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Value distribution for difference operator of meromorphic functions with maximal deficiency sum

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Abstract

The main purpose of this paper is to investigate the relationship between the characteristic function of a meromorphic function $f(z)$ with maximal deficiency sum and that of the exact difference $\Delta_c f = f(z+c) - f(z)$. As an application, the author also establishes an inequality on the zeros and poles for $\Delta_c f$ and gives an example to show that the upper bound of the inequality is accurate.

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1 Introduction

If $f(z)$ is a meromorphic function in the complex plane $\mathbb{C}$ and $a \in \mathbb{C}$, we use the following notations frequently used in Nevanlinna theory (see [1–3]): $m(r,f)$, $N(r,f)$, $m(r,a) = m(r, \frac{1}{f-a})$, $N(r,a) = N(r, \frac{1}{f-a}), \dots$. Denote by $S(r,f)$ any quantity such that $S(r,f) = o(T(r,f))$, $r \to \infty$ without restriction if $f(z)$ is of finite order and otherwise except possibly for a set of values of r of finite linear measure. The Nevanlinna deficiency of f with respect to a finite complex number a is defined by

$$\delta(a,f) = \liminf_{r\to\infty} \frac{m(r,a)}{T(r,f)} = 1 - \limsup_{r\to\infty} \frac{N(r,a)}{T(r,f)}.$$

If $a = \infty$, then one should replace $m(r,a)(N(r,a))$ in the above formula by $m(r,f)(N(r,f))$. The classical second fundamental theorem of Nevanlinna theory asserts that the total deficiency of any meromorphic function $f(z)$ satisfies the inequality

$$\sum_{a\in\mathbb{C}} \delta(a,f) + \delta(\infty,f) \le 2.$$

If the above equality holds, then we say that f has maximal deficiency sum. The Valiron-Mo'honko identity states that if the function $R(z,f)$ is rational in f and has small meromorphic coefficients, then

$$T\bigl(r, R(z,f)\bigr) = \deg_f(R) T(r,f) + S(r,f). \tag{1.1}$$

Certain relationship between the characteristic function of a meromorphic function $f(z)$ with maximal deficiency sum and that of derivative $f'(z)$ plays a key role in the study of a conjecture of Nevanlinna (see [4]). The main contribution of this paper is to study the relationship between the characteristic function of a meromorphic function $f(z)$ with maximal deficiency sum and that of the exact difference $\Delta_c f = f(z+c) - f(z)$, where $c \neq 0$ (see [5]).

In 1956, Shan and Singh [6] proved the following theorem.

Theorem A [6] *Suppose that $f(z)$ is a transcendental meromorphic function of finite order and $\sum_{a\in\mathbb{C}} \delta(a,f) = 2$. Then*

$$T(r,f') \sim 2T(r,f), \quad r \to +\infty.$$

After that, Edrei [7] and Weitsman [4] proved the following theorem, respectively.

Theorem B [4, 7] *Suppose that $f(z)$ is a transcendental meromorphic function of finite order with maximal deficiency sum. Then*

$$\lim_{r\to+\infty} \frac{T(r,f')}{T(r,f)} = 2 - \delta(\infty,f),$$

and

$$\lim_{r\to+\infty} \frac{N(r,\frac{1}{f'})}{T(r,f')} = 0.$$

Under the condition of Theorem B, Singh and Gopalakrishna [8] proved that

$$\lim_{r\to+\infty} \frac{N(r,a)}{T(r,f)} = 1 - \delta(a,f) \tag{1.2}$$

holds for every $a \in \mathbb{C}$.

Let $f(z)$ be a transcendental meromorphic function of order less than one. Bergweiler and Langley [9] proved that $\Delta_c f(z) \sim f'(z)$ outside some exceptional set. Motivated by this result, we extend Theorem B to the exact difference $\Delta_c f$ and prove the following theorem.

Theorem 1.1 (main) *Suppose that $f(z)$ is a transcendental meromorphic function of order less than one with maximal deficiency sum. Then we have*

(1) $\lim_{r\to+\infty} \frac{T(r,\Delta_c f)}{T(r,f)} = 2 - \delta(\infty,f)$.

(2) $\lim_{r\to+\infty} \frac{N(r,\frac{1}{\Delta_c f})}{T(r,\Delta_c f)} = 0$.

