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Schrödinger type operators on generalized Morrey spaces

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Abstract

In this paper we introduce a class of generalized Morrey spaces associated with the Schrödinger operator $L = -\Delta + V$. Via a pointwise estimate, we obtain the boundedness of the operators $V^{\beta_2}(-\Delta + V)^{-\beta_1}$ and their dual operators on these Morrey spaces.

MSC: Primary 42B35; 42B20

Keywords: generalized Morrey spaces; Schrödinger operator; commutator; reverse Hölder class

1 Introduction

The investigation of Schrödinger operators on the Euclidean space $\mathbb{R}^n$ with nonnegative potentials which belong to the reverse Hölder class has attracted attention of many authors. Shen [1] studied the Schrödinger operator $L = -\Delta + V$, assuming the nonnegative potential V belongs to the reverse Hölder class B_q, $q \geq \frac{n}{2}$. In [1], Shen proved the L^p-boundedness of the operators $(-\Delta + V)^{i\gamma}$, $\nabla^2(-\Delta + V)^{-1}$, $\nabla(-\Delta + V)^{-1/2}$ and $\nabla(-\Delta + V)^{-1}\nabla$. For further information, we refer the reader to Guo *et al.* [2], Liu [3], Liu *et al.* [4, 5], Tang and Dong [6], Yang *et al.* [7, 8] and the references therein.

The purpose of this paper is to generalize the results of Shen [1] and Sugano [9] to a class of Morrey spaces associated with L, denoted by $L^{p,q,\lambda}_{\alpha,\theta,V}(\mathbb{R}^n)$. See Definition 2.8 below. The significance of these spaces is that for particular choices of the parameters p, q, λ, θ and α, one obtains many classical function spaces (see Table 1).

In Section 3, let T be one of the Schrödinger type operators $\nabla(-\Delta + V)^{-1}\nabla$, $\nabla(-\Delta + V)^{-1/2}$ and $(-\Delta + V)^{-1/2}\nabla$. With the help of the L^p-boundedness of T, it is easy to verify that T is bounded on $L^{p,q,\lambda}_{\alpha,\theta,V}(\mathbb{R}^n)$. For $b \in BMO(\mathbb{R}^n)$, we can also obtain the boundedness of the commutator $[b, T]$ on $L^{p,q,\lambda}_{\alpha,\theta,V}(\mathbb{R}^n)$. See Theorems 3.2 and 3.3. For $\theta = 0$, $p = q$ and $0 < \lambda < 1$, $L^{p,p,\lambda}_{\alpha,0,V}(\mathbb{R}^n)$ becomes the spaces $L^{p,\lambda}_{\alpha,V}(\mathbb{R}^n)$ introduced by Tang and Dong [6]. Hence, the results are generalizations of Theorems 1 and 2 in [6].

Table 1 Special cases of $L^{p,q,\lambda}_{\alpha,\beta,V}$

$\theta = 0, \alpha = 0, p = q, 0 < \lambda < 1$	Morrey space $L^{p,\lambda}(\mathbb{R}^n)$ [10]
$\theta = 0, p = q, 0 < \lambda < 1$	Morrey type space $L^{p,\lambda}_{\alpha,V}(\mathbb{R}^n)$ [6]
$\alpha = \lambda = 0, \theta \in \mathbb{R}, 0 < p, q < \infty$,	Herz spaces $K^{\theta,q}_p$ [11]
$\alpha = 0, \lambda \geq 0, \theta \in \mathbb{R}, 0 < p, q < \infty$	Morrey-Herz spaces $MK^{\theta,\lambda}_{p,q}$ [12, 13]

In recent years, the fractional integral operator $I_\alpha = (-\Delta + V)^{-\alpha}$ has been studied extensively. We refer to Duong and Yan [14], Jiang [15], Tang and Dong [6] and Yang *et al.* [7] for details. Suppose that $V \in B_s$, $s \ge \frac{n}{2}$. For $0 \le \beta_2 \le \beta_1 < \frac{n}{2}$, let

$$\begin{cases} T_{\beta_1,\beta_2} =: V^{\beta_2}(-\Delta + V)^{-\beta_1}, \\ T^*_{\beta_1,\beta_2} =: (-\Delta + V)^{-\beta_1} V^{\beta_2}. \end{cases}$$

Sugano [9] obtained the weighted estimates for T_{β_1,β_2}, $T^*_{\beta_1,\beta_2}$, $0 < \beta_2 \le \beta_1 < 1$. If $\beta_2 = 0$, we can see that $T_{\beta_1,0} = I_{\beta_1}$. So T_{β_1,β_2} and $T^*_{\beta_1,\beta_2}$ can be seen as generalizations of I_α. Moreover, for $(\beta_1, \beta_2) = (1,1)$ and $(1/2, 1/2)$, $T^*_{1,1} = (-\Delta + V)^{-1} V$ and $T^*_{1/2,1/2} = (-\Delta + V)^{-1/2} V^{1/2}$, respectively, which are studied by Shen [1] thoroughly. In Section 4, assume that $1 < p_1 < \infty$, $1 < p_2 < s/\beta_2$ and $1 < q < \infty$. If the index $(q, \beta_1, \beta_2, \lambda, \alpha, \theta)$ satisfies

