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An algorithm with strong convergence for the split common fixed point problem of total asymptotically strict pseudocontraction mappings

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Abstract

The purpose of this paper is to propose an algorithm to solve the split common fixed point problems for total asymptotically strict pseudocontraction mappings in Hilbert spaces. Without the assumption of semi-compactness on the mappings, the iterative scheme is shown to converge strongly to a split common fixed point of such mappings. The results presented in the paper improve and extend some recent corresponding results.
MSC: 47H09; 47J25

Keywords: split common fixed point problem; split feasibility problem; total asymptotically strict pseudocontraction mapping; Hilbert space

1 Introduction

Let H_1 and H_2 be two real Hilbert spaces, C and Q be nonempty closed convex subsets of H_1 and H_2, respectively. The split feasibility problem is formulated as finding a point $q \in H_1$ with the properties

$$q \in C \quad \text{and} \quad Aq \in Q, \tag{1.1}$$

where $A : H_1 \to H_2$ is a bounded linear operator. Assuming that SFP (1.1) is consistent (*i.e.*, (1.1) has a solution), it is not hard to see that $x \in C$ solve (1.1) if and only if it solves the following fixed point equation:

$$x = P_C\big(I - \gamma A^*(I - P_Q)A\big)x, \quad x \in C, \tag{1.2}$$

where P_C and P_Q are the (orthogonal) projections onto C and Q, respectively, $\gamma > 0$ is any positive constant, and A^* denotes the adjoint of A. The split feasibility problem in finite dimensional Hilbert spaces was introduced by Censor and Elfving [1] in 1994 for modeling inverse problems which arise from phase retrievals and in medical image reconstruction [2]. Recently, it has been found that split feasibility problems can be used in various disciplines, such as image restoration, computer tomography and radiation therapy treatment planning [3–7]. The split feasibility problem in Hilbert space can be found

in [2, 4, 6, 8, 9]. Also the convex feasibility formalism is at the core of the modeling of many inverse problems and has been used to model significant real-world problems. The split common fixed point problem is a generalization of the split feasibility problem and the convex feasibility problem. If C and Q are the sets of fixed points of two nonlinear mappings, respectively, and C and Q are nonempty closed convex subsets, then q is said to be a split common fixed point for the two nonlinear mappings. The split common fixed point problem (SCFP) for mappings S and T is to find a point $q \in H_1$ with the properties

$$q \in F(S) \quad \text{and} \quad Aq \in F(T), \tag{1.3}$$

we use Γ to denote the set of solutions of SCFP (1.3), that is, $\Gamma = \{q \in F(S) : Aq \in F(T)\}$.

Let H be a real Hilbert space, C be a nonempty and closed convex subset of H. A mapping $T : C \to C$ is said to be a k-strictly pseudocontractive mapping, if there exists $k \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in C. \tag{1.4}$$

A mapping $T : C \to C$ is said to be $(k, \{k_n\})$-asymptotically strictly pseudocontractive, if there exist a constant $k \in [0, 1)$ and a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ such that

$$\|T^n x - T^n y\|^2 \leq k_n\|x - y\|^2 + k\|(I - T^n)x - (I - T^n)y\|^2, \quad \forall x, y \in C. \tag{1.5}$$

A mapping $T : C \to C$ is said to be $(k, \{\mu_n\}, \{\xi_n\}, \phi)$-totally asymptotically strictly pseudocontractive, if there exist a constant $k \in [0, 1)$ and sequences $\{\mu_n\} \subset [0, \infty)$ and $\{\xi_n\} \subset [0, \infty)$ with $\mu_n \to 0$ and $\xi_n \to 0$ such that for all $x, y \in C$,

$$\|T^n x - T^n y\|^2 \leq \|x - y\|^2 + k\|(I - T^n)x - (I - T^n)y\|^2$$
$$+ \mu_n \phi(\|x - y\|) + \xi_n, \quad \forall n \geq 1, \tag{1.6}$$

where $\phi : [0, \infty) \to [0, \infty)$ is a continuous and strictly increasing function with $\phi(0) = 0$.

Now, we give two examples of $(k, \{\mu_n\}, \{\xi_n\}, \phi)$-total asymptotically strict pseudocontraction mappings.

Example 1.1 Let C be a unit ball in a real Hilbert space l^2 and let $S : C \to C$ be a mapping defined by

$$S : (x_1, x_2, \ldots) \to (0, x_1^2, a_2 x_2, a_3 x_3, \ldots),$$

where $\{a_i\}$ is a sequence in $(0, 1)$ such that $\prod_{i=2}^{\infty} a_i = \frac{1}{2}$.

It is proved by Goebel and Kirk [10] that

(i) $\|Sx - Sy\| \leq 2\|x - y\|, \forall x, y \in C$;

(ii) $\|S^n x - S^n y\| \leq 2 \prod_{j=2}^{n} a_j \|x - y\|, \forall x, y \in C, \forall n \geq 2$.

Denote by $k_1^{\frac{1}{2}} = 2$, $k_n^{\frac{1}{2}} = 2 \prod_{j=2}^{n} a_j$, $n \geq 2$, then

$$\lim_{n \to \infty} k_n = \lim_{n \to \infty} \left(2 \prod_{j=2}^{n} a_j\right)^2 = 1.$$

Letting $\mu_n = (k_n - 1)$, $\forall n \geq 1$, $\phi(t) = t^2$, $\forall t \geq 0$, $k = 0$, and $\{\xi_n\}$ be a nonnegative real sequence with $\xi_n \to 0$, then $\forall x, y \in C$, $n \geq 1$, we have

$$\|S^n x - S^n y\|^2 \leq \|x - y\|^2 + \mu_n \phi(\|x - y\|) + k\|x - y - (S^n x - S^n y)\|^2 + \xi_n.$$

Especially, as $a_{n+1} = 2^{-\frac{1}{2^n}}$, $\prod_{j=2}^{n} a_j = \frac{1}{2}$.