Consequently, we have that the deficiency of $\Delta_c f$ with respect to 0 is 1, i.e.,

$$\delta(0,\Delta_c f) = 1 - \limsup_{r\to+\infty} \frac{N(r,\frac{1}{\Delta_c f})}{T(r,\Delta_c f)} = 1.$$

For the zeros and poles involving the derivative of a transcendental meromorphic function of finite order with maximal deficiency sum, Singh and Kulkarni [10] proved the following theorem.

Theorem C [10] *Suppose that $f(z)$ is a transcendental meromorphic function of finite order with maximal deficiency sum. Then*

$$\frac{1-\delta(\infty,f)}{2-\delta(\infty,f)} \le K(f') \le \frac{2(1-\delta(\infty,f))}{2-\delta(\infty,f)},$$

where

$$K(f') = \limsup_{r\to+\infty} \frac{N(r,f') + N(r,\frac{1}{f'})}{T(r,f')}.$$

In 2000, Fang [11] proved the following theorem.

Theorem D [11] *Suppose that $f(z)$ is a transcendental meromorphic function of finite order with maximal deficiency sum. Then*

$$K(f') = \frac{2(1-\delta(\infty,f))}{2-\delta(\infty,f)}.$$

In fact, Fang [11] proved that Theorem D is valid for higher order derivatives of $f(z)$. In this paper, we shall extend Theorem D to the exact difference $\Delta_c f$ and prove the following theorem.

Theorem 1.2 (main) *Suppose that $f(z)$ is a transcendental meromorphic function of order less than one with maximal deficiency sum. Then*

$$K(\Delta_c f) \le \frac{2(1-\delta(\infty,f))}{2-\delta(\infty,f)},$$

where

$$K(\Delta_c f) = \limsup_{r\to+\infty} \frac{N(r,\Delta_c f) + N(r,\frac{1}{\Delta_c f})}{T(r,\Delta_c f)}.$$

The following example shows that the upper bound of the inequality in Theorem 1.2 is accurate.

Example 1.3 Let $f(z) = \frac{1}{e^z-1}$, then $\Delta_c f = \frac{(1-e^c)e^z}{(e^c e^z-1)(e^z-1)}$. Then $\Delta_c f \not\equiv 0$, $\delta(0,f) = 1$, $\delta(-1,f) = 1$, $\delta(\infty,f) = 0$. Thus $f(z)$ is a meromorphic function with maximal deficiency sum. It is obvious that $\delta(0,e^z) = 1$, $\delta(\infty,e^z) = 1$ and $N(r,\Delta_c f) = N(r,e^z = 1) + N(r,e^z = e^{-c})$. It follows from (1.2) that

$$N(r,e^z = 1) = N(r,e^z = e^{-c}) \sim T(r,e^z), \quad r \to +\infty$$

and from Valiron-Mo'honko identity (1.1) that

$$T(r,\Delta_c f) \sim 2T(r,e^z), \quad r \to +\infty.$$

Therefore, $K(\Delta_c f) = \frac{2(1-\delta(\infty,f))}{2-\delta(\infty,f)} = 1$.

By Theorem D and Example 1.3, we pose the following question.

Question 1.4 Under the condition of Theorem 1.2, can we replace $K(\Delta_c f) \leq \frac{2(1-\delta(\infty,f))}{2-\delta(\infty,f)}$ by $K(\Delta_c f) = \frac{2(1-\delta(\infty,f))}{2-\delta(\infty,f)}$?

Corollary 1.5 *Let $f(z)$ be a transcendental meromorphic function of order less than one with maximal deficiency sum, and assume $\delta(\infty,f) = 1$. Then*

$$\lim_{r\to+\infty} \frac{N(r,\Delta_c f)}{T(r,\Delta_c f)} = 0.$$

Consequently, we have that the deficiency of $\Delta_c f$ with respect to ∞ is 1, i.e.,

$$\delta(\infty,\Delta_c f) = 1 - \limsup_{r\to+\infty} \frac{N(r,\Delta_c f)}{T(r,\Delta_c f)} = 1.$$

As the end of this paper, we shall prove the following theorem.

