Yang and Wang *Journal of Inequalities and Applications* (2015) 2015:56
DOI 10.1186/s13660-015-0572-0

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH **Open Access**

Generalized Finsler-Hadwiger type inequalities for simplices and applications

Shiguo Yang[1,2] and Wen Wang[1*]

*Correspondence: wenwang1985@163.com
[1] School of Mathematics and Statistics, Hefei Normal University, Hefei, Anhui 230601, P.R. China
Full list of author information is available at the end of the article

Abstract
For an n-dimensional simplex in the Euclidean space E^n, we establish some generalized Finsler-Hadwiger inequalities. By using these inequalities, generalizations of some well-known important inequalities for simplices are obtained.

MSC: 51M20; 52A40

Keywords: simplex; volume; circumradius; inradius; inequality

1 Introduction

Let ABC be a triangle (in the Euclidean plane) of area S and sides a, b, c. The following inequality [1, 2] is well known:

$$a^2 + b^2 + c^2 \geq 4\sqrt{3}S + (a-b)^2 + (b-c)^2 + (c-a)^2. \tag{1.1}$$

Equality holds iff this triangle is regular.

Let Ω_n be an n-dimensional simplex (in the n-dimensional Euclidean space E^n) of volume V and edge lengths a_i ($i = 1, 2, \ldots, \frac{1}{2}n(n+1)$). Let A_i ($i = 0, 1, \ldots, n$) be the vertices of Ω_n, F_i be the $(n-1)$-dimensional volume of the ith face $f_i = A_0 \cdots A_{i-1}A_{i+1} \cdots A_{n+1}$ ($(n-1)$-dimensional simplex) for $i = 0, 1, \ldots, n$. In 1989, Chen and Ma extended the inequality (1.1) to an n-simplex and established Finsler-Hadwiger type inequality for the edge length and the volume of an n-simplex as follows [3, 4]:

$$\sum_{i=1}^{\frac{1}{2}n(n+1)} a_i^2 \geq n(n+1)\left[\frac{(n!)^2}{n+1}\right]^{\frac{1}{n}} V^{\frac{2}{n}} + \frac{2}{n(n-1)} \sum_{1\leq i<j\leq \frac{1}{2}n(n+1)} (a_i - a_j)^2. \tag{1.2}$$

Equality holds iff Ω_n is regular.

In 1997, Leng and Tang established another kind of Finsler-Hadwiger type inequality for the face areas and the volume of an n-simplex as follows [4, 5]:

$$\sum_{i=0}^{n} F_i^2 \geq \frac{n^3(n+1)^{\frac{1}{n}}}{(n!)^{\frac{2}{n}}} V^{\frac{2(n-1)}{n}} + \frac{1}{n-1} \sum_{0\leq i<j\leq n} (F_i - F_j)^2. \tag{1.3}$$

Equality holds iff Ω_n is regular.

In this paper, we will discuss problem for generalized Finsler-Hadwiger type inequalities for k-dimensional face areas and the volume of an n-simplex, and establish generalized

Springer

Finsler-Hadwiger type inequalities for an n-simplex. From these inequalities, generalizations of some inequalities for an n-simplex are obtained.

2 Main results

Let R and r be the circumradius and the inradius of the simplex Ω_n, respectively. Let V_0 denote the volume of the n-dimensional regular simplex with circumradius R. For $k = 0,1,\ldots,n-1$, we put $\mu_{n,k} = \binom{n+1}{k+1} = \frac{(n+1)!}{(k+1)!(n-k)!}$. Let $V_i(k)$ $(i = 1,2,\ldots,\mu_{n,k})$ denote the k-dimensional volumes of the k-dimensional subsimplices of Ω_n. Our main results are the following theorems.

Theorem 1 *Let Ω_n be an n-simplex, nature number $k \in [1,n-1]$, real numbers $\alpha \in (0,1]$ and $\lambda \in (0,n+1-k]$. Then we have*

$$\sum_{i=1}^{\mu_{n,k}} V_i^{2\alpha}(k) \ge \left(\frac{R}{nr}\right)^{\frac{2k\alpha}{n(n+1)(n-1)^2}} \cdot \mu_{n,k} \cdot \left[\frac{k+1}{(k!)^2}\left(\frac{(n!)^2}{n+1}\right)^{\frac{k}{n}}\right]^{\alpha} V^{\frac{2k\alpha}{n}} + \frac{1}{\mu_{n,k}-\lambda}\sum_{1\le i<j\le\mu_{n,k}}\left(V_i^{\alpha}(k)-V_j^{\alpha}(k)\right)^2. \tag{2.1}$$

Equality holds iff Ω_n is regular.

Theorem 2 *Let Ω_n be an n-simplex, nature number $k \in [1,n-1]$, real numbers $\alpha \in (0,1]$ and $\lambda \in (0,n+1-k]$. Then we have*

$$\sum_{i=1}^{\mu_{n,k}} V_i^{2\alpha}(k) \ge \left(\frac{V_0}{V}\right)^{\frac{2k\alpha}{n(n+1)(n-1)^2}} \cdot \mu_{n,k} \cdot \left[\frac{k+1}{(k!)^2}\left(\frac{(n!)^2}{n+1}\right)^{\frac{k}{n}}\right]^{\alpha} V^{\frac{2k\alpha}{n}} + \frac{1}{\mu_{n,k}-\lambda}\sum_{1\le i<j\le\mu_{n,k}}\left(V_i^{\alpha}(k)-V_j^{\alpha}(k)\right)^2. \tag{2.2}$$

Equality holds iff Ω_n is regular.

Notice From the inequality $R \ge nr$ and Lemma 3, we have $\frac{R}{nr} \ge 1$ and $\frac{V_0}{V} \ge 1$, respectively. Further, according to the conditions of equalities hold, we now consider a class of n-simplices φ inscribed in an $(n-1)$-dimensional sphere of radius R (where R is constant number). If the radius r of an n-simplex in φ is sufficiently small, then $\frac{R}{nr}$ is large enough. Similarly, if the volume V of an n-simplex in φ is sufficiently small, then $\frac{V_0}{V}$ is sufficiently large.

Take $\lambda = n+1-k$, from Theorem 1 or Theorem 2 we derive an inequality for a simplex as follows.