Example 1.2 Let D be an orthogonal subspace of R^n with the norm $\|x\| = \sqrt{\sum_{i=1}^{n} x_i^2}$ and the inner product $\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i$ for $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$. For each $x = (x_1, x_2, \ldots, x_n) \in D$, we define a mapping $T : D \to D$ by

$$Tx = \begin{cases} (x_1, x_2, \ldots, x_n), & \text{if } \prod_{i=1}^{n} x_i < 0, \\ (-x_1, -x_2, \ldots, -x_n), & \text{if } \prod_{i=1}^{n} x_i \geq 0. \end{cases} \tag{1.7}$$

Next we prove that T is a $(k, \{\mu_n\}, \{\xi_n\}, \phi)$-total asymptotically strict pseudocontraction mapping.

In fact, for any $x, y \in D$, let $\mu_n = (k_n - 1)$, $\forall n \geq 1$, $\phi(t) = t^2$, $\forall t \geq 0$, $k = 0$, and letting $\{\xi_n\}$ be a nonnegative real sequence with $\xi_n \to 0$, we have the following:

Case 1. If $\prod_{i=1}^{n} x_i < 0$ and $\prod_{i=1}^{n} y_i < 0$, then we have $T^n x = x$, $T^n y = y$, and so inequality (1.6) holds.

Case 2. If $\prod_{i=1}^{n} x_i < 0$, and $\prod_{i=1}^{n} y_i \geq 0$, then we have $T^n x = x$, $T^n y = (-1)^n y$. This implies that

$$\begin{cases} \|T^n x - T^n y\|^2 = \|x - (-1)^n y\|^2 = \|x\|^2 + \|y\|^2, \\ k_n \|x - y\|^2 = k_n(\|x\|^2 + \|y\|^2), \\ \|x - T^n x - (y - T^n y)\|^2 = [1 - (-1)^n]^2 \|y\|^2. \end{cases}$$

Therefore the inequality (1.6) holds.

Case 3. If $\prod_{i=1}^{n} x_i \geq 0$ and $\prod_{i=1}^{n} y_i < 0$, then we have $T^n x = (-1)^n x$, $T^n y = y$. Therefore we obtain

$$\begin{cases} \|T^n x - T^n y\|^2 = \|(-1)^n x - y\|^2 = \|x\|^2 + \|y\|^2, \\ k_n \|x - y\|^2 = k_n(\|x\|^2 + \|y\|^2), \\ \|x - T^n x - (y - T^n y)\|^2 = [1 - (-1)^n]^2 \|x\|^2. \end{cases}$$

So the inequality (1.6) holds.

Case 4. If $\prod_{i=1}^{n} x_i \geq 0$ and $\prod_{i=1}^{n} y_i \geq 0$, then we have $T^n x = (-1)^n x$, $T^n y = (-1)^n y$. Hence we have

$$\begin{cases} \|T^n x - T^n y\|^2 = \|(-1)^n x - (-1)^n y\|^2 = \|x - y\|^2 = \|x\|^2 + \|y\|^2, \\ k_n \|x - y\|^2 = k_n(\|x\|^2 + \|y\|^2), \\ \|x - T^n x - (y - T^n y)\|^2 = [1 - (-1)^n]^2 \|x - y\|^2 = [1 - (-1)^n]^2 (\|x\|^2 + \|y\|^2). \end{cases}$$

Thus the inequality (1.6) still holds. Therefore the mapping T defined by (1.7) is a $(k, \{\mu_n\}, \{\xi_n\}, \phi)$-total asymptotically strict pseudocontraction mapping.

Remark 1.3 If $\phi(\lambda) = \lambda^2$, $\lambda \geq 0$, and $\xi_n = 0$, then a $(k, \{\mu_n\}, \{\xi_n\}, \phi)$-total asymptotically strictly pseudocontractive mapping is an $(k, \{k_n\})$-asymptotically strict pseudocontractive mapping, where $k_n = 1 + \mu_n$.

A mapping $T : C \to C$ is said to be L-Lipschitzian, if there exists a constant $L > 0$, such that

$$\|Tx - Ty\| \leq L\|x - y\|, \quad \forall x, y \in C \text{ and } n \geq 1. \tag{1.8}$$

A mapping $T : C \to C$ is said to be uniformly L-Lipschitzian, if there exists a constant $L > 0$, such that

$$\left\|T^n x - T^n y\right\| \leq L\|x - y\|, \quad \forall x, y \in C \text{ and } n \geq 1. \tag{1.9}$$

Recently, Chang *et al.* [11] proposed the following iterative algorithm for solving a split common fixed point problem for total asymptotically strict pseudocontraction mappings in the framework of infinite-dimensional Hilbert spaces:

$$\begin{cases} x_1 \in H_1 \quad \text{chosen arbitrary,} \\ u_n = x_n + \gamma A^*(T^n - I)A x_n, \\ x_{n+1} = (1 - \alpha_n)u_n + \alpha_n S^n(u_n), \quad n \in N, \end{cases}$$

they proved that $\{x_n\}$ converges weakly to a split common fixed point x^* of the mappings S and T, where $S : H_1 \to H_1$ and $T : H_2 \to H_2$ are two total asymptotically strict pseudocontraction mappings, $A : H_1 \to H_2$ is a bounded linear operator. In addition, they also show that $\{x_n\}$ converges strongly to a split common fixed point x^* for mappings S and T when S is semi-compact.