$$\begin{cases} 1/p_2 = 1/p_1 - 2(\beta_1 - \beta_2)/n, \\ \alpha \in (-\infty, 0] \quad \text{and} \quad \lambda \in (0, n), \\ \lambda/q - 1/p_1 + 2\beta_1/n < \theta < \lambda/q + 1 - 1/p_1, \end{cases}$$

we prove that T_{β_1,β_2} is bounded from $L^{p_1,q,\lambda}_{\alpha,\theta,V}(\mathbb{R}^n)$ to $L^{p_2,q,\lambda}_{\alpha,\theta,V}(\mathbb{R}^n)$. Specially, we know that $(-\Delta + V)^{-1} V$ and $(-\Delta + V)^{-1/2} V^{1/2}$ are bounded on $L^{p,q,\lambda}_{\alpha,\theta,V}(\mathbb{R}^n)$. See Theorems 4.7 and 4.8 for details.

In the research of harmonic analysis and partial differential equations, the commutators play an important role. If T is a Calderón-Zygmund operator, $b \in BMO(\mathbb{R}^n)$, the L^p-boundedness of $[b, T]$ was first discovered by Coifman *et al.* [16]. Later, Strömberg [14] gave a simple proof, adopting the idea of relating commutators with the sharp maximal operator of Fefferman and Stein. In 2008, Guo *et al.* [2] introduced a condition $H(m)$ and obtained L^p-boundedness of the commutator of Riesz transforms associated with L, where $b \in BMO(\mathbb{R}^n)$. For further information, we refer to Liu [17], Liu *et al.* [4, 5], Yang *et al.* [8] and the references therein.

In Section 5, by the boundedness of I_α and $(-\Delta + V)^{-\beta} V^\beta$, we can deduce that the commutators $[b, T_{\beta_1,\beta_2}]$ and $[b, T^*_{\beta_1,\beta_2}]$ are bounded from $L^{p_1}(\mathbb{R}^n)$ to $L^{p_2}(\mathbb{R}^n)$ (see Theorem 5.1). Theorem 5.1 together with Lemmas 4.1 and 2.7 can be used to prove that the commutators $[b, T_{\beta_1,\beta_2}]$ and $[b, T^*_{\beta_1,\beta_2}]$ are bounded from $L^{p_1,q,\lambda}_{\alpha,\theta,V}(\mathbb{R}^n)$ to $L^{p_2,q,\lambda}_{\alpha,\theta,V}(\mathbb{R}^n)$, respectively (see Theorems 5.2 and 5.3).

Remark 1.1 Unlike the setting of the Lebesgue spaces, it is well known that the dual of $L^{p,\lambda}(\mathbb{R}^n)$ is not $L^{p',-\lambda}(\mathbb{R}^n)$. Hence, after obtaining Theorem 4.7, we cannot deduce Theorem 4.8 via the method of duality used by Guo *et al.* [2].

2 Preliminaries

2.1 Schrödinger operator and the auxiliary function

In this paper, we consider the Schrödinger differential operator $L = -\Delta + V$ on $\mathbb{R}^n$, $n \ge 3$, where V is a nonnegative potential belonging to the reverse Hölder class B_s, $s \ge \frac{n}{2}$, which is defined as follows.

Definition 2.1 Let V be a nonnegative function.

(i) We say $V \in B_s$, $s > 1$, if there exists $C > 0$ such that for every ball $B \subset \mathbb{R}^n$, the reverse Hölder inequality

$$\left(\frac{1}{|B|}\int_B V^s(x)\,dx\right)^{\frac{1}{s}} \lesssim \left(\frac{1}{|B|}\int_B V(x)\,dx\right)$$

holds.

(ii) We say $V \in B_\infty$ if there exists a constant C such that for every ball $B \subset \mathbb{R}^n$,

$$\|V\|_{L^\infty(B)} = \frac{1}{|B|}\int_B V(x)\,dx.$$

Remark 2.2 Assume $V \in B_s, 1 < s < \infty$. Then $V(y)\,dy$ is a doubling measure. Namely, there exists a constant C_0 such that for any $r > 0$ and $y \in \mathbb{R}^n$,

$$\int_{B(x,2r)} V(y)\,dy \lesssim C_0 \int_{B(x,r)} V(y)\,dy. \tag{2.1}$$

Definition 2.3 (Shen [1]) For $x \in \mathbb{R}^n$, the function $m_V(x)$ is defined as

$$\frac{1}{m_V(x)} =: \sup\left\{r > 0 : \frac{1}{r^{n-2}}\int_{B(x,r)} V(y)\,dy \le 1\right\}.$$

Remark 2.4 The function m_V reflects the scale of V essentially, but behaves better. It is deeply studied in Shen [1] and plays a crucial role in our proof. We list a property of m_V which will be used in the sequel and refer the reader to Guo *et al.* [2] for the details.

We state some notations and properties of m_V.