Corollary 1 *Let Ω_n be an n-simplex and real number $\alpha \in (0,1]$. Then*

$$\sum_{i=1}^{\mu_{n,k}} V_i^{2\alpha}(k) \ge \mu_{n,k} \cdot \left[\frac{k+1}{(k!)^2}\left(\frac{(n!)^2}{n+1}\right)^{\frac{k}{n}}\right]^{\alpha} V^{\frac{2k\alpha}{n}} + \frac{1}{\mu_{n,k}-(n+1-k)}\sum_{1\le i<j\le\mu_{n,k}}\left(V_i^{\alpha}(k)-V_j^{\alpha}(k)\right)^2. \tag{2.3}$$

Equality holds iff Ω_n is regular.

By taking $k = 1$ and $\alpha = 1$ in the inequality (2.3) we can obtain the inequality (1.2). Take $k = n - 1$ and $\alpha = 1$ in the inequality (2.3) we can obtain the inequality (1.3).

Take $k = 1$, $\alpha = 1$ and $\lambda = n$ in Theorem 1 and Theorem 2, we get generalizations of the inequality (1.2) as follows.

Corollary 2 *For an n-simplex Ω_n, we have*

$$\sum_{i=1}^{\frac{1}{2}n(n+1)} a_i^2 \geq \left(\frac{R}{nr}\right)^{\frac{2}{n(n+1)(n-1)^2}} \cdot n(n+1) \cdot \left[\frac{(n!)^2}{n+1}\right]^{\frac{1}{n}} V^{\frac{2}{n}} + \frac{2}{n(n-1)} \sum_{1 \leq i < j \leq \frac{1}{2}n(n+1)} (a_i - a_j)^2, \tag{2.4}$$

$$\sum_{i=1}^{\frac{1}{2}n(n+1)} a_i^2 \geq \left(\frac{V_0}{V}\right)^{\frac{2}{n(n+1)(n-1)^2}} \cdot n(n+1) \cdot \left[\frac{(n!)^2}{n+1}\right]^{\frac{1}{n}} V^{\frac{2}{n}} + \frac{2}{n(n-1)} \sum_{1 \leq i < j \leq \frac{1}{2}n(n+1)} (a_i - a_j)^2. \tag{2.5}$$

Equality holds iff Ω_n is regular.

Take $k = n - 1$, $\alpha = 1$, and $\lambda = 2$ in Theorem 1 and Theorem 2; we get generalizations of the inequality (1.3) as follows.

Corollary 3 *Let Ω_n be an n-simplex. Then we have*

$$\sum_{i=0}^{n} F_i^2 \geq \left(\frac{R}{nr}\right)^{\frac{2}{n(n^2-1)}} \cdot \frac{n^3(n+1)^{\frac{1}{n}}}{(n!)^{\frac{2}{n}}} V^{\frac{2(n-1)}{n}} + \frac{1}{n-1} \sum_{0 \leq i < j \leq n} (F_i - F_j)^2, \tag{2.6}$$

$$\sum_{i=0}^{n} F_i^2 \geq \left(\frac{V_0}{V}\right)^{\frac{2}{n(n^2-1)}} \cdot \frac{n^3(n+1)^{\frac{1}{n}}}{(n!)^{\frac{2}{n}}} V^{\frac{2(n-1)}{n}} + \frac{1}{n-1} \sum_{0 \leq i < j \leq n} (F_i - F_j)^2. \tag{2.7}$$

Equality holds iff Ω_n is regular.

To prove the above theorems, we need some lemmas as follows.

Lemma 1 [4, 6] *For an n-simplex Ω_n, we have*

$$\left(\prod_{i=0}^{n} F_i\right)^{n-1} \geq \left[\frac{n^{3(n-1)}}{2(n+1)^{n-2}(n!)^2}\right]^{\frac{n+1}{2}} V^{n^2-n-2} \left(\prod_{i=1}^{\frac{1}{2}n(n+1)} a_i\right)^{\frac{2}{n}}, \tag{2.8}$$

with equality iff Ω_n is regular.

Lemma 2 [4, 7] *Let Ω_n be an n-simplex with edge lengths $a_{ij} = |A_iA_j|$ $(0 \leq i < j \leq n)$, then we have*

$$\prod_{0 \leq i < j \leq n} a_{ij} \geq \left(\frac{2^{n+1}}{n}\right)^{\frac{n}{4}} (n!)^{\frac{n}{2}} (VR)^{\frac{n}{2}}, \tag{2.9}$$

with equality iff Ω_n is regular.

Lemma 3 [4, 8] *For an n-simplex Ω_n, we have*

$$V \leq \left[\frac{(n+1)^{n+1}}{(n!)^2 n^n}\right]^{\frac{1}{2}} R^n, \tag{2.10}$$

with equality iff Ω_n is regular.

Lemma 4 [4, 6] *For an n-simplex Ω_n, we have*

$$\left(\prod_{i=0}^{n} F_i\right)^{n-1} \geq \left[n^{3n^2-4}(n+1)^{-(n+1)(n-2)}\right]^{\frac{1}{2}} (n!)^{-n} V^{n^2-n-1} R. \tag{2.11}$$

Equality holds iff Ω_n is regular.

Lemma 5 [4, 9] *Let Ω_n be an n-simplex and natural number $k \in [1, n-1)$. Then*

$$\frac{k!}{\sqrt{k+1}}\left(\prod_{i=1}^{\mu_{n,k}} V_i(k)\right)^{\frac{1}{\mu_{n,k}}} \geq \left[\frac{(n-1)!}{\sqrt{n}}\left(\prod_{i=0}^{n} F_i\right)^{\frac{1}{n+1}}\right]^{\frac{k}{n-1}}, \tag{2.12}$$

with equality iff Ω_n is regular.

Lemma 6 *For an n-simplex Ω_n, we have*

$$V \geq \frac{(n+1)^{\frac{n+1}{2}} n^{\frac{n^2-1}{2n}}}{n!} r^{\frac{n^2-1}{n}} R^{\frac{1}{n}}, \tag{2.13}$$

with equality iff Ω_n is regular.