Inspired and motivated by the recent work of Chang *et al.* [11], Moudafi [12, 13], *etc.*, the purpose of this paper is to propose an algorithm to solve the split common fixed point problems for total asymptotically strict pseudocontraction mappings in Hilbert spaces. Under suitable conditions on the control parameters and without the assumption of semi-compactness on the mappings, a strong convergence theorem is established. The results presented in the paper improve and extend some recent corresponding results in [4, 5, 8, 11–15].

2 Preliminaries

Throughout this paper, let H be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. We denote the strong convergence and weak convergence of a sequence $\{x_n\}$ to a point $x \in H$ by $x_n \to x$, $x_n \rightharpoonup x$, respectively.

Let H be a Hilbert space. A mapping $T : H \to H$ is said to be demi-closed at origin, if for any sequence $\{x_n\} \subset H$ with $x_n \rightharpoonup x^*$ and $\|(I - T)x_n\| \to 0$, then $x^* = Tx^*$.

A mapping $T : C \to C$ is said to be semi-compact, if for any bounded sequence $\{x_n\} \subset C$ with $\lim_{n \to \infty} \|x_n - Tx_n\| = 0$, then there exists a subsequence $\{x_{n_i}\} \subset \{x_n\}$ such that $\{x_{n_i}\}$ converges strongly to some point $x^* \in C$.

For every point $x \in H$, there exists a unique nearest point of C, denoted by $P_C x$, such that $\|x - P_C x\| \leq \|x - y\|$ for all $y \in C$. Such a P_C is called the metric projection from H onto C. It is well known that P_C is a firmly nonexpansive mapping from H to C, *i.e.*,

$$\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y \rangle, \quad \forall x, y \in H.$$

Further, for any $x \in H$ and $z \in C$, $z = P_C x$ if and only if

$$\langle x - z, z - y \rangle \geq 0, \quad \forall y \in C. \tag{2.1}$$

Lemma 2.1 ([16]) *Let C be a nonempty closed convex subset of a real Hilbert space H and $P_C : H \to C$ be the metric projection from H onto C. Then the following inequality holds*:

$$\left\| y - P_C(x) \right\|^2 + \left\| x - P_C(x) \right\|^2 \leq \|x - y\|^2, \quad \forall y \in C, \forall x \in H. \tag{2.2}$$

Lemma 2.2 ([17]) *Let H be a real Hilbert space, then the following equalities hold*:
(i) $\|\lambda x + (1 - \lambda)y\|^2 = \lambda \|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|^2$, $\forall x, y \in H$, $\forall \lambda \in R$;
(ii) $2\langle x, y \rangle = \|x\|^2 + \|y\|^2 - \|x - y\|^2$, $\forall x, y \in H$.

Lemma 2.3 ([18]) *Let H be a real Hilbert space. If $\{x_n\}$ is a sequence in H, weakly convergent to z, then*

$$\limsup_{n \to \infty} \|x_n - y\|^2 = \limsup_{n \to \infty} \|x_n - z\|^2 + \|z - y\|^2, \quad \forall y \in H.$$

Lemma 2.4 ([12]) *Let $T : C \to C$ be a $(\rho, \{\mu_n\}, \{\xi_n\}, \phi)$-total asymptotically strictly pseudocontractive mapping. If $F(T) \neq \emptyset$, then for each $q \in F(T)$ and for each $x \in C$, the following equivalent inequalities hold*:

$$\left\| T^n x - q \right\|^2 \leq \|x - q\|^2 + \rho \left\| x - T^n x \right\|^2 + \mu_n \phi\left(\|x - q\| \right) + \xi_n, \tag{2.3}$$

$$\langle x - T^n x, x - q \rangle \geq \frac{1 - \rho}{2} \left\| x - T^n x \right\|^2 - \frac{\mu_n}{2} \phi\left(\|x - q\| \right) - \frac{\xi_n}{2}, \tag{2.4}$$

$$\langle x - T^n x, q - T^n x \rangle \leq \frac{\rho + 1}{2} \left\| T^n x - x \right\|^2 + \frac{\mu_n}{2} \phi\left(\|x - q\| \right) + \frac{\xi_n}{2}. \tag{2.5}$$

Lemma 2.5 ([12]) *Let H be a real Hilbert space and let $T : H \to H$ be a uniformly L-Lipschitzian and $(k, \{\mu_n\}, \{\xi_n\}, \phi)$-total asymptotically strictly pseudocontractive mapping. Then the demi-closedness principle holds for T in the sense that if $\{x_n\}$ is a sequence in H such that $x_n \rightharpoonup x^*$, and $\limsup_{m \to \infty} \limsup_{n \to \infty} \|x_n - T^m x_n\| = 0$, then $(I - T)x^* = 0$. In particular, if $x_n \rightharpoonup x^*$, and $\|(I - T)x_n\| \to 0$, then $(I - T)x^* = 0$, i.e., T is demi-closed at the origin.*