Lemma 2.5 (Lemma 1.4 in [1]) *Suppose that $V \in B_s$ with $s \ge \frac{n}{2}$. Then there exist positive constants C and k_0 such that*

(a) *if* $|x-y| \le \frac{C}{m_V(x)}$, $m_V(x) \sim m_V(y)$;
(b) $m_V(y) \lesssim (1 + |x-y| m_V(x))^{k_0} m_V(x)$;
(c) $m_V(y) \ge C m_V(x)/\{1 + |x-y| m_V(x)\}^{k_0/(k_0+1)}$.

Lemma 2.6 (Lemma 1.2 in [1]) *Suppose that $V \in B_s$, $s > \frac{n}{2}$. There exists a constant C such that for $0 < r < R < \infty$,*

$$\frac{1}{r^{n-2}}\int_{B(x,r)} V(y)\,dy \lesssim \left(\frac{R}{r}\right)^{\frac{n}{s}-2} \cdot \frac{1}{R^{n-2}}\int_{B(x,R)} V(y)\,dy.$$

Lemma 2.7 (Lemma 2.3 in [2]) *Suppose $V \in B_s$, $s > \frac{n}{2}$. Then, for any $N > \log_2 C_0 + 1$, there exists a constant C_N such that for any $x \in \mathbb{R}^n$ and $r > 0$,*

$$\frac{1}{(1 + r m_V(x))^N}\int_{B(x,r)} V(y)\,dy \lesssim C_N r^{n-2}.$$

2.2 Generalized Morrey spaces associated with *L*

Suppose that $V \in B_s$, $s > 1$. Let $L = -\Delta + V$ be the Schrödinger operator. Now we introduce a class of generalized Morrey spaces associated with L. For $k \in \mathbb{Z}$, let $E_k = B(x_0, 2^k r)\backslash B(x_0, 2^{k-1} r)$ and χ_k be the characteristic function of E_k.

Definition 2.8 Suppose that $V \in B_s$, $s > 1$. Let $p \in [1, +\infty)$, $q \in [1, +\infty)$, $\alpha \in (-\infty, +\infty)$ and $\lambda \in (0, n)$, $\theta \in (-\infty, +\infty)$. For $f \in L^q_{\mathrm{loc}}(\mathbb{R}^n)$, we say $f \in L^{p,q,\lambda}_{\alpha,\theta,V}(\mathbb{R}^n)$ provided that

$$\|f\|^q_{L^{p,q,\lambda}_{\alpha,\theta,V}(\mathbb{R}^n)} = \sup_{B(x_0,r)\subset\mathbb{R}^n} \frac{(1 + rm_V(x_0))^\alpha}{r^{\lambda n}} \sum_{k=-\infty}^{0} |E_k|^{\theta q} \|\chi_k f\|^q_{L^p(\mathbb{R}^n)} < \infty,$$

where $B(x_0, r)$ denotes a ball centered at x_0 and with radius r.

Proposition 2.9
(i) *For* $\alpha_1 > \alpha_2$, $L^{p,q,\lambda}_{\alpha_1,\theta,V}(\mathbb{R}^n) \subseteq L^{p,\lambda,q}_{\alpha_2,\theta,V}(\mathbb{R}^n)$.
(ii) *If* $\theta = 0$, $p = q$ *and* $\alpha < 0$, $L^{p,\lambda}(\mathbb{R}^n) \subset L^{p,q,\lambda}_{\alpha,\theta,V}(\mathbb{R}^n)$.
(iii) *If* $\theta = 0$, $p = q$ *and* $\alpha > 0$, $L^{p,q,\lambda}_{\alpha,\theta,V}(\mathbb{R}^n) \subset L^{p,\lambda}(\mathbb{R}^n)$.

2.3 Calderón-Zygmund operators

We say that an operator T taking $C^\infty_c(\mathbb{R}^n)$ into $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ is called a Calderón-Zygmund operator if
(a) T extends to a bounded linear operator on $L^2(\mathbb{R}^n)$;
(b) there exists a kernel K such that for every $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$,

$$Tf(x) = \int_{\mathbb{R}^n} K(x,y)f(y)\,dy \quad \text{a.e. on } \{\operatorname{supp} f\}^c;$$

(c) the kernel $K(x, y)$ satisfies the Calderón-Zygmund estimate

$$\bigl|K(x,y)\bigr| \le \frac{C}{|x-y|^n};$$
$$\bigl|K(x+h,y) - K(x,y)\bigr| + \bigl|K(x,y+h) - K(x,y)\bigr| \le \frac{C|h|^\delta}{|x-y|^{n+\delta}}$$

for $x, y \in \mathbb{R}^n$, $|h| < \frac{|x-y|}{2}$ and for some $\delta > 0$.

Shen [1] obtained the following result.

Theorem 2.10 (Theorem 0.8 in [1]) *Suppose* $V \in B_n$. *Then*

$$\nabla(-\Delta + V)^{-1}\nabla, \qquad \nabla(-\Delta + V)^{-\frac{1}{2}} \quad \textit{and} \quad (-\Delta + V)^{-\frac{1}{2}}\nabla$$

are Calderón-Zygmund operators.