Proof Let P be an interior point of Ω_n and d_i the distance from the point P to the ith face f_i of Ω_n for $i = 0, 1, \ldots, n$. Obviously, we have $\sum_{i=0}^{n} d_i F_i = nV$. Then

$$\sum_{i=0}^{n} \frac{d_i F_i}{nV} = 1.$$

Using the arithmetic-geometric means inequality we get

$$\prod_{i=0}^{n} \frac{d_i F_i}{nV} \leq \left(\frac{1}{n+1}\sum_{i=0}^{n}\frac{d_i F_i}{nV}\right)^{n+1} = \frac{1}{(n+1)^{n+1}},$$

namely

$$\left(\prod_{i=0}^{n} d_i\right)^{n-1} \leq \frac{(nV)^{n^2-1}}{(n+1)^{n^2-1}(\prod_{i=0}^{n} F_i)^{n-1}}. \tag{2.14}$$

Substituting (2.11) into (2.14) we get

$$\left(\prod_{i=0}^{n} d_i\right)^{n-1} \leq (n!)^n \left[(n+1)^{n(n+1)} n^{n^2-2}\right]^{-\frac{1}{2}} \frac{V^n}{R}. \tag{2.15}$$

We now take the point P is the incenter of Ω_n, then $d_i = r$ for $i = 0,1,\dots,n$. Thus from (2.15) we can obtain the inequality (2.13). It is easy to see that equality holds in (2.13) iff Ω_n is regular. □

Lemma 7 *For an n-simplex Ω_n and $k = 1,2,\dots,n-1$, we have*

$$\left(\prod_{i=1}^{\mu_{n,k}} V_i(k)\right)^{\frac{1}{\mu_{n,k}}} \geq \left(\frac{R}{nr}\right)^{\frac{k}{n(n+1)(n-1)^2}} \frac{\sqrt{k+1}}{k!}\left(\frac{n!}{\sqrt{n+1}}V\right)^{\frac{k}{n}}, \tag{2.16}$$

$$\left(\prod_{i=1}^{\mu_{n,k}} V_i(k)\right)^{\frac{1}{\mu_{n,k}}} \geq \left(\frac{V_0}{V}\right)^{\frac{k}{n(n+1)(n-1)^2}} \frac{\sqrt{k+1}}{k!}\left(\frac{n!}{\sqrt{n+1}}V\right)^{\frac{k}{n}}. \tag{2.17}$$

Equality holds iff Ω_n is regular.

Proof We have the well-known inequality [4, 10, 11]

$$V \leq \frac{(n+1)^{\frac{n+1}{2}}}{n^{\frac{n-2}{2}} n!} R^{n-1} r. \tag{2.18}$$

Equality holds iff Ω_n is regular.

Using (2.18) and (2.11) we get

$$\left(\prod_{i=0}^{n} F_i\right)^{n-1} \geq \left[\frac{n^{3n^2-4}}{(n+1)^{n^2-n-2}}\right]^{\frac{1}{2}} \frac{V^{\frac{(n^2-1)(n-1)}{n}} R}{(n!^n) V^{\frac{1}{n}}}. \tag{2.19}$$

From (2.19) and (2.18) we have

$$\left(\prod_{i=0}^{n} F_i\right)^{\frac{1}{n+1}} \geq \left(\frac{R}{nr}\right)^{\frac{1}{n(n^2-1)}} \frac{n^{\frac{3}{2}}}{(n!)^{\frac{1}{n}} (n+1)^{\frac{n-1}{2n}}} V^{\frac{n-1}{n}}. \tag{2.20}$$

Thus, inequality (2.16) is valid for $k = n-1$. We now prove that inequality (2.16) is valid for $1 \leq k < n-1$. Substituting (2.20) into (2.12) we can obtain inequality (2.16). So the inequality (2.16) is valid for $1 \leq k < n-1$. It is easy to see that equality holds in (2.16) iff Ω_n is regular.

By Lemma 3 we have

$$V_0 = \left[\frac{(n+1)^{n+1}}{(n!)^2 n^n}\right]^{\frac{1}{2}} R^n. \tag{2.21}$$

Substituting (2.21) into the right of (2.9) we get

$$\prod_{i=1}^{\frac{1}{2}n(n+1)} a_i \geq \left(\frac{V_0}{V}\right)^{\frac{1}{2}} \left(\frac{2^n}{n+1}\right)^{\frac{n+1}{4}} (n!V)^{\frac{n+1}{2}}. \tag{2.22}$$

From (2.8) and (2.22) we have

$$\left(\prod_{i=0}^{n} F_i\right)^{\frac{1}{n+1}} \geq \left(\frac{V_0}{V}\right)^{\frac{1}{n(n^2-1)}} \frac{n^{\frac{3}{2}}}{(n!)^{\frac{1}{n}} (n+1)^{\frac{n-1}{2n}}} V^{\frac{n-1}{n}}. \tag{2.23}$$

Thus the equality (2.17) is valid for $k = n-1$. Now we prove that inequality (2.17) is valid for $1 \le k < n-1$. Substituting (2.23) into the right of (2.12) we can obtain inequality (2.17). So the inequality (2.17) is valid for $1 \le k < n-1$. It is easy to see that equality holds in (2.16) iff Ω_n is regular. □

Lemma 8 [4, 5] *Let Ω_n be an n-simplex and real number $\alpha \in (0,1]$. Then*

$$2\sum_{0\le i<j\le n} F_i^\alpha F_j^\alpha - \sum_{i=0}^{n} F_i^{2\alpha} \ge (n^2-1)\left[\frac{n^3}{n+1}\left(\frac{n+1}{(n!)^2}\right)^{\frac{1}{n}}\right]^\alpha V^{\frac{2(n-1)\alpha}{n}}, \tag{2.24}$$

with equality iff Ω_n is regular.

Proof of Theorem 1 *and Theorem* 2 The inequality (2.1) can be written as

$$\begin{aligned}&\sum_{i=1}^{\mu_{n,k}} V_i^{2\alpha}(k) + 2\sum_{1\le i<j\le \mu_{n,k}} V_i^\alpha(k)V_j^\alpha(k) - \lambda\sum_{i=1}^{\mu_{n,k}} V_i^{2\alpha}(k)\\ &\quad\ge \left(\frac{R}{nr}\right)^{\frac{2k\alpha}{n(n+1)(n-1)^2}}\cdot \mu_{n,k}(\mu_{n,k}-\lambda)\cdot\left[\frac{\sqrt{k+1}}{k!}\left(\frac{n!}{\sqrt{n+1}}\right)^{\frac{k}{n}}\right]^{2\alpha} V^{\frac{2k\alpha}{n}}.\end{aligned} \tag{2.25}$$

Now we prove that the inequality (2.25) is valid.