3 Main results

Theorem 3.1 *Let H_1 and H_2 be two real Hilbert spaces, and $A : H_1 \to H_2$ be a bounded linear operator, $S : H_1 \to H_1$ be a uniformly L-Lipschitzian and $(\rho, \{\mu_n^{(1)}\}, \{\xi_n^{(1)}\}, \phi_1)$-total asymptotically strict pseudocontraction mapping and $T : H_2 \to H_2$ be a uniformly $\tilde{L}$-Lipschitzian and $(k, \{\mu_n^{(2)}\}, \{\xi_n^{(2)}\}, \phi_2)$-total asymptotically strict pseudocontraction mapping satisfying $F(S) \neq \emptyset$ and $F(T) \neq \emptyset$, respectively. Let $\{x_n\}$ be a sequence generated by $x_1 \in C_1 = H_1$,*

$$\begin{cases} y_n = (1 - \alpha_n)z_n + \alpha_n S^n(z_n), \\ z_n = x_n + \gamma A^*(T^n - I)Ax_n, \quad \forall n \geq 1, \\ C_{n+1} = \{v \in C_n : \|y_n - v\|^2 \leq (1 + \mu_n M^*)\|z_n - v\|^2 + \mu_n \phi(M) + \xi_n, \\ \qquad \|z_n - v\|^2 \leq (1 + \gamma \mu_n M^* \|A\|^2)\|x_n - v\|^2 + \gamma \mu_n \phi(M) + \gamma \xi_n\}, \\ x_{n+1} = P_{C_{n+1}}(x_1), \end{cases} \tag{3.1}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$, γ is a positive constant, $\{\mu_n\}$, $\{\xi_n\}$, and ϕ satisfy the following conditions:

(i) $\alpha_n \in (\delta, 1 - \rho)$, $\forall n \geq 1$ *and* $\gamma \in (0, \frac{1-k}{\|A\|^2})$, *where δ is a constant in* $(0, 1 - \rho)$;

(ii) $\mu_n = \max\{\mu_n^{(1)}, \mu_n^{(2)}\}$, $\xi_n = \max\{\xi_n^{(1)}, \xi_n^{(2)}\}$, $n \geq 1$, *and* $\sum_{n=1}^{\infty} \mu_n < \infty$, $\sum_{n=1}^{\infty} \xi_n < \infty$;

(iii) $\phi = \max\{\phi_1, \phi_2\}$ *and there exist two positive constants M and M^* such that*
$\phi(\lambda) \leq M^* \lambda^2$ *for all* $\lambda \geq M$.

If $\Gamma = \{p \in F(S) : Ap \in F(T)\} \neq \emptyset$, then the sequence $\{x_n\}$ converges strongly to a split common fixed point $x^* \in \Gamma$.

Proof We shall divide the proof into five steps.

Step 1. We first show that C_n is closed and convex for each $n \geq 1$.

Since $C_1 = H_1$, C_1 is closed and convex. Suppose that C_n is closed and convex for some $n > 1$. Since for any $v \in C_n$, we have

$$\|y_n - v\|^2 \leq (1 + \mu_n M^*)\|z_n - v\|^2 + \mu_n \phi(M) + \xi_n$$

$$\Leftrightarrow \quad 2\langle(1 + \mu_n M^*)z_n - y_n - \mu_n M^* v, v\rangle$$

$$\leq (1 + \mu_n M^*)\|z_n\|^2 - \|y_n\|^2 + \mu_n \phi(M) + \xi_n \tag{3.2}$$

and

$$\|z_n - v\|^2 \leq (1 + \gamma \mu_n M^* \|A\|^2)\|x_n - v\|^2 + \gamma \mu_n \phi(M) + \gamma \xi_n$$

$$\Leftrightarrow \quad 2\langle(1 + \gamma \mu_n M^* \|A\|^2)x_n - z_n - \gamma \mu_n M^* \|A\|^2 v, v\rangle$$

$$\leq (1 + \gamma \mu_n M^* \|A\|^2)\|x_n\|^2 - \|z_n\|^2 + \gamma \mu_n \phi(M) + \gamma \xi_n, \tag{3.3}$$

hence the set C_{n+1} is closed and convex. Therefore C_n is closed and convex for each $n \geq 1$, and $P_{C_{n+1}} x_1$ is well defined.

Step 2. We show that $\Gamma \subset C_n$ for all $n \geq 1$.

In fact, since ϕ is a continuous and increasing function, $\phi(\lambda) \leq \phi(M)$, if $\lambda \leq M$, and $\phi(\lambda) \leq M^* \lambda^2$, if $\lambda \geq M$. In either case, we can obtain

$$\phi(\lambda) \leq \phi(M) + M^* \lambda^2, \quad \forall \lambda \geq 0. \tag{3.4}$$

For any given $p \in \Gamma$, then $p \in F(S)$ and $Ap \in F(T)$. It follows from (3.1) that

$$\|z_n - p\|^2 = \|x_n - p + \gamma A^*(T^n - I)Ax_n\|^2$$

$$= \|x_n - p\|^2 + \gamma^2\|A^*(T^n - I)Ax_n\|^2 + 2\gamma\langle x_n - p, A^*(T^n - I)Ax_n\rangle, \tag{3.5}$$

where

$$2\gamma\langle x_n - p, A^*(T^n - I)Ax_n\rangle$$

$$= 2\gamma\langle Ax_n - Ap, (T^n - I)Ax_n\rangle$$

$$= 2\gamma\langle Ax_n - Ap + (T^n - I)Ax_n - (T^n - I)Ax_n, (T^n - I)Ax_n\rangle$$

$$= 2\gamma\{\langle T^n Ax_n - Ap, T^n Ax_n - Ax_n\rangle - \|(T^n - I)Ax_n\|^2\}$$

$$\leq 2\gamma \left[\frac{1+k}{2} \left\| (T^n - I) A x_n \right\|^2 + \frac{\mu_n}{2} \phi \left(\|A x_n - A p\| \right) + \frac{\xi_n}{2} - \left\| (T_n^n - I) A x_n \right\|^2 \right]$$

$$\leq \gamma (k-1) \left\| y (T_n^n - I) A x_n \right\|^2 + \gamma \mu_n M^* \|A\|^2 \|x_n - p\|^2 + \gamma \mu_n \phi(M) + \gamma \xi_n. \tag{3.6}$$