Corollary 2.11 *Suppose that* $V \in B_n$ *and* $b \in BMO(\mathbb{R}^n)$. *The commutator* $[b, T]$ *is bounded on* $L^p(\mathbb{R}^n)$.

In particular, let K denote the kernel of one of the above operators. Then K satisfies the following estimate:

$$\bigl|K(x,y)\bigr| \le \frac{C_N}{(1 + |x-y|m_V(x))^N} \frac{1}{|x-y|^n} \tag{2.2}$$

for any $N \in \mathbb{N}$. See (6.5) of Shen [1] for details.

Suppose $V \in B_s$ for $s \geq \frac{n}{2}$. Let $L = -\Delta + V$. The semigroup generated by L is defined as

$$T_t f(x) = e^{-tL} f(x) = \int_{\mathbb{R}^n} K_t(x,y) f(y)\, dy, \quad f \in L^2(\mathbb{R}^n), t > 0, \tag{2.3}$$

where K_t is the kernel of e^{-tL}.

Lemma 2.12 ([18]) *Let $K_t(x,y)$ be as in* (2.3). *For every nonnegative integer k, there is a constant C_k such that*

$$0 \leq K_t(x,y) \leq C_k t^{-\frac{n}{2}} \exp\bigl(-|x-y|^2/5t\bigr)\bigl(1 + \sqrt{t} m_V(x) + \sqrt{t} m_V(y)\bigr)^{-k}.$$

Some notations Throughout the paper, c and C will denote unspecified positive constants, possibly different at each occurrence. The constants are independent of the functions. $\mathsf{U} \approx \mathsf{V}$ represents that there is a constant $c > 0$ such that $c^{-1}\mathsf{V} \leq \mathsf{U} \leq c\mathsf{V}$ whose right inequality is also written as $\mathsf{U} \lesssim \mathsf{V}$. Similarly, if $\mathsf{V} \geq c\mathsf{U}$, we denote $\mathsf{V} \gtrsim \mathsf{U}$.

3 Riesz transforms and the commutators on $L^{p,q,\lambda}_{\alpha,\theta,V}(\mathbb{R}^n)$

Throughout this paper, for $p \in (1,\infty)$, denote by p' the conjugate of p, that is, $\frac{1}{p} + \frac{1}{p'} = 1$. Let $V \in B_n$. In this section, we assume that T is one of the Schrödinger type operators $\nabla(-\Delta + V)^{-1}\nabla$, $\nabla(-\Delta + V)^{-1/2}$ and $(-\Delta + V)^{-1/2}\nabla$. We study the boundedness on $L^{p,q,\lambda}_{\alpha,\theta,V}(\mathbb{R}^n)$ of T and its commutator $[b,T]$ with $b \in BMO(\mathbb{R}^n)$. The bounded mean oscillation space $BMO(\mathbb{R}^n)$ is defined as follows.

Definition 3.1 A locally integrable function b is said to belong to $BMO(\mathbb{R}^n)$ if

$$\|b\|_{BMO} =: \sup_B \frac{1}{|B|} \int_B \bigl|b(x) - b_B\bigr|\, dx < \infty,$$

where the supremum is taken over all balls B in $\mathbb{R}^n$. Here $b_B = \frac{1}{|B|}\int_B b(x)\, dx$ stands for the mean value of b over the ball B and $|B|$ means the measure of B.

We first prove that T is bounded on $L^{p,q,\lambda}_{\alpha,\theta,V}(\mathbb{R}^n)$.

Theorem 3.2 *Suppose that $\alpha \in (-\infty, 0]$, $\lambda \in (0,n)$ and $1 < q < \infty$. If $1 < p < \infty$, $\frac{\lambda}{q} - \frac{1}{p} < \theta < \frac{\lambda}{q} + 1 - \frac{1}{p}$, then the operators T are bounded on $L^{p,q,\lambda}_{\alpha,\theta,V}(\mathbb{R}^n)$.*

Proof For any ball $B(x_0,r)$, write

$$f(y) = \sum_{j=-\infty}^{\infty} f(y)\chi_j(y) = \sum_{j=-\infty}^{\infty} f_j(y),$$

where $E_j = B(x_0, 2^j r)\backslash B(x_0, 2^{j-1} r)$. Hence, we have

$$\begin{aligned}
&\bigl(1 + r m_V(x_0)\bigr)^{\alpha} r^{\lambda n} \sum_{k=-\infty}^{0} |E_k|^{\theta q} \|\chi_k Tf\|^q_{L^p(\mathbb{R}^n)} \\
&\quad \lesssim \bigl(1 + r m_V(x_0)\bigr)^{\alpha} r^{-\lambda n} \sum_{k=-\infty}^{0} |E_k|^{\theta q} \Biggl(\sum_{j=-\infty}^{k-2} \|\chi_k T f_j\|_{L^p(\mathbb{R}^n)}\Biggr)^q
\end{aligned}$$

$$
\begin{aligned}
&\quad + \bigl(1 + rm_V(x_0)\bigr)^{\alpha} r^{-\lambda n} \sum_{k=-\infty}^{0} |E_k|^{\theta q} \Biggl(\sum_{j=k-1}^{k+1} \|\chi_k Tf_j\|_{L^p(\mathbb{R}^n)} \Biggr)^q \\
&\quad + \bigl(1 + rm_V(x_0)\bigr)^{\alpha} r^{-\lambda n} \sum_{k=-\infty}^{0} |E_k|^{\theta q} \Biggl(\sum_{j=k+2}^{\infty} \|\chi_k Tf_j\|_{L^p(\mathbb{R}^n)} \Biggr)^q \\
&= A_1 + A_2 + A_3.
\end{aligned}
$$