Let $p(k,\alpha,\delta)$ denote the left of the inequality (2.25). Then

$$\begin{aligned}p(k,\alpha,\delta) &= 2\sum_{1\le i<j\le \mu_{n,k}} V_i^\alpha(k)V_j^\alpha(k) - (n-k)\sum_{i=1}^{\mu_{n,k}} V_i^{2\alpha}(k) + (n+1-k-\lambda)\sum_{i=1}^{\mu_{n,k}} V_i^{2\alpha}(k)\\ &= \sum_{(i_1,i_2,\dots,i_{k+2})\in\sigma}\left(2\sum_{1\le p<q\le k+2} V_{i_p}^\alpha(k)V_{i_q}^\alpha(k) - \sum_{p=1}^{k+2} V_{i_p}^{2\alpha}(k)\right)\\ &\quad + 2\sum_{i_p,i_q\in\tau} V_{i_p}^\alpha(k)V_{i_q}^\alpha(k) + (n+1-k-\lambda)\sum_{i=1}^{\mu_{n,k}} V_i^{2\alpha}(k),\end{aligned} \tag{2.26}$$

where $\sigma = \{(i_1,i_2,\dots,i_{k+2})|$ there exists a $(k+1)$-subsimplex $A_{i_1}A_{i_2}\cdots A_{i_{k+2}}$ of Ω_n, such that its $k+2$ side areas are $V_{i_1}(k), V_{i_2}(k),\dots,V_{i_{k+2}}(k)$ $(1\le i_1<i_2<\cdots<i_{k+2}\mu_{n,k})\}$, and $\tau = \{(i_p,i_q)|$ there is not a $(k+1)$-subsimplex of Ω_n, such that its two side areas are $V_{i_p}(k)$, $V_{i_q}(k)\}$.

On the other hand, we easily get

$$|\sigma| = \binom{n+1}{k+2},\qquad |\tau| = \binom{\mu_{n,k}}{2} - \binom{n+1}{k+2} = \frac{1}{2}\left[\mu_{n,k} - (n-k)(k+1) - 1\right].$$

We put $m = |\sigma|$, then $m = \binom{n+1}{k+2} = \frac{n-k}{k+2}\cdot\mu_{n,k}$. If $(i_1,i_2,\dots,i_{k+2})\in\sigma$, we use $V_i(k+1)$ to denote the volume of $k+1$-subsimplex with side areas $V_{i_1}(k), V_{i_2}(k),\dots,V_{i_{k+2}}(k)$. By Lemma 8 we have

$$\begin{aligned}&2\sum_{1\le p<q\le k+2} V_{i_p}^\alpha(k)V_{i_q}^\alpha(k) - \sum_{p=1}^{k+2} V_{i_p}^{2\alpha}(k)\\ &\quad\ge k(k+2)\left[\frac{(k+1)^3}{k+2}\left(\frac{k+2}{(k+1)!^2}\right)^{\frac{1}{k+1}}\right]^\alpha \left(V_i(k+1)\right)^{\frac{2k\alpha}{k+1}}.\end{aligned} \tag{2.27}$$

Equality holds iff the simplex $A_{i_1}A_{i_2}\cdots A_{i_{k+2}}$ is regular.

Using the arithmetic-geometric means inequality, (2.27), and (2.16) we get

$$\begin{aligned}
I_1 &= \sum_{(i_1,i_2,\dots,i_{k+2})\in\sigma}\left(2\sum_{1\le p<q\le k+2} V_{i_p}^{\alpha}(k)V_{i_q}^{\alpha}(k) - \sum_{p=1}^{k+2} V_{i_p}^{2\alpha}(k)\right) \\
&\ge \sum_{(i_1,1_2,\dots,i_{k+2})\in\sigma} k(k+2)\left[\frac{(k+1)^3}{k+2}\left(\frac{k+2}{(k+1)!^2}\right)^{\frac{1}{k+1}}\right]^{\alpha}\bigl(V_i(k+1)\bigr)^{\frac{2k\alpha}{k+1}} \\
&\ge \mu_{n,k+1}k(k+2)\left[\frac{(k+1)^3}{k+2}\left(\frac{k+2}{(k+1)!^2}\right)^{\frac{1}{k+1}}\right]^{\alpha}\left(\prod_{i=1}^{\mu_{n,k+1}} V_i(k+1)\right)^{\frac{2k\alpha}{(k+1)\mu_{n,k+1}}} \\
&\ge \left(\frac{R}{nr}\right)^{\frac{2k\alpha}{n(n+1)(n-1)^2}} k(k+2)\left[\frac{(k+1)^3}{k+2}\left(\frac{k+2}{(k+1)!^2}\right)^{\frac{1}{k+1}}\right]^{\alpha} \\
&\quad\cdot \mu_{n,k+1}\left[\frac{\sqrt{k+2}}{(k+1)!}\left(\frac{n!}{\sqrt{n+1}}\right)^{\frac{k+1}{n}}\right]^{\frac{2k\alpha}{k+1}} V^{\frac{2k\alpha}{n}} \\
&= \left(\frac{R}{nr}\right)^{\frac{2k\alpha}{n(n+1)(n-1)^2}} \mu_{n,k+1}k(n-k)\left[\frac{\sqrt{k+1}}{(k)!}\left(\frac{n!}{\sqrt{n+1}}\right)^{\frac{k}{n}}\right]^{2\alpha} V^{\frac{2k\alpha}{n}}.
\end{aligned} \tag{2.28}$$

By the arithmetic-geometric means inequality, and by (2.16), we get

$$\begin{aligned}
I_2 &= \sum_{(i_p,i_q)\in\tau} 2V_{i_p}^{\alpha}(k)V_{i_q}^{\alpha}(k) + (n+1-k-\lambda)\sum_{i=1}^{\mu_{n,k}} V_i^{2\alpha}(k) \\
&\ge \mu_{n,k}\bigl[\mu_{n,k}-(n-k)(k+1)-1\bigr]\left(\prod_{i=1}^{\mu_{n,k}} V_i(k)\right)^{\frac{2\alpha}{\mu_{n,k}}} + (n+1-k-\lambda)\mu_{n,k}\left(\prod_{i=1}^{\mu_{n,k}} V_i(k)\right)^{\frac{2\alpha}{\mu_{n,k}}} \\
&\ge \left(\frac{R}{nr}\right)^{\frac{2k\alpha}{n(n+1)(n-1)^2}} \mu_{n,k}\bigl[\mu_{n,k}-(n-k)k-\lambda\bigr]\left[\frac{\sqrt{k+1}}{(k)!}\left(\frac{n!}{\sqrt{n+1}}\right)^{\frac{k}{n}}\right]^{2\alpha} V^{\frac{2k\alpha}{n}}.
\end{aligned} \tag{2.29}$$

From (2.26), (2.28), and (2.29) we get

$$\begin{aligned}
p(k,\alpha,\lambda) &= I_1 + I_2 \\
&\ge \left(\frac{R}{nr}\right)^{\frac{2k\alpha}{n(n+1)(n-1)^2}} \mu_{n,k}[\mu_{n,k}-\lambda]\left[\frac{\sqrt{k+1}}{(k)!}\left(\frac{n!}{\sqrt{n+1}}\right)^{\frac{k}{n}}\right]^{2\alpha} V^{\frac{2k\alpha}{n}}.
\end{aligned} \tag{2.30}$$

Thus the equality (2.25) is valid. It is easy to prove that the equality holds iff Ω_n is regular. □

Similarly, we can prove that (2.2) is true by the same method.