Substituting (3.6) into (3.5), we have

$$\|z_n - p\|^2 = \left\| x_n - p + \gamma A^* (T^n - I) A x_n \right\|^2$$

$$\leq \|x_n - p\|^2 + \gamma^2 \left\| A^* (T^n - I) A x_n \right\|^2 + \gamma (k-1) \left\| (T^n - I) A x_n \right\|^2$$

$$+ \gamma \mu_n M^* \|A\|^2 \|x_n - p\|^2 + \gamma \mu_n \phi(M) + \gamma \xi_n$$

$$\leq \|x_n - p\|^2 + \gamma^2 \|A\|^2 \left\| T^n A x_n - A x_n \right\|^2 + \gamma (k-1) \left\| (T^n - I) A x_n \right\|^2$$

$$+ \gamma \mu_n M^* \|A\|^2 \|x_n - p\|^2 + \gamma \mu_n \phi(M) + \gamma \xi_n$$

$$= \left(1 + \gamma \mu_n M^* \|A\|^2 \right) \|x_n - p\|^2 - \gamma \left(1 - k - \gamma \|A\|^2 \right) \left\| T^n A x_n - A x_n \right\|^2$$

$$+ \gamma \mu_n \phi(M) + \gamma \xi_n. \tag{3.7}$$

On the other hand, since

$$\|y_n - p\|^2 = \left\| z_n - p - \alpha_n (z_n - S^n z_n) \right\|^2$$

$$= \|z_n - p\|^2 - 2\alpha_n \langle z_n - p, z_n - S^n z_n \rangle + \alpha_n^2 \|z_n - S^n z_n\|^2$$

$$\leq \|z_n - p\|^2 - \alpha_n (1 - \rho) \|z_n - S^n z_n\|^2$$

$$+ \alpha_n \mu_n \phi \left(\|z_n - p\| \right) + \alpha_n \xi_n + \alpha_n^2 \|z_n - S^n z_n\|^2$$

$$\leq \|z_n - p\|^2 - \alpha_n (1 - \rho - \alpha_n) \|z_n - S^n z_n\|^2$$

$$+ \alpha_n \mu_n \left(\phi(M) + M^* \left(\|z_n - p\| \right)^2 \right) + \alpha_n \xi_n$$

$$= \left(1 + \alpha_n \mu_n M^* \right) \|z_n - p\|^2 - \alpha_n (1 - \rho - \alpha_n) \|z_n - S^n z_n\|^2$$

$$+ \alpha_n \mu_n \phi(M) + \alpha_n \xi_n, \tag{3.8}$$

so, it follows from (3.7) and (3.8) that $p \in C_n$, and then $\Gamma \subset C_n$ for any $n \geq 1$.

Step 3. We prove that $\{x_n\}$ is a Cauchy sequence.

From the definition of C_{n+1}, we know that $x_n = P_{C_n} x_1$. Since $\Gamma \subset C_{n+1} \subset C_n$, and $x_{n+1} \in C_{n+1} \subset C_n$, $\forall n > 1$, we have

$$\|x_n - x_1\| \leq \|x_{n+1} - x_1\| \tag{3.9}$$

and

$$\|x_n - x_1\| \leq \|p - x_1\|, \quad \forall n \in N \text{ and } p \in \Gamma. \tag{3.10}$$

It means that $\{x_n\}$ is nondecreasing and bounded. So, $\lim_{n \to \infty} \|x_n - x_1\|$ exists. For $m > n$, by the definition of C_n, we have $x_m = P_{C_m} x_1 \in C_m \subset C_n$, it from Lemma 2.1 that

$$\|x_m - x_n\|^2 + \|x_1 - x_n\|^2 = \|x_m - P_{C_n} x_1\|^2 + \|x_1 - P_{C_n} x_1\|^2 \leq \|x_m - x_1\|^2. \tag{3.11}$$

Since $\lim_{n\to\infty} \|x_n - x_1\|$ exists, from (3.11), we obtain $\lim_{n\to\infty} \|x_n - x_m\| = 0$. Therefore $\{x_n\}$ is a Cauchy sequence.

Step 4. We prove that $\lim_{n\to\infty} \|z_n - Sz_n\| = \lim_{n\to\infty} \|Ax_n - TAx_n\| = 0$.

Since $x_{n+1} = P_{C_{n+1}} x_1 \in C_{n+1} \subset C_n$, we obtain

$$
\begin{aligned}
\|z_n - x_n\|^2 &\leq \|z_n - x_{n+1}\|^2 + \|x_{n+1} - x_n\|^2 + 2\|z_n - x_{n+1}\| \cdot \|x_{n+1} - x_n\| \\
&\leq \left[\left(1 + \gamma\mu_n M^* \|A\|^2\right) + 1\right]\|x_{n+1} - x_n\|^2 + \gamma\mu_n\phi(M) + \gamma\xi_n \\
&\quad + 2\sqrt{\left(1 + \gamma\mu_n M^* \|A\|^2\right)\|x_{n+1} - x_n\|^2 + \gamma\mu_n\phi(M) + \gamma\xi_n} \\
&\quad \times \|x_{n+1} - x_n\|,
\end{aligned}
\tag{3.12}
$$