For A_2, by Theorem 2.10, we have

$$
\begin{aligned}
A_2 &\lesssim \bigl(1 + rm_V(x_0)\bigr)^{\alpha} r^{-\lambda n} \sum_{k=-\infty}^{0} |E_k|^{\theta q} \Biggl(\sum_{j=k-1}^{k+1} \|Tf_j\|_{L^p(\mathbb{R}^n)} \Biggr)^q \\
&\lesssim \bigl(1 + rm_V(x_0)\bigr)^{\alpha} r^{-\lambda n} \sum_{k=-\infty}^{0} |E_k|^{\theta q} \Biggl(\sum_{j=k-1}^{k+1} \|f_j\|_{L^p(\mathbb{R}^n)} \Biggr)^q \\
&\lesssim \|f\|^q_{L^{p,q,\lambda}_{\alpha,\theta,V}}.
\end{aligned}
$$

We first estimate the term E_1. Note that if $x \in E_k$, $y \in E_j$ and $j \le k-2$, then $|x-y| \sim 2^k r$. By Lemma 2.5 and (2.2), we can get

$$
\begin{aligned}
\|\chi_k Tf_j\|_{L^p(\mathbb{R}^n)} &\lesssim \biggl(\int_{E_k} \biggl| \int_{\mathbb{R}^n} \frac{1}{(1 + |x-y| m_V(x))^N} \frac{1}{|x-y|^n} \bigl|f_j(y)\bigr|\,dy \biggr|^p dx \biggr)^{\frac{1}{p}} \\
&\lesssim \frac{1}{(1 + 2^k rm_V(x_0))^{N/k_0+1}} \frac{1}{(2^k r)^n} |E_k|^{\frac{1}{p}} \int_{E_j} \bigl|f(y)\bigr|\,dy \\
&\lesssim \frac{1}{(1 + 2^k rm_V(x_0))^{N/k_0+1}} |E_k|^{\frac{1}{p}-1} |E_j|^{\frac{1}{p'}} \biggl(\int_{E_j} \bigl|f(y)\bigr|^p\,dy \biggr)^{\frac{1}{p}},
\end{aligned}
$$

where $\frac{1}{p} + \frac{1}{p'} = 1$. Since $-\frac{1}{p} + \frac{\lambda}{q} < \theta < (1 - \frac{1}{p}) + \frac{\lambda}{q}$, we obtain

$$
\begin{aligned}
A_1 &\lesssim \bigl(1 + rm_V(x_0)\bigr)^{\alpha} r^{-\lambda n} \sum_{k=-\infty}^{0} |E_k|^{\theta q} \Biggl(\sum_{j=-\infty}^{k-2} \frac{|E_k|^{\frac{1}{p}-1} |E_j|^{\frac{1}{p'}} \|\chi_j f\|_{L^p(\mathbb{R}^n)}}{(1 + 2^k rm_V(x_0))^{N/k_0+1}} \Biggr)^q \\
&\lesssim \bigl(1 + rm_V(x_0)\bigr)^{\alpha} r^{-\lambda n} \sum_{k=-\infty}^{0} |E_k|^{\theta q} \Biggl(\sum_{j=-\infty}^{k-2} \frac{2^{\frac{n(j-k)}{p'}} (1 + 2^j rm_V(x_0))^{-\frac{\alpha}{q}}}{(1 + 2^k rm_V(x_0))^{N/k_0+1}} \\
&\quad \times \bigl(2^j r\bigr)^{\frac{\lambda n}{q}} |E_j|^{-\theta} \bigl(1 + 2^j rm_V(x_0)\bigr)^{\frac{\alpha}{q}} \bigl(2^j r\bigr)^{-\frac{\lambda n}{q}} \bigl(|E_j|^{\theta q} \|\chi_j f\|^q_{L^p(\mathbb{R}^n)}\bigr)^{\frac{1}{q}} \Biggr)^q \\
&\lesssim \bigl(1 + rm_V(x_0)\bigr)^{\alpha} r^{-\lambda n} \sum_{k=-\infty}^{0} |E_k|^{\lambda} \Biggl(\sum_{j=-\infty}^{k-2} 2^{\frac{n(j-k)}{p'}} |E_k|^{\theta - \frac{\lambda}{q}} |E_j|^{\frac{\lambda}{q} - \theta} \Biggr)^q \|f\|^q_{L^{p,q,\lambda}_{\alpha,\theta,V}(\mathbb{R}^n)} \\
&\lesssim \bigl(1 + rm_V(x_0)\bigr)^{\alpha} r^{-\lambda n} \sum_{k=-\infty}^{0} |E_k|^{\lambda} \Biggl(\sum_{j=-\infty}^{k-2} 2^{(j-k)n(1 - \frac{1}{p} + \frac{\lambda}{q} - \theta)} \Biggr)^q \|f\|^q_{L^{p,q,\lambda}_{\alpha,\theta,V}(\mathbb{R}^n)} \\
&\lesssim \|f\|^q_{L^{p,q,\lambda}_{\alpha,\theta,V}(\mathbb{R}^n)}.
\end{aligned}
$$