Theorem 1.6 (main) *Let $f(z)$ be a transcendental meromorphic function of order less than one, and assume $\delta(\infty,f) = 1$. Then*

$$\sum_{a\in\mathbb{C}} \delta(a,f) \leq \delta(0,\Delta_c f).$$

2 Some lemmas

Lemma 2.1 [12] *Let $f(z)$ be a meromorphic function of finite order σ, and let c be a non-zero complex number. Then, for each $\varepsilon > 0$, we have*

$$m\left(r,\frac{f(z+c)}{f(z)}\right) = O\left(r^{\sigma-1+\varepsilon}\right).$$

Lemma 2.2 *Let $f(z)$ be a transcendental meromorphic function of order σ (< 1), and let c be a non-zero complex number. Then*

$$m\left(r,\frac{f(z+c)}{f(z)}\right) = o\left(T(r,f)\right) = S(r,f).$$

Proof Since the order of $f(z)$ is less than one, then, for any $0 < \varepsilon < 1 - \sigma$, it follows from Lemma 2.1 that

$$m\left(r,\frac{f(z+c)}{f(z)}\right) = O\left(r^{\sigma-1+\varepsilon}\right) = O(1).$$

Therefore, we have

$$m\left(r,\frac{f(z+c)}{f(z)}\right) = O(1) = o\left(T(r,f)\right) = S(r,f). \qquad \square$$

Lemma 2.3 [12] *Let $f(z)$ be a meromorphic function with the exponent of convergence of poles $\lambda(\frac{1}{f}) = \lambda < +\infty$, and let c be a non-zero complex number. Then, for each $\varepsilon > 0$, we have*

$$N\left(r,f(z+c)\right) = N(r,f) + O\left(r^{\lambda-1+\varepsilon}\right) + O(\log r).$$

From Lemma 2.3, using a similar method as that in the proof of Lemma 2.2, we can prove the following lemma.

Lemma 2.4 *Let $f(z)$ be a transcendental meromorphic function of order less than one, and let c be a non-zero complex number. Then*

$$N\big(r,f(z+c)\big)=N(r,f)+S(r,f).$$

3 Proof of Theorem 1.1

Proof By combining the first main theorem of Nevanlinna theory and Lemmas 2.2, 2.4, we have

$$\begin{aligned}T(r,\Delta_c f)&=m(r,\Delta_c f)+N(r,\Delta_c f)\\&=m\left(r,\frac{f\Delta_c f}{f}\right)+N(r,\Delta_c f)\\&\leq m\left(r,\frac{\Delta_c f}{f}\right)+m(r,f)+N(r,f)+N\big(r,f(z+c)\big)+O(1)\\&=T(r,f)+N(r,f)+S(r,f).\end{aligned}$$

Hence,

$$\limsup_{r\to\infty}\frac{T(r,\Delta_c f)}{T(r,f)}\leq 1+\limsup_{r\to\infty}\frac{N(r,f)}{T(r,f)}=2-\delta(\infty,f). \tag{3.1}$$

Let $\{a_i\}$ be a sequence of distinct complex numbers in $\mathbb{C}$ containing all the finite deficient values of $f(z)$. For any positive q, define

$$F(z)=\sum_{i=1}^{q}\frac{1}{f-a_i}.$$

Since $T(r,f(z)-a_i)=T(r,f(z))+O(1)$ and $\Delta_c(f(z)-a_i)=\Delta_c f(z)$, we deduce from Lemma 2.2 that

$$m\big(r,F(z)\Delta_c f(z)\big)\leq\sum_{i=1}^{q}m\left(r,\frac{\Delta_c(f(z)-a_i)}{f(z)-a_i}\right)+\log q=S(r,f).$$

This relation yields

$$m\big(r,F(z)\big)=m\left(r,F(z)\Delta_c f(z)\frac{1}{\Delta_c f(z)}\right)\leq m\left(r,\frac{1}{\Delta_c f}\right)+S(r,f). \tag{3.2}$$

By combining the first main theorem of Nevanlinna theory, (3.2) and Valiron-Mo'honko identity (1.1), we have

$$\begin{aligned}&qT(r,f)+N\left(r,\frac{1}{\Delta_c f}\right)\\&\quad=T\big(r,F(z)\big)+N\left(r,\frac{1}{\Delta_c f}\right)+O(1)\end{aligned}$$

$$\begin{aligned}
&= m\bigl(r, F(z)\bigr) + N\bigl(r, F(z)\bigr) + N\left(r, \frac{1}{\Delta_c f}\right) + O(1) \\
&\le m\bigl(r, F(z)\bigr) + N\left(r, \frac{1}{\Delta_c f}\right) + \sum_{i=1}^{q} N(r, a_i) + O(1) \\
&\le m\left(r, \frac{1}{\Delta_c f}\right) + N\left(r, \frac{1}{\Delta_c f}\right) + \sum_{i=1}^{q} N(r, a_i) + S(r,f) \\
&= T(r, \Delta_c f) + \sum_{i=1}^{q} N(r, a_i) + S(r,f).
\end{aligned}$$