since $\sum_{n=1}^{\infty} \mu_n < \infty$, $\sum_{n=1}^{\infty} \xi_n < \infty$, and $\lim_{n\to\infty} \|x_n - x_m\| = 0$, therefore

$$
\lim_{n\to\infty} \|z_n - x_n\| = 0.
\tag{3.13}
$$

And

$$
\begin{aligned}
\|y_n - x_n\|^2 &\leq \|y_n - x_{n+1}\|^2 + \|x_{n+1} - x_n\|^2 + 2\|y_n - x_{n+1}\|\|x_{n+1} - x_n\| \\
&\leq \left(1 + \gamma\mu_n M^*\right)\|z_n - x_{n+1}\|^2 + \mu_n\phi(M) + \xi_n + \|x_{n+1} - x_n\|^2 \\
&\quad + 2\|y_n - x_{n+1}\|\|x_{n+1} - x_n\| \\
&\leq \left[\left(1 + \gamma\mu_n M^*\right)\left(1 + \gamma\mu_n M^* \|A\|^2\right) + 1\right]\|x_{n+1} - x_n\|^2 \\
&\quad + \left[\left(1 + \gamma\mu_n M^*\right)\gamma + 1\right]\mu_n\phi(M) + \left[\left(1 + \gamma\mu_n M^*\right)\gamma + 1\right]\xi_n \\
&\quad + 2\sqrt{\left(1 + \gamma\mu_n M^*\right)\|z_n - x_{n+1}\|^2 + \mu_n\phi(M) + \xi_n}\|x_{n+1} - x_n\|,
\end{aligned}
$$

by $\sum_{n=1}^{\infty} \mu_n < \infty$, $\sum_{n=1}^{\infty} \xi_n < \infty$, and $\lim_{n\to\infty} \|x_n - x_m\| = 0$, we have

$$
\lim_{n\to\infty} \|y_n - x_n\| = 0.
\tag{3.14}
$$

Further,

$$
\|z_n - y_n\| \leq \|z_n - x_n\| + \|x_n - y_n\| \to 0.
\tag{3.15}
$$

It follows from (3.7) that

$$
\begin{aligned}
\gamma\left(1 - k - \gamma\|A\|^2\right)\left\|T^n Ax_n - Ax_n\right\|^2 \\
\leq \left(1 + \gamma\mu_n M^* \|A\|^2\right)\|x_n - p\|^2 - \|z_n - p\|^2 + \gamma\mu_n\phi(M) + \gamma\xi_n \\
\leq \|x_n - z_n\|\left(\|x_n - p\| + \|z_n - p\|\right) + \gamma\mu_n M^* \|A\|^2 \|x_n - p\|^2 \\
+ \gamma\mu_n\phi(M) + \gamma\xi_n.
\end{aligned}
\tag{3.16}
$$

Since $\sum_{n=1}^{\infty} \mu_n < \infty$, $\sum_{n=1}^{\infty} \xi_n < \infty$, $\gamma(1 - k - \gamma\|A\|^2) > 0$, and $\{x_n\}$ is bounded, by (3.13) and (3.16), we get

$$
\lim_{n\to\infty} \left\|T^n Ax_n - Ax_n\right\| = 0.
\tag{3.17}
$$

On the other hand, from (3.8), we have

$$\alpha_n(1 - \rho - \alpha_n)\|z_n - S^n z_n\|^2$$

$$\leq (1 + \alpha_n \mu_n M^*)\|z_n - p\|^2 - \|y_n - p\|^2 + \alpha_n \mu_n \phi(M) + \alpha_n \xi_n$$

$$\leq \|z_n - y_n\|(\|z_n - p\| + \|y_n - p\|) + \alpha_n \mu_n M^*\|z_n - p\|^2 + \alpha_n \mu_n \phi(M) + \alpha_n \xi_n. \qquad (3.18)$$

This together with the conditions (i), (ii), and $\{z_n\}$ being bounded, from (3.15) and (3.18), we have

$$\lim_{n \to \infty} \|z_n - S^n z_n\| = 0. \qquad (3.19)$$

In addition, since $\|z_{n+1} - z_n\| \leq \|z_{n+1} - x_{n+1}\| + \|x_{n+1} - x_n\| + \|x_n - z_n\|$, this means that

$$\lim_{n \to \infty} \|z_{n+1} - z_n\| = 0. \qquad (3.20)$$

Since S is uniformly L-Lipschitzian continuous,

$$\|z_n - S z_n\| \leq \|z_n - S^n z_n\| + \|S^n z_n - S z_n\|$$

$$\leq \|z_n - S^n z_n\| + L\|S^{n-1} z_n - z_n\|$$

$$\leq \|z_n - S^n z_n\| + L(\|S^{n-1} z_n - S^{n-1} z_{n-1}\| + \|S^{n-1} z_{n-1} - z_n\|)$$

$$\leq \|z_n - S^n z_n\| + L^2\|z_n - z_{n-1}\| + L\|S^{n-1} z_{n-1} - z_{n-1} + z_{n-1} - z_n\|$$

$$\leq \|z_n - S^n z_n\| + L(1 + L)\|z_n - z_{n-1}\| + L\|z_{n-1} - S^{n-1} z_{n-1}\|. \qquad (3.21)$$

It follows from (3.19), (3.20), and (3.21) that

$$\lim_{n \to \infty} \|z_n - S z_n\| = 0. \qquad (3.22)$$

Similarly, in the same way as above, from (3.17), we can also obtain

$$\lim_{n \to \infty} \|A x_n - T A x_n\| = 0. \qquad (3.23)$$

Step 5. We prove that $\{x_n\}$ converges strongly to $x^* \in \Gamma$.