For A_3, we can see that when $x \in E_k$, $y \in E_j$, then $|x-y| \sim 2^j r$ for $j \geq k+2$. Similar to E_1, we have

$$\begin{aligned}
\|\chi_k Tf_j\|_{L^p(\mathbb{R}^n)} &\lesssim \frac{1}{(1+2^j rm_V(x_0))^{N/k_0+1}} \frac{1}{(2^j r)^n} |E_k|^{\frac{1}{p}} \int_{E_j} |f(y)|\,dy \\
&\lesssim \frac{1}{(1+2^j rm_V(x_0))^{N/k_0+1}} \frac{1}{(2^j r)^n} |E_k|^{\frac{1}{p}} |E_j|^{\frac{1}{p'}} \left(\int_{E_j} |f(y)|^p\,dy\right)^{\frac{1}{p}} \\
&\lesssim \frac{1}{(1+2^j rm_V(x_0))^{N/k_0+1}} |E_k|^{\frac{1}{p}} |E_j|^{-\frac{1}{p}} \|\chi_j f\|_{L^p(\mathbb{R}^n)}.
\end{aligned}$$

Since $-\frac{1}{p}+\frac{\lambda}{q} < \theta < (1-\frac{1}{p})+\frac{\lambda}{q}$, choosing N large enough, we obtain

$$\begin{aligned}
A_3 &\lesssim \left(1+rm_V(x_0)\right)^{\alpha} r^{-\lambda n} \sum_{k=-\infty}^{0} |E_k|^{\theta q} \left(\sum_{j=k+2}^{\infty} \frac{|E_k|^{\frac{1}{p}} |E_j|^{-\frac{1}{p}} \|\chi_j f\|_{L^p(\mathbb{R}^n)}}{(1+2^j rm_V(x_0))^{N/k_0+1}}\right)^q \\
&\lesssim \left(1+rm_V(x_0)\right)^{\alpha} r^{-\lambda n} \sum_{k=-\infty}^{0} |E_k|^{\theta q} \\
&\quad \times \left\{\sum_{j=k+2}^{\infty} \frac{(1+2^j rm_V(x_0))^{-\frac{\alpha}{q}} (2^j r)^{\frac{\lambda n}{q}} |E_j|^{-\alpha}}{(1+2^j rm_V(x_0))^{N/k_0+1}}\right. \\
&\quad \left. \times 2^{(k-j)\frac{n}{p}} \left(1+2^j rm_V(x_0)\right)^{\frac{\alpha}{q}} \left(2^j r\right)^{-\frac{\lambda n}{q}} \left(|E_j|^{\theta q} \|\chi_j f\|^q_{L^p(\mathbb{R}^n)}\right)^{\frac{1}{q}}\right\}^q \\
&\lesssim \left(1+rm_V(x_0)\right)^{\alpha} r^{-\lambda n} \sum_{k=-\infty}^{0} |E_k|^{\theta q} \left(\sum_{j=k+2}^{\infty} 2^{(k-j)\frac{n}{p}} |E_j|^{\frac{\lambda}{q}-\theta}\right)^q \|f\|^q_{L^{p,q,\lambda}_{\alpha,\theta,V}(\mathbb{R}^n)} \\
&\lesssim \|f\|^q_{L^{p,q,\lambda}_{\alpha,\theta,V}(\mathbb{R}^n)}.
\end{aligned}$$

Let $N = [-\frac{\alpha}{q}+1](k_0+1)$. Finally, $\|Tf\|_{L^{p,q,\lambda}_{\alpha,\theta,V}(\mathbb{R}^n)} \lesssim \|f\|_{L^{p,q,\lambda}_{\alpha,\theta,V}(\mathbb{R}^n)}$. This completes the proof of Theorem 3.2. □

Suppose that $b \in BMO(\mathbb{R}^n)$ and $V \in B_n$. Let T be one of the Schrödinger type operators $\nabla(-\Delta+V)^{-1}\nabla$, $\nabla(-\Delta+V)^{-1/2}$ and $(-\Delta+V)^{-1/2}\nabla$. The commutator $[b,T]$ is defined as

$$[b,T]f = bT(f) - T(bf).$$

Theorem 3.3 *Suppose that $V \in B_n$ and $b \in BMO(\mathbb{R}^n)$. Let $1 < p < \infty$, $1 < q < \infty$, $\alpha \in (-\infty, 0]$, $\lambda \in (0,n)$. If the index (p,q,θ,λ) satisfies $\frac{\lambda}{q}-\frac{1}{p} < \theta < \frac{\lambda}{q}+1-\frac{1}{p}$, then*

$$\left\|[b,T]f\right\|_{L^{p,q,\lambda}_{\alpha,\theta,V}} \leq C\|f\|_{L^{p,q,\lambda}_{\alpha,\theta,V}} \|b\|_{BMO}.$$