Hence,

$$\begin{aligned}
q &\le \liminf_{r\to\infty} \frac{T(r, \Delta_c f)}{T(r,f)} + \sum_{i=1}^{q} \limsup_{r\to\infty} \frac{N(r, a_i)}{T(r,f)} \\
&= \liminf_{r\to\infty} \frac{T(r, \Delta_c f)}{T(r,f)} + \sum_{i=1}^{q} \bigl\{1 - \delta(a_i, f)\bigr\}.
\end{aligned}$$

Thus

$$\liminf_{r\to\infty} \frac{T(r, \Delta_c f)}{T(r,f)} \ge \sum_{i=1}^{q} \delta(a_i, f).$$

Since q is arbitrary, we have

$$\liminf_{r\to\infty} \frac{T(r, \Delta_c f)}{T(r,f)} \ge \sum_{a\in\mathbb{C}} \delta(a, f) = 2 - \delta(\infty, f).$$

Then

$$\lim_{r\to+\infty} \frac{T(r, \Delta_c f)}{T(r,f)} = 2 - \delta(\infty, f). \tag{3.3}$$

On the other hand, by combining the first main theorem of Nevanlinna theory and (3.2), we have

$$\begin{aligned}
&\sum_{i=1}^{q} m(r, a_i) + N\left(r, \frac{1}{\Delta_c f}\right) \\
&\quad\le m\bigl(r, F(z)\bigr) + N\left(r, \frac{1}{\Delta_c f}\right) + O(1) \\
&\quad\le T(r, \Delta_c f) + S(r,f).
\end{aligned}$$

Thus

$$\sum_{i=1}^{q} \frac{m(r, a_i)}{T(r, \Delta_c f)} + \frac{N(r, \frac{1}{\Delta_c f})}{T(r, \Delta_c f)} \le 1 + \frac{S(r,f)}{T(r, \Delta_c f)}.$$

We derive from (3.3) that

$$\begin{aligned}
&\sum_{i=1}^{q} \liminf_{r\to+\infty} \frac{m(r,a_i)}{T(r,\Delta_c f)} + \limsup_{r\to+\infty} \frac{N(r,\frac{1}{\Delta_c f})}{T(r,\Delta_c f)} \\
&\quad \le 1 + \limsup_{r\to+\infty} \frac{S(r,f)}{T(r,\Delta_c f)} \\
&\quad \le 1 + \limsup_{r\to+\infty} \frac{S(r,f)}{T(r,f)} \frac{T(r,f)}{T(r,\Delta_c f)} \\
&\quad = 1.
\end{aligned}$$

Thus

$$\begin{aligned}
1 &\ge \limsup_{r\to+\infty} \frac{N(r,\frac{1}{\Delta_c f})}{T(r,\Delta_c f)} + \sum_{i=1}^{q} \liminf_{r\to+\infty} \frac{m(r,a_i)}{T(r,\Delta_c f)} \\
&\ge \limsup_{r\to+\infty} \frac{N(r,\frac{1}{\Delta_c f})}{T(r,\Delta_c f)} + \sum_{i=1}^{q} \liminf_{r\to+\infty} \frac{m(r,a_i)}{T(r,f)} \liminf_{r\to+\infty} \frac{T(r,f)}{T(r,\Delta_c f)}.
\end{aligned}$$

It follows from (3.3) that

$$1 \ge \limsup_{r\to+\infty} \frac{N(r,\frac{1}{\Delta_c f})}{T(r,\Delta_c f)} + \frac{\sum_{i=1}^{q} \delta(a_i,f)}{2-\delta(\infty,f)}.$$

Since q is arbitrary, we have

$$1 \ge \limsup_{r\to+\infty} \frac{N(r,\frac{1}{\Delta_c f})}{T(r,\Delta_c f)} + 1.$$