Since $\{x_n\}$ is a Cauchy sequence, we may assume that $x_n \to x^*$. Thus we have $z_n \to x^*$ from (3.13), which implies that $z_n \rightharpoonup x^*$. So it follows from (3.22) and Lemma 2.5 that $x^* \in F(S)$.

On the other hand, since A is a bounded linear operator, we know that $\lim_{n \to \infty} \|A x_n - A x^*\| = 0$. Therefore, it follows from Lemma 2.5 and (3.23) that $A x^* \in F(T)$. This means that $x^* \in \Gamma$ and $\{x_n\}$ converges strongly to $x^* \in \Gamma$. The proof is completed. $\qquad \square$

The following result can be obtained from Theorem 3.1 immediately.

Corollary 3.2 *Let H_1 and H_2 be two real Hilbert spaces, $A : H_1 \to H_2$ be a bounded linear operator, $S : H_1 \to H_1$ be a uniformly L-Lipschitzian and $(\rho, \{k_n^{(1)}\})$-asymptotically strict*

pseudocontraction mapping and $T : H_2 \to H_2$ be a uniformly $\tilde{L}$-Lipschitzian and $(k, \{k_n^{(2)}\})$-asymptotically strict pseudocontraction mapping satisfying $F(S) \neq \emptyset$ and $F(T) \neq \emptyset$, respectively. Let $\{x_n\}$ be a sequence defined as follows: $x_1 \in C_1 = H_1$,

$$
\begin{cases}
y_n = (1 - \alpha_n)z_n + \alpha_n S^n(z_n), \\
z_n = x_n + \gamma A^*(T^n - I)Ax_n, \quad \forall n \geq 1, \\
C_{n+1} = \{v \in C_n : \|y_n - v\|^2 \leq (1 + (k_n - 1)\alpha_n)\|z_n - v\|^2, \\
\qquad \|z_n - v\|^2 \leq (1 + (k_n - 1)\gamma\|A\|^2)\|x_n - v\|^2\}, \\
x_{n+1} = P_{C_{n+1}}(x_1),
\end{cases}
\tag{3.24}
$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$, γ is a positive constant and $\{k_n\}$ satisfy the following conditions:

(i) $k_n = \max\{k_n^{(1)}, k_n^{(2)}\}$, *and* $\sum_{n=1}^{\infty}(k_n - 1) < \infty$;

(ii) $\alpha_n \in (\delta, 1 - \rho)$, $\forall n \geq 1$ *and* $\gamma \in (0, \frac{1-k}{\|A\|^2})$, *where δ is a constant in $(0, 1 - \rho)$*.

If $\Gamma \neq \emptyset$, then the sequence $\{x_n\}$ converges strongly to a split common fixed point $x^* \in \Gamma$.

Remark 3.3 Theorem 3.1 extends and improves the result of Chang *et al.* [11, 15] from weak convergence to strong convergence by using the modified iterative scheme that we propose.

Remark 3.4 In Theorem 3.1, as S and T are two nonexpansive mappings, demi-contractive mappings or asymptotically strict pseudocontraction mappings, we can also obtain similar results.

Example 3.5 Let C and S be the same as in Example 1.1, and D and T be the same as in Example 1.2. It is obvious that $F(T) = \{(0, 0, \ldots, 0)\} \cup \{(x_1, x_2, \ldots, x_n) : \prod_{i=1}^{n} x_i < 0\}$, $F(S) = \{(0, 0, \ldots, 0, \ldots)\}$, C and D are nonempty closed convex subsets of l^2 and R^n, respectively. Let $A : C \to D$ be defined by $Ax = (x_1, x_2, \ldots, x_n)$ for $x = (x_1, x_2, \ldots) \in C$. Then A is a bounded linear operator with adjoint operator $A^* z = (x_1, x_2, \ldots, x_n, 0, 0, \ldots)$ for $z = (x_1, x_2, \ldots, x_n) \in D$. Clearly, $\|A\| = \|A^*\| = 1$. By using algorithm (3.1) with $\frac{1}{4} < \alpha_n < \frac{3}{4}$ and $\gamma \in (0, 1)$. We can verify $x_n \to (0, 0, \ldots) \in F(S)$ and $A(0, 0, \ldots) = (0, 0, \ldots, 0) \in F(T)$.

4 Applications

4.1 Application to hierarchical variational inequality problem

Let H be a real Hilbert space, T and S be two nonexpansive mappings from H to H such that $F(T) \neq \emptyset$ and $F(S) \neq \emptyset$.

The so-called hierarchical variational inequality problem for nonexpansive mapping T with respect to nonexpansive mapping $S : H \to H$ is to find a point $x^* \in F(S)$ such that

$$
\langle x^* - Tx^*, x^* - x \rangle \leq 0, \quad \forall x \in F(S).
\tag{4.1}
$$

It is easy to see that (4.1) is equivalent to the following fixed point problem:

$$
\text{find } x^* \in F(S) \text{ such that } x^* = P_{F(S)} Tx^*,
\tag{4.2}
$$

where $P_{F(S)}$ is the metric projection from H onto $F(S)$. Letting $C := F(S)$ and $Q := F(P_{F(S)}T)$ (the fixed point set of the mapping $P_{F(S)}T$) and $A = I$ (the identity mapping on H), then

the problem (4.2) is equivalent to the following split feasibility problem:

$$\text{find } x^* \in C \text{ such that } Ax^* \in Q. \tag{4.3}$$

Hence from Theorem 3.1 we have the following theorem.

