i\theta})\mathrm{d}\rho| + 2|f''(0)| + N_3|z| \\ &\leqq 3N_3|z| + 2|f''(0)| \\ &< 3N_3 + 2|f''(0)|. \end{aligned} \tag{3.4}$$

We know that $f(z) \in \mathcal{K}$ if and only if $zf'(z) \in \mathcal{S}^*$. Therefore, if

$$3N_3 + 2\left|f''(0)\right| \leqq \frac{2}{\sqrt{5}}, \tag{3.5}$$

then $zf'(z) \in \mathcal{S}^*$ by means of Lemma 1.3. Thus, if

$$|f'''(z)| \leqq N_3 \leqq \frac{2}{3}\left(\frac{1}{\sqrt{5}} - |f''(0)|\right) \quad (z \in \mathbb{U}), \tag{3.6}$$

then $f(z) \in \mathcal{K}$. This shows that the theorem is true for $j = 3$.

Next, we assume that theorem is true for $j = 3, 4, 5, \ldots, (k - 1)$. Then, letting $|f^{(k)}(z)| \leqq N_k (z \in \mathbb{U})$, we obtain that

$$\begin{aligned} \left|(zf'(z))^{(k-1)}\right| &= |(k-1)f^{(k-1)}(z) + zf^{(k)}(z)| \\ &= \left|(k-1)\left(\int_0^z f^{(k)}(t)\mathrm{d}t + f^{(k-1)}(0)\right) + zf^{(k)}(z)\right| \\ &\leqq (k-1)\left(\int_0^{|z|} |f^{(k)}(\rho \mathrm{e}^{i\theta})\mathrm{d}\rho| + |f^{(k-1)}(0)|\right) + |z|\left|f^{(k)}(z)\right|. \end{aligned} \tag{3.7}$$

Now, we consider $|f^{(k)}(z)| \leqq N_k (z \in \mathbb{U})$,. Then, (3.7) implies that

$$\begin{aligned} \left|(zf'(z))^{(k-1)}\right| &\leqq kN_k|z| + (k-1)\left|f^{(k-1)}(0)\right| \\ &< kN_k + (k-1)\left|f^{(k-1)}(0)\right|. \end{aligned} \tag{3.8}$$

Since, if

$$\left|\left(zf'(z)\right)^{(k-1)}\right| \leqq \frac{1}{(k-1)!}\left(\frac{4}{\sqrt{5}} - \sum_{n=2}^{k-2} n \cdot n! \left|f^{(n)}(0)\right|\right),$$

then $f(z) \in \mathcal{K}$ (or $zf'(z) \in \mathcal{S}^*$), if $f(z)$ satisfies that

$$kN_k + (k-1)\left|f^{(k-1)}(0)\right| \leqq \frac{1}{(k-1)!}\left(\frac{4}{\sqrt{5}} - \sum_{n=2}^{k-2} n \cdot n! \left|f^{(n)}(0)\right|\right), \tag{3.9}$$

that is, that

$$N_k \leqq \frac{1}{k!}\left(\frac{4}{\sqrt{5}} - \sum_{n=2}^{k-1} n \cdot n! \left|f^{(n)}(0)\right|\right), \tag{3.10}$$

then $f(z) \in \mathcal{K}$. Thus, the result is true for $j = k$. Using the mathematical induction, we complete the proof the theorem.

Example 3.1 We consider the function

$$f(z) = z + a_2 z^2 + a_3 z^3 + a_4 z^4.$$

Then, if $f(z)$ satisfies

$$24|a_4| \leqq \frac{1}{24}\left(\frac{4}{\sqrt{5}} - 8|a_2| - 108|a_3|\right),$$

then $f(z) \in \mathcal{K}$. Since

$$2\sqrt{5}|a_2| + 27\sqrt{5}|a_3| + 144\sqrt{5}|a_4| \leqq 1,$$

we consider

$$a_2 = \frac{e^{i\theta_1}}{4\sqrt{5}}, \quad a_3 = \frac{e^{i\theta_2}}{81\sqrt{5}}, \quad a_4 = \frac{e^{i\theta_3}}{864\sqrt{5}}.$$

With this conditions, the function

$$f(z) = z + \frac{e^{i\theta_1}}{4\sqrt{5}} z^2 + \frac{e^{i\theta_2}}{81\sqrt{5}} z^3 + \frac{e^{i\theta_3}}{864\sqrt{5}} z^4$$

belongs to the class $\mathcal{K}$.

If we use the same technique as in the proof of Theorem 2.1 applying Lemma 1.4, then we have

Theorem 3.2 *If* $f(z) \in \mathcal{A}$*satisfies*

$$|f^{(j)}(z)| \leqq \frac{1}{\sqrt{5}} - M \quad (z \in \mathbb{U}) \tag{3.11}$$

for some j ($j = 2, 3, 4, \ldots$), *then* $f(z) \in \mathcal{K}$, *where* M *is given by* (2.2).

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