Then

$$\limsup_{r\to+\infty} \frac{N(r,\frac{1}{\Delta_c f})}{T(r,\Delta_c f)} = 0.$$

Therefore,

$$\lim_{r\to+\infty} \frac{N(r,\frac{1}{\Delta_c f})}{T(r,\Delta_c f)} = 0.$$

□

4 Proof of Theorem 1.2

Proof It follows from Lemma 2.4 that

$$N(r,\Delta_c f) \le 2N(r,f) + S(r,f).$$

The above inequality implies that

$$\frac{N(r,\Delta_c f)}{T(r,\Delta_c f)} \frac{T(r,\Delta_c f)}{T(r,f)} \le 2\frac{N(r,f)}{T(r,f)} + \frac{S(r,f)}{T(r,f)}.$$

By Theorem 1.1(1), we have

$$\bigl(2-\delta(\infty,f)\bigr)\limsup_{r\to+\infty}\frac{N(r,\Delta_c f)}{T(r,\Delta_c f)}\le 2\bigl(1-\delta(\infty,f)\bigr).$$

Therefore,

$$\limsup_{r\to+\infty}\frac{N(r,\Delta_c f)}{T(r,\Delta_c f)}\le\frac{2(1-\delta(\infty,f))}{2-\delta(\infty,f)}.$$

This relation and Theorem 1.1(2) together yield

$$K(\Delta_c f)\le\frac{2(1-\delta(\infty,f))}{2-\delta(\infty,f)}. \qquad \square$$

5 Proof of Theorem 1.6

Proof If $\sum_{a\in\mathbb{C}}\delta(a,f)=0$, Theorem 1.6 is valid in this case. In the following, we assume that $\sum_{a\in\mathbb{C}}\delta(a,f)>0$. Let $\{a_\mu\}$ be a sequence of distinct complex numbers in $\mathbb{C}$ containing all the finite deficient values of $f(z)$. For any positive integer q, as we did in the proof of Theorem 1.1(2), we can get that

$$\sum_{\mu=1}^{q}m(r,a_\mu)+N\left(r,\frac{1}{\Delta_c f}\right)\le T(r,\Delta_c f)+S(r,f)$$

holds for any q finite complex numbers in $\{a_\mu\}$. Therefore, we have

$$\frac{N(r,\frac{1}{\Delta_c f})}{T(r,\Delta_c f)}+\frac{T(r,f)}{T(r,\Delta_c f)}\left(\frac{\sum_{\mu=1}^{q}m(r,a_\mu)}{T(r,f)}-o(1)\right)\le 1,\quad r\to+\infty.$$

Hence, from (3.1) we can get

$$\begin{aligned}
1&\ge\limsup_{r\to+\infty}\left[\frac{N(r,\frac{1}{\Delta_c f})}{T(r,\Delta_c f)}+\frac{T(r,f)}{T(r,\Delta_c f)}\left(\frac{\sum_{\mu=1}^{q}m(r,a_\mu)}{T(r,f)}-o(1)\right)\right]\\
&\ge\limsup_{r\to+\infty}\frac{N(r,\frac{1}{\Delta_c f})}{T(r,\Delta_c f)}+\liminf_{r\to+\infty}\frac{T(r,f)}{T(r,\Delta_c f)}\left(\frac{\sum_{\mu=1}^{q}m(r,a_\mu)}{T(r,f)}-o(1)\right)\\
&\ge\limsup_{r\to+\infty}\frac{N(r,\frac{1}{\Delta_c f})}{T(r,\Delta_c f)}+\liminf_{r\to+\infty}\frac{T(r,f)}{T(r,\Delta_c f)}\liminf_{r\to+\infty}\frac{\sum_{\mu=1}^{q}m(r,a_\mu)}{T(r,f)}\\
&\ge\limsup_{r\to+\infty}\frac{N(r,\frac{1}{\Delta_c f})}{T(r,\Delta_c f)}+\frac{\sum_{j=1}^{q}\delta(a_\mu,f)}{2-\delta(\infty,f)}.
\end{aligned}$$

Since q is arbitrary and $\delta(\infty,f)=1$, we have

$$\sum_{a\in\mathbb{C}}\delta(a)\le\delta(0,\Delta_c f). \qquad \square$$

Competing interests
The author declares that he has no competing interests.

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