3 Some applications

Geometric inequalities for simplices which are the simplest and useful polytopes have been a very attractive subject for a long time. Mitrinović, Pečarić, Volenec, Zhang and Yang and other authors have obtained a great number of elegant results (see [8]). Specially, we mention the well-known Euler inequality for a simplex, inequalities for the width of a simplex, and inequalities related the interior point of a simplex. In this section we shall improve these inequalities by using the results in the above section.

LF Tóth extended the Euler inequality $R \geq 2r$ for a triangle to an n-dimensional simplex and obtained the Euler inequality for an n-simplex as follows [8, 10]:

$$R \geq nr. \tag{3.1}$$

Equality holds iff Ω_n is regular.

Let O and I denote the circumcenter and the incenter of an n-simplex Ω_n, respectively. Let G be the barycenter of Ω_n. Klamkin [8, 12] and Yang and Wang obtained the following generalizations of (3.1):

$$R^2 - |OI|^2 \geq (nr)^2, \tag{3.2}$$

$$R^2 - |OG|^2 \geq (nr)^2. \tag{3.3}$$

Equality holds iff Ω_n is regular.

We shall use these results in the above section to study related inequalities, and we obtain a strengthened Euler inequality as follows.

Theorem 3 *For an n-simplex Ω_n and $k = 1, 2, \ldots, n-1$, we have*

$$\bigl(R^2 - |OG|^2\bigr)^k \geq \left(\frac{R}{nr}\right)^{\frac{2k}{n(n+1)(n-1)^2}} (nr)^{2k} + \frac{1}{\mu_{n,k} - (n+1-k)} \sum_{1 \leq i<j \leq \mu_{n,k}} \bigl(V_i(k) - V_j(k)\bigr)^2, \tag{3.4}$$

$$\bigl(R^2 - |OG|^2\bigr)^k \geq \left(\frac{V_0}{V}\right)^{\frac{2k}{n(n+1)(n-1)^2}} (nr)^{2k} + \frac{1}{\mu_{n,k} - (n+1-k)} \sum_{1 \leq i<j \leq \mu_{n,k}} \bigl(V_i(k) - V_j(k)\bigr)^2. \tag{3.5}$$

Equality holds iff Ω_n is regular.

If take $k = 1$ and $k = n-1$ in (3.4) and (3.5), we get two corollaries as follows.

Corollary 4 *For an n-simplex Ω_n and $k = 1, 2, \ldots, n-1$, we have*

$$R^2 - |OG|^2 \geq \left(\frac{R}{nr}\right)^{\frac{2}{n(n+1)(n-1)^2}} (nr)^2 + \frac{2}{n(n-1)} \sum_{1 \leq i<j \leq \frac{1}{2}n(n+1)} (a_i - a_j)^2, \tag{3.6}$$

$$R^2 - |OG|^2 \geq \left(\frac{V_0}{V}\right)^{\frac{2}{n(n+1)(n-1)^2}} (nr)^2 + \frac{2}{n(n-1)} \sum_{1 \leq i<j \leq \frac{1}{2}n(n+1)} (a_i - a_j)^2, \tag{3.7}$$

with equality iff Ω_n is regular.

Corollary 5 *For an n-simplex Ω_n and $k = 1, 2, \ldots, n-1$, we have*

$$\bigl(R^2 - |OG|^2\bigr)^{n-1} \geq \left(\frac{R}{nr}\right)^{\frac{2}{n(n^2-1)}} (nr)^{2(n-1)} + \frac{1}{n-1} \sum_{0 \leq i<j \leq n} (F_i - F_j)^2, \tag{3.8}$$

$$\left(R^2-|OG|^2\right)^{n-1}\geq\left(\frac{V_0}{V}\right)^{\frac{2}{n(n^2-1)}}(nr)^{2(n-1)}+\frac{1}{n-1}\sum_{0\leq i<j\leq n}(F_i-F_j)^2. \tag{3.9}$$

Equality holds iff Ω_n is regular.

Proof of Theorem 3 If take $\alpha=1$ and $\lambda=n+1-k$ in (2.1) and (2.2), we obtain

$$\begin{aligned}\sum_{i=1}^{\mu_{n,k}}V_i^2(k)\geq{}&\left(\frac{R}{nr}\right)^{\frac{2}{n(n+1)(n-1)^2}}\mu_{n,k}\cdot\frac{k+1}{(k!)^2}\left(\frac{(n!)^2}{n+1}\right)^{\frac{k}{n}}V^{\frac{2k}{n}}\\&+\frac{1}{\mu_{n,k}-(n+1-k)}\sum_{1\leq i<j\leq\mu_{n,k}}\left(V_i(k)-V_j(k)\right)^2,\end{aligned} \tag{3.10}$$

$$\begin{aligned}\sum_{i=1}^{\mu_{n,k}}V_i^2(k)\geq{}&\left(\frac{V_0}{V}\right)^{\frac{2}{n(n+1)(n-1)^2}}\mu_{n,k}\cdot\frac{k+1}{(k!)^2}\left(\frac{(n!)^2}{n+1}\right)^{\frac{k}{n}}V^{\frac{2k}{n}}\\&+\frac{1}{\mu_{n,k}-(n+1-k)}\sum_{1\leq i<j\leq\mu_{n,k}}\left(V_i(k)-V_j(k)\right)^2.\end{aligned} \tag{3.11}$$

We use the well-known equality [8, 12]

$$\sum_{i=1}^{\frac{1}{2}n(n+1)}a_i^2=(n+1)^2R^2 \tag{3.12}$$

and the well-known inequality [4, 13]

$$\sum_{i=1}^{\mu_{n,k}}V_i^2(k)\leq\frac{n!}{n^k(k!)^3(n-k)!(n+1)^{k-1}}\left(\sum_{i=1}^{\frac{1}{2}n(n+1)}a_i^2\right)^k. \tag{3.13}$$

Equality holds iff Ω_n is regular. When $k=1$, then (3.13) is the identity.