Proof For any ball $B = B(x_0,r)$, we can get

$$f(y) = \sum_{j=-\infty}^{\infty} f(y)\chi_{E_j}(y) = \sum_{j=-\infty}^{\infty} f_j(y),$$

where $E_j = B(x_0, 2^j r)\backslash B(x_0, 2^{j-1}r)$. Hence, we have

$$\begin{aligned}
&\bigl(1+rm_V(x_0)\bigr)^{\alpha} r^{-\lambda n} \sum_{k=-\infty}^{0} |E_k|^{\theta q} \bigl\| \chi_k [b,T]f \bigr\|_{L^p(\mathbb{R}^n)}^q \\
&\quad\lesssim \bigl(1+rm_V(x_0)\bigr)^{\alpha} r^{-\lambda n} \sum_{k=-\infty}^{0} |E_k|^{\theta q} \Biggl(\sum_{j=-\infty}^{k-2} \bigl\| \chi_k [b,T]f_j \bigr\|_{L^p(\mathbb{R}^n)} \Biggr)^q \\
&\qquad+ \bigl(1+rm_V(x_0)\bigr)^{\alpha} r^{-\lambda n} \sum_{k=-\infty}^{0} |E_k|^{\theta q} \Biggl(\sum_{j=k-1}^{k+1} \bigl\| \chi_k [b,T]f_j \bigr\|_{L^p(\mathbb{R}^n)} \Biggr)^q \\
&\qquad+ \bigl(1+rm_V(x_0)\bigr)^{\alpha} r^{-\lambda n} \sum_{k=-\infty}^{0} |E_k|^{\theta q} \Biggl(\sum_{j=k+2}^{\infty} \bigl\| \chi_k [b,T]f_j \bigr\|_{L^p(\mathbb{R}^n)} \Biggr)^q \\
&\quad=: B_1 + B_2 + B_3.
\end{aligned}$$

For B_2, by Corollary 2.11, we have

$$\begin{aligned}
B_2 &\lesssim \bigl(1+rm_V(x_0)\bigr)^{\alpha} r^{-\lambda n} \sum_{k=-\infty}^{0} |E_k|^{\theta q} \Biggl(\sum_{j=k-1}^{k+1} \bigl\| [b,T]f_j \bigr\|_{L^p(\mathbb{R}^n)} \Biggr)^q \\
&\lesssim \bigl(1+rm_V(x_0)\bigr)^{\alpha} r^{-\lambda n} \sum_{k=-\infty}^{0} |E_k|^{\theta q} \Biggl(\sum_{j=k-1}^{k+1} \|f_j\|_{L^p(\mathbb{R}^n)} \Biggr)^q \|b\|_{BMO}^q \\
&\lesssim \|f\|_{L^{p,q,\lambda}_{\alpha,\theta,V}}^q \|b\|_{BMO}^q.
\end{aligned}$$

Denote by $b_{2^k r}$ the mean value of b on the ball $B(x_0, 2^k r)$. For B_1, by Lemma 2.5 and (2.2), we have

$$\begin{aligned}
&\bigl\| \chi_k [b,T]f_j \bigr\|_{L^p(\mathbb{R}^n)} \\
&\quad\lesssim \frac{1}{(1+2^k rm_V(x_0))^{N/k_0+1}} \frac{1}{(2^k r)^n} \\
&\qquad\times \biggl[\int_{E_k} \biggl(\int_{E_j} \bigl|b(x)-b(y)\bigr| \bigl|f(y)\bigr|\,dy \biggr)^p dx \biggr]^{\frac{1}{p}} \\
&\quad\lesssim \frac{1}{(1+2^k rm_V(x_0))^{N/k_0+1}} \frac{1}{(2^k r)^n} \biggl[\biggl(\int_{E_k} \bigl|b(x)-b_{2^k r}\bigr|^p\,dx \biggr)^{\frac{1}{p}} \int_{E_j} \bigl|f(y)\bigr|\,dy \\
&\qquad+ |E_k|^{\frac{1}{p}} \int_{E_j} \bigl|b(y)-b_{2^k r}\bigr| \bigl|f(y)\bigr|\,dy \biggr] \\
&\quad\lesssim \frac{1}{(1+2^k rm_V(x_0))^{N/k_0+1}} \frac{1}{(2^k r)^n} \biggl[|E_k|^{\frac{1}{p}} |E_j|^{1-\frac{1}{p}} \|b\|_{BMO} \|f_j\|_{L^p(\mathbb{R}^n)} \\
&\qquad+ |E_k|^{\frac{1}{p}} \|f_j\|_{L^p(\mathbb{R}^n)} \biggl(\int_{E_j} \bigl|b(y)-b_{2^k r}\bigr|^{p'}\,dx \biggr)^{\frac{1}{p'}} \biggr] \\
&\quad\lesssim \frac{1}{(1+2^k rm_V(x_0))^{N/k_0+1}} \frac{|E_j|^{1-\frac{1}{p}}}{|E_k|^{1-\frac{1}{p}}} (k-j) \|f_j\|_{L^p(\mathbb{R}^n)} \|b\|_{BMO},
\end{aligned}$$