Theorem 4.1 *Let H, S, T, C, and Q be the same as above. Let $\{x_n\}$ be a sequence generated by $x_1 \in C_1 = H_1$,*

$$\begin{cases} y_n = (1 - \alpha_n)z_n + \alpha_n S z_n, \\ z_n = x_n + \gamma(T - I)x_n, \quad \forall n \geq 1, \\ C_{n+1} = \{v \in C_n : \|y_n - v\| \leq \|z_n - v\| \leq \|x_n - v\|\}, \\ x_{n+1} = P_{C_{n+1}}(x_1), \end{cases} \tag{4.4}$$

where $\gamma \in (0,1)$ and $\{\alpha_n\}$ is a sequence in $(0,1)$ satisfying $\liminf_{n \to \infty} \alpha_n(1 - \alpha_n) > 0$. If $C \cap Q \neq \emptyset$, then the sequence $\{x_n\}$ converges strongly to a solution of the hierarchical variational inequality problem (4.1).

Proof Since S is nonexpansive, it is uniformly L-Lipschitzian continuous and $(\rho, \{\mu_n^{(1)}\}, \{\xi_n^{(1)}\}, \phi_1)$-total asymptotically strict pseudocontractive with $L = 1$, $\mu_n^{(1)} = 0$, $\xi_n^{(1)} = 0$, $\phi_1 = 0$. Again since T is nonexpansive, it is uniformly $\tilde{L}$-Lipschitzian continuous and $(k, \{\mu_n^{(2)}\}, \{\xi_n^{(2)}\}, \phi_2)$-total asymptotically strict pseudocontractive with $\tilde{L} = 1$, $\mu_n^{(2)} = 0$, $\xi_n^{(2)} = 0$, $\phi_2 = 0$. Therefore, all conditions in Theorem 3.1 are satisfied. The conclusions of Theorem 4.1 can be obtained from Theorem 3.1. □

4.2 Application to quadratic minimization problem over a fixed point set

Let $K : H \to H$ be a linear bounded η-strongly positive operator with $\eta > 1$, *i.e.*,

$$\langle Kx, x \rangle \geq \eta \|x\|^2.$$

Let $f : H \to H$ be a ϱ-contraction with $\varrho \in (0,1)$ and $\gamma \in (\varrho^{-1}(\eta - \eta^{-1}), \frac{\eta}{\varrho})$, $S : H \to H$ be a nonexpansive mapping with $F(S) \neq \emptyset$ and $T := I - \eta(K - \gamma f)$ be a mapping from H to H.

Lemma 4.2 ([19]) *Assume A is a strongly positive linear bounded operator on a Hilbert space H with coefficient $\bar{\gamma} > 0$ and $0 < \rho \leq \|A\|^{-1}$. Then $\|I - \rho A\| \leq 1 - \rho\bar{\gamma}$.*

Now we prove that $T : H \to H$ is a nonexpansive mapping. In fact, for $\forall x, y \in H$, $\gamma \in (\varrho^{-1}(\eta - \eta^{-1}), \frac{\eta}{\varrho})$, from Lemma 4.2, we have

$$\begin{aligned} \|Tx - Ty\| &\leq \|(I - \eta K)x - (I - \eta K)y\| + \eta\gamma \|f(x) - f(y)\| \\ &\leq \|I - \eta K\| \|x - y\| + \eta\gamma\varrho \|x - y\| \\ &\leq (1 - \eta^2 + \eta\gamma\varrho) \|x - y\| \\ &\leq \|x - y\|. \end{aligned} \tag{4.5}$$

Then the hierarchical variational inequality problem (4.1) reduces to finding $x^* \in F(S)$ such that

$$\langle (K - \gamma f)x^*, x - x^* \rangle \geq 0, \quad \forall x \in F(S). \tag{4.6}$$

It is easy to see that (4.6) is exactly the optimality condition of the following quadratic minimization problem:

$$\min_{x \in F(S)} \frac{1}{2}\big\{\langle Kx, x\rangle - h(x)\big\}, \tag{4.7}$$

where h is the potential for γf, i.e., $h'(x) = \gamma f$. Let $C = F(S)$, $Q = F(P_{F(S)}(I - \eta(K - \gamma f)))$ and $A = I$, then the quadratic minimization problem (4.7) is equivalent to the following split feasibility problem:

find $x^* \in C$ and such that $Ax^* \in Q$.

Hence from Theorem 3.1 we have the following result.

Theorem 4.3 *Let H, K, f, S, T, C, and Q be the same as above. Let $\{x_n\}$ be a sequence generated by $x_1 \in C_1 = H_1$,*

$$\begin{cases} y_n = (1 - \alpha_n)z_n + \alpha_n S z_n, \\ z_n = x_n + \gamma(T - I)x_n, \quad \forall n \geq 1, \\ C_{n+1} = \{v \in C_n : \|y_n - v\| \leq \|z_n - v\| \leq \|x_n - v\|\}, \\ x_{n+1} = P_{C_{n+1}}(x_1), \end{cases} \tag{4.8}$$

where $\gamma \in (0,1)$ and $\{\alpha_n\}$ is a sequence in $(0,1)$ satisfying $\liminf_{n \to \infty} \alpha_n(1 - \alpha_n) > 0$. If $C \cap Q \neq \emptyset$, then the sequence $\{x_n\}$ converges strongly to a solution of problem (4.7).

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