By (3.12) and (3.13) we get

$$\sum_{i=1}^{\mu_{n,k}}V_i^2(k)\leq\frac{(n+1)^{k+1}n!}{(k!)^3n^k(n-k)!}R^{2k}. \tag{3.14}$$

We use the well-known inequality in [8]

$$V\geq\frac{n^{\frac{n}{2}}(n+1)^{\frac{n+1}{2}}}{n!}r^n, \tag{3.15}$$

with equality iff Ω_n is regular.

From (3.10), (3.14), and (3.15) we can obtain (3.4). By (3.11), (3.14), and (3.15) we can obtain (3.5). It is easy to see that equality holds in (3.4) or (3.5) iff Ω_n is regular. □

Let ω be the width of Ω_n. We put

$$\beta_n=\frac{n[\frac{n+1}{2}](n+1-[\frac{n+1}{2}])}{(n+1)^2}.$$

In 1977, Alexander [8, 14] proved Sallee's conjecture and obtained the inequality

$$\omega^{-2} \geq \beta_n R^{-2}. \tag{3.16}$$

Equality holds iff Ω_n is regular.

In 1983, Yang and Zhang [8, 15] improved (3.16) and established an inequality as follows:

$$\omega^{-2} \geq \beta_n \frac{(n+1)^{\frac{n+1}{n}}}{n \cdot (n!)^{\frac{2}{n}}} V^{-\frac{2}{n}}, \tag{3.17}$$

with equality iff Ω_n is regular.

In this section, we shall give generalizations of the equalities (3.16) and (3.17) by using (2.4) and (2.5) in the above section.

Theorem 4 *For an n-simplex Ω_n and $k = 1, 2, \dots, n-1$, we have*

$$\omega^{-2} \geq \left(\frac{R}{nr}\right)^{\frac{2}{n(n^2-1)}} \beta_n \frac{(n+1)^{\frac{n+1}{n}}}{n \cdot (n!)^{\frac{2}{n}}} V^{-\frac{2}{n}} + \beta_n \frac{(n+1)V^{-2}}{n^4(n-1)} \sum_{0 \leq i < j \leq n} (F_i - F_j)^2, \tag{3.18}$$

$$\omega^{-2} \geq \left(\frac{V_0}{V}\right)^{\frac{2}{n(n^2-1)}} \beta_n \frac{(n+1)^{\frac{n+1}{n}}}{n \cdot (n!)^{\frac{2}{n}}} V^{-\frac{2}{n}} + \beta_n \frac{(n+1)V^{-2}}{n^4(n-1)} \sum_{0 \leq i < j \leq n} (F_i - F_j)^2. \tag{3.19}$$

Equality holds iff Ω_n is regular.

Proof We use the well-known inequality in [15] as follows:

$$\omega \leq \left[\frac{n^3(n+1)}{[\frac{n+1}{2}](n+1-[\frac{n+1}{2}])}\right]^{\frac{1}{2}} \frac{V}{(\sum_{i=0}^n F_i^2)^{\frac{1}{2}}}.$$

Equality holds iff Ω_n is regular.

The above inequality can be written as

$$\omega^{-2} \geq \beta_n \cdot \frac{n+1}{n^4} \frac{\sum_{i=0}^n F_i^2}{V^2}. \tag{3.20}$$

By (3.20) and (2.6) we can get (3.18), and from (3.20) and (2.7) we can get (3.19). □

Substituting (2.10) into (3.18) and (3.19) we obtain generalizations of the inequality (3.16) as follows.

Corollary 6 *For an n-simplex Ω_n and $k = 1, 2, \dots, n-1$, we have*

$$\omega^{-2} \geq \left(\frac{R}{nr}\right)^{\frac{2}{n(n^2-1)}} \beta_n R^{-2} + \beta_n \frac{(n+1)V^{-2}}{n^4(n-1)} \sum_{0 \leq i < j \leq n} (F_i - F_j)^2, \tag{3.21}$$

$$\omega^{-2} \geq \left(\frac{V_0}{V}\right)^{\frac{2}{n(n^2-1)}} \beta_n R^{-2} + \beta_n \frac{(n+1)V^{-2}}{n^4(n-1)} \sum_{0 \leq i < j \leq n} (F_i - F_j)^2. \tag{3.22}$$

Equality holds iff Ω_n is regular.

For $i = 0, 1, \ldots, n$, let h_i be the altitude of the ith face of the simplex Ω_n. The following inequality is well known (see [8]):

$$V^2 \geq \frac{n^n}{(n!)^2(n+1)^{n-1}} \left(\prod_{i=0}^{n} h_i \right)^{\frac{2n}{n+1}}. \tag{3.23}$$

Equality holds iff Ω_n is regular.

By using the results in Section 2 we can obtain generalizations of the inequality (3.23) as follows.

Theorem 5 *For an n-simplex Ω_n and $k = 1, 2, \ldots, n-1$, we have*

$$\begin{aligned} V^{2(n^2-1)k} \geq{}& \left(\frac{R}{nr}\right)^{\frac{2k\alpha}{n(n+1)(n-1)^2}} \cdot \left[\frac{n^n}{(n!)^2(n+1)^{n-1}} \left(\prod_{i=0}^{n} h_i \right)^{\frac{2n}{n+1}} \right]^{(n^2-1)k} \\ &+ \frac{(nr)^{2(n^3-n-1)k}}{\mu_{n,k} - (n+1-k)} \left[\frac{(n+1)^{n+1}}{n^n \cdot (n!)^2} \right]^{(n^2-1)k} \sum_{1 \leq i < j \leq \mu_{n,k}} \bigl(V_i(k) - V_j(k)\bigr)^2, \end{aligned} \tag{3.24}$$

$$\begin{aligned} V^{2(n^2-1)k} \geq{}& \left(\frac{V_0}{V}\right)^{\frac{2k\alpha}{n(n+1)(n-1)^2}} \cdot \left[\frac{n^n}{(n!)^2(n+1)^{n-1}} \left(\prod_{i=0}^{n} h_i \right)^{\frac{2n}{n+1}} \right]^{(n^2-1)k} \\ &+ \frac{(nr)^{2(n^3-n-1)k}}{\mu_{n,k} - (n+1-k)} \left[\frac{(n+1)^{n+1}}{n^n \cdot (n!)^2} \right]^{(n^2-1)k} \sum_{1 \leq i < j \leq \mu_{n,k}} \bigl(V_i(k) - V_j(k)\bigr)^2. \end{aligned} \tag{3.25}$$

Equality holds iff Ω_n is regular.