where in the third inequality, we have used John-Nirenberg's inequality [19]. Since $-\frac{1}{p}+\frac{\lambda}{q}<\theta<(1-\frac{1}{p})+\frac{\lambda}{q}$, we obtain

$$\begin{aligned}
B_1 &\lesssim \frac{(1+rm_V(x_0))^\alpha}{r^{\lambda n}}\sum_{k=-\infty}^{0}|E_k|^{\theta q}\left(\sum_{j=-\infty}^{k-2}\frac{(k-j)\|f_j\|_{L^p(\mathbb{R}^n)}}{(1+2^k rm_V(x_0))^{N/k_0+1}}\frac{|E_j|^{1-\frac{1}{p}}}{|E_k|^{1-\frac{1}{p}}}\right)^q\|b\|_{BMO}^q \\
&\lesssim \frac{(1+rm_V(x_0))^\alpha}{r^{\lambda n}}\sum_{k=-\infty}^{0}|E_k|^{\theta q}\Bigg[\sum_{j=-\infty}^{k-2}\frac{(1+2^j rm_V(x_0))^{-\frac{\alpha}{q}}}{(1+2^k rm_V(x_0))^{N/k_0+1}}\|b\|_{BMO}^q \\
&\quad\times (k-j)\big(2^j r\big)^{\frac{\lambda n}{q}}|E_j|^{-\theta}\frac{|E_j|^{1-\frac{1}{p}}}{|E_k|^{1-\frac{1}{p}}}\Bigg]^q\|f\|_{L^{p,q,\lambda}_{\alpha,\theta,V}}^q \\
&\lesssim \frac{(1+rm_V(x_0))^\alpha}{r^{\lambda n}}\sum_{k=-\infty}^{0}|E_k|^{\lambda}\left(\sum_{j=-\infty}^{k-2}(k-j)2^{(k-j)n(\theta-\frac{\alpha}{q}+\frac{1}{p}-1)}\right)^q\|f\|_{L^{p,q,\lambda}_{\alpha,\theta,V}}^q\|b\|_{BMO}^q \\
&\lesssim \|f\|_{L^{p,q,\lambda}_{\alpha,\theta,V}}^q\|b\|_{BMO}^q.
\end{aligned}$$

For B_3, similar to B_1, we have

$$\begin{aligned}
&\big\|\chi_k[b,T]f_j\big\|_{L^p(\mathbb{R}^n)} \\
&\quad\lesssim \frac{1}{(1+2^j rm_V(x_0))^{N/k_0+1}}\frac{1}{(2^j r)^n}\left(\int_{E_k}\left|\int_{E_j}\big|\big(b(x)-b(y)\big)f(y)\big|\,dy\right|^p dx\right)^{\frac{1}{p}} \\
&\quad\lesssim \frac{j-k}{(1+2^j rm_V(x_0))^{N/k_0+1}}|E_k|^{\frac{1}{p}}|E_j|^{-\frac{1}{p}}\|f_j\|_{L^p(\mathbb{R}^n)}\|b\|_{BMO}.
\end{aligned}$$

Since $-\frac{1}{p}+\frac{\lambda}{q}<\theta<(1-\frac{1}{p})+\frac{\lambda}{q}$, choosing N large enough, we obtain

$$\begin{aligned}
B_3 &\lesssim \big(1+rm_V(x_0)\big)^\alpha r^{-\lambda n}\sum_{k=-\infty}^{0}|E_k|^{\theta q}\left(\sum_{j=k+2}^{\infty}\frac{|E_k|^{\frac{1}{p}}|E_j|^{-\frac{1}{p}}(j-k)\|f_j\|_{L^p(\mathbb{R}^n)}}{(1+2^j rm_V(x_0))^{N/k_0+1}}\right)^q\|b\|_{BMO}^q \\
&\lesssim \big(1+rm_V(x_0)\big)^\alpha r^{-\lambda n}\sum_{k=-\infty}^{0}|E_k|^{\theta q}\Bigg[\sum_{j=k+2}^{\infty}\frac{(1+2^j rm_V(x_0))^{-\frac{\alpha}{q}}}{(1+2^j rm_V(x_0))^{N/k_0+1}} \\
&\quad\times (j-k)\big(2^j r\big)^{\frac{\lambda n}{q}}|E_j|^{-\theta}|E_k|^{\frac{1}{p}}|E_j|^{-\frac{1}{p}}\Bigg]^q\|f\|_{L^{p,q,\lambda}_{\alpha,\theta,V}(\mathbb{R}^n)}^q\|b\|_{BMO}^q \\
&\lesssim \big(1+rm_V(x_0)\big)^\alpha r^{-\lambda n}\sum_{k=-\infty}^{0}|E_k|^{\lambda}\left(\sum_{j=k+2}^{\infty}2^{(k-j)n(\frac{1}{p}-\frac{\lambda}{