Proof We take $k = n-1$ in (2.16) and (2.17), and get

$$\prod_{i=0}^{n} F_i \geq \left(\frac{R}{nr}\right)^{\frac{1}{n(n-1)}} \frac{n^{\frac{3(n+1)}{2}}}{(n!)^{\frac{n+1}{n}} (n+1)^{\frac{n^2-1}{2n}}} V^{\frac{n^2-1}{n}}, \tag{3.26}$$

$$\prod_{i=0}^{n} F_i \geq \left(\frac{V_0}{V}\right)^{\frac{1}{n(n-1)}} \frac{n^{\frac{3(n+1)}{2}}}{(n!)^{\frac{n+1}{n}} (n+1)^{\frac{n^2-1}{2n}}} V^{\frac{n^2-1}{n}}. \tag{3.27}$$

Using the formula $F_i = \frac{nV}{h_i}$ (see [8]) and (3.26) we get

$$V^{2(n^2-1)k} \geq \frac{R^{2k}}{(nr)^{2k}} \left[\frac{n^n}{(n!)^2(n+1)^{n-1}} \left(\prod_{i=0}^{n} h_i \right)^{\frac{2n}{n+1}} \right]^{(n^2-1)k}. \tag{3.28}$$

By Theorem 3 we get

$$R^{2k} \geq \left(\frac{R}{nr}\right)^{\frac{2k}{n(n+1)(n-1)^2}} (nr)^{2k} + \frac{1}{\mu_{n,k} - (n+1-k)} \sum_{1 \leq i < j \leq \mu_{n,k}} \bigl(V_i(k) - V_j(k)\bigr)^2. \tag{3.29}$$

Substituting (3.29) into (3.28) we get

$$V^{2(n^2-1)k} \geq \left(\frac{R}{nr}\right)^{\frac{2k\alpha}{n(n+1)(n-1)^2}} \cdot \left[\frac{n^n}{(n!)^2(n+1)^{n-1}}\left(\prod_{i=0}^{n} h_i\right)^{\frac{2n}{n+1}}\right]^{(n^2-1)k}$$
$$+ \frac{1}{(nr)^{2k}[\mu_{n,k}-(n+1-k)]} \cdot \left[\frac{n^n}{(n!)^2(n+1)^{n-1}}\left(\prod_{i=0}^{n} h_i\right)^{\frac{2n}{n+1}}\right]^{(n^2-1)k}$$
$$\cdot \sum_{1\leq i<j\leq \mu_{n,k}} \big(V_i(k)-V_j(k)\big)^2. \tag{3.30}$$

We use the well-known inequality in [4, 8] as follows:

$$\prod_{i=0}^{n} h_i \geq (n+1)^{n+1} r^{n+1}, \tag{3.31}$$

with equality holding iff Ω_n is regular.

From (3.30) and (3.31) we can obtain (3.24). It is easy to see that equality holds iff Ω_n is regular.

By a similar method we can prove that inequality (3.25) is also valid. □

Let P be an interior point of Ω_n and d_i the distance from the point P to the ith face f_i of Ω_n for $i = 0, 1, \ldots, n$. In [8], Gerber established the following inequality:

$$\left(\prod_{i=0}^{n} d_i\right)^{-2} \geq \left[\frac{n^{n(n+1)}(n+1)^{(n+1)^2}}{(n!)^{2(n+1)}}\right]^{\frac{1}{n}} V^{-\frac{2(n+1)}{n}}. \tag{3.32}$$

For any natural number $m > 0$, Leng and Ma [16] obtained an inequality as follows:

$$\left[\sum_{0\leq i<j\leq n} (d_i d_i)^{-m} - (\mu_{n,1})^{1-m} \cdot 2 \cdot r^{-2m}\right]^{\frac{n}{m}} \geq \left[(\mu_{n,1})^{1-m}\big((\mu_{n,1})^m - 1\big) n^{2m} R^{-2m}\right]^{\frac{n}{m}}. \tag{3.33}$$

Equalities hold in (3.32) and (3.33) iff Ω_n is regular and P is the center of Ω_n.

By using the results in the Section 2 we can obtain generalizations of (3.32) and (3.33) as follows.

Theorem 6 *For an n-simplex Ω_n, we have*

$$\left(\prod_{i=0}^{n} d_i\right)^{-2} \geq \left(\frac{R}{nr}\right)^{\frac{2}{n(n^2-1)}} \cdot \left[\frac{n^{n+1}(n+1)^{(n+1)^2}}{(n!)^{2(n+1)}}\right]^{\frac{1}{n}} V^{-\frac{2(n+1)}{n}}$$
$$+ \frac{n^{n-2}(n+1)^{n+2} V^{-4}}{(n!)^2(n-1)} \sum_{0\leq i<j\leq n} (F_i - F_j)^2, \tag{3.34}$$

$$\left(\prod_{i=0}^{n} d_i\right)^{-2} \geq \left(\frac{V_0}{V}\right)^{\frac{2}{n(n^2-1)}} \cdot \left[\frac{n^{n+1}(n+1)^{(n+1)^2}}{(n!)^{2(n+1)}}\right]^{\frac{1}{n}} V^{-\frac{2(n+1)}{n}}$$
$$+ \frac{n^{n-2}(n+1)^{n+2} V^{-4}}{(n!)^2(n-1)} \sum_{0\leq i<j\leq n} (F_i - F_j)^2. \tag{3.35}$$

Equality holds in (3.32) *or* (3.33) *iff Ω_n is regular and P is the center of Ω_n.*

Proof By (2.14), we have

$$\left(\prod_{i=0}^{n} d_i\right)^{-2} \geq \frac{(n+1)^{2(n+1)}}{n^{2(n+1)}} V^{-2(n+1)} \prod_{i=0}^{n} F_i^2. \tag{3.36}$$

We use the well-known inequality in [7] as follows:

$$\prod_{i=0}^{n} F_i^2 \geq \frac{n^{3n}}{(n!)^2(n+1)^n} V^{2(n-1)} \sum_{i=0}^{n} F_i^2. \tag{3.37}$$

Equality holds iff Ω_n is regular.

Using (3.37) and (2.6) we get

$$\begin{aligned}\prod_{i=0}^{n} F_i^2 \geq{}& \left(\frac{R}{nr}\right)^{\frac{2}{n(n^2-1)}} \frac{n^{3(n+1)}}{(n!)^{\frac{2(n+1)}{n}}(n+1)^{\frac{n^2-1}{n}}} V^{\frac{2(n^2-1)}{n}} \\ &+ \frac{n^{3n}V^{2(n-1)}}{(n!)^2(n+1)^n(n-1)} \sum_{0\leq i<j\leq n} (F_i-F_j)^2.\end{aligned} \tag{3.38}$$

From (3.36) and (3.38) we can obtain (3.34). It is easy to see that equality holds in (3.34) iff Ω_n is regular and P is the center of Ω_n.

By a similar method we can prove that (3.35) is valid. □

Theorem 7 *Let Ω_n be an n-simplex and $m > 0$ a natural number, then we have*

$$\begin{aligned}&\left[\sum_{0\leq i<j\leq n} (d_id_j)^{-m} - (\mu_{n,1})^{1-m} r^{-2m}\right]^{\frac{n}{m}} \\ &\quad\geq \left(\frac{R}{nr}\right)^{\frac{2}{n(n+1)(n-1)^2}} \left[(\mu_{n,1})^{1-m}\left((\mu_{n,1})^m-1\right)n^{2m}R^{-2m}\right]^{\frac{n}{m}} \\ &\quad\quad\cdot \gamma(n,m)(Rr)^{-2} \sum_{1\leq i<j\leq \mu_{n,1}} (a_i-a_j)^2,\end{aligned} \tag{3.39}$$

$$\begin{aligned}&\left[\sum_{0\leq i<j\leq n} (d_id_j)^{-m} - (\mu_{n,1})^{1-m} r^{-2m}\right]^{\frac{n}{m}} \\ &\quad\geq \left(\frac{V_0}{V}\right)^{\frac{2}{n(n+1)(n-1)^2}} \left[(\mu_{n,1})^{1-m}\left((\mu_{n,1})^m-1\right)n^{2m}R^{-2m}\right]^{\frac{n}{m}} \\ &\quad\quad\cdot \gamma(n,m)(Rr)^{-2} \sum_{1\leq i<j\leq \mu_{n,1}} (a_i-a_j)^2.\end{aligned} \tag{3.40}$$

Equality holds iff Ω_n is regular and P is the center of Ω_n, where

$$\gamma_{n,m} = \left[(\mu_{n,1})^{1-m}\left((\mu_{n,1})^m-1\right)\right]^{\frac{n}{m}} \cdot \frac{2}{n^{2n+3}(n-1)}.$$

Proof We use the following well-known inequality (3.8) in [16]:

$$\sum_{0\leq i<j\leq n} (d_id_j)^{-m} \geq (\mu_{n,1})^{1-m}\left[r^{-2m} + (n+1)^m n^m \left(\frac{n+1}{n!^2}\right)^{\frac{n}{m}} \cdot \frac{(\mu_{n,1})^m-1}{V^{\frac{2m}{n}}}\right],$$

i.e.

$$\begin{aligned}&\left[\sum_{0\le i<j\le n}(d_i d_j)^{-m}-(\mu_{n,1})^{1-m}r^{-2m}\right]\\&\quad\ge(\mu_{n,1})^{1-m}(n+1)^m n^m\left(\frac{n+1}{n!^2}\right)\cdot\left[(\mu_{n,1})^m-1\right]^{\frac{2m}{n}}\frac{1}{V^2}.\end{aligned}\tag{3.41}$$

Equality holds iff Ω_n is regular and P is the center of Ω_n.

From (2.18) we have

$$\frac{1}{V^2}\ge\frac{n!^2\cdot n^n}{(n+1)^{n+1}R^{2n}(nr)^2}R^2.\tag{3.42}$$

Substituting (3.42) into (3.42) and using (3.6) we can obtain (3.39). It is easy to see that equality holds in (3.39) iff Ω_n is regular and P is the center of Ω_n.

The proof of (3.4) is similar. □

In fact, the strengthening of some well-known inequalities for simplices can be derived from Theorem 1 and Theorem 2. In this paper, we omit the details.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

The authors co-authored this paper together. All authors read and approved the final manuscript.

Author details

[1]School of Mathematics and Statistics, Hefei Normal University, Hefei, Anhui 230601, P.R. China. [2]Department of Mathematics and Physics, Anhui Xinhua University, Hefei, 230088, P.R. China.

Acknowledgements

This work is supported by the Doctoral Programs Foundation of Education Ministry of China (20113401110009) and Foundation of Anhui higher school (KJ2013A220); Natural Science Research Project of Hefei Normal University (2012kj11). The authors are grateful for the help.

Received: 6 July 2014 Accepted: 22 January 2015 Published online: 19 February 2015

References

1. Finsler, P, Hadwiger, H: Inequalities for triangles. Comment. Math. Helv. **10**, 316-326 (1937/1938)
2. Bottema, O: Geometric Inequalities. Springer, Berlin (1989)
3. Chen, J, Ma, Y: A class of inequalities involving two simplices. J. Math. Res. Expo. **9**(2), 282-284 (1989) (in Chinese)
4. Shen, WX: Introduction on Simplices. Hunan Normal University Press, ChangSha (2000) (in Chinese)
5. Leng, GS, Tang, LH: Some generalizations to several dimensions of the Peode inequality with applications. Acta Math. Sin. **40**(1), 14-21 (1997) (in Chinese)
6. Su, HM: An inequality concerning the volume, edge-lengths and side faces of a simplex. J. Math. **13**, 453-454 (1993) (in Chinese)
7. Zhang, JZ, Yang, L: A class of geometric inequalities concerning to mass-point system. J. Univ. Sci. Technol. China **11**, 1-8 (1981) (in Chinese)
8. Mitrinović, DS, Pečarić, JE, Volenec, V: Recent Advances in Geometric Inequalities. Kluwer Academic, Dordrecht (1989)
9. Lindensttauss, J, Milman, VD: Geometric Aspects of Functional Analysis. Springer, Berlin (1991)
10. Tóth, LF: Extremum properties of regular polytopes. Acta Math. Acad. Sci. Hung. **6**, 143-146 (1995)
11. Kuang, JC: Applied Inequalities. Shandong Science and Technology Press, Jinan (2004) (in Chinese)
12. Yang, SG, Wang, J: Improvement of *n*-dimensional Euler inequality. J. Geom. **51**, 190-195 (1995)
13. Yang, L, Zhang, JZ: A class of geometric inequalities for the point system. Acta Math. Sin. **23**, 740-749 (1980) (in Chinese)
14. Alexander, R: The width and diameter of a simplex. Geom. Dedic. **6**, 87-94 (1977)
15. Yang, L, Zhang, JZ: Metric equation applied to Sallee's conjecture. Acta Math. Sin. **26**(4), 488-493 (1983) (in Chinese)
16. Leng, GS, Ma, Y: Inequalities for a simplex and interior point. Geom. Dedic. **85**, 1-10 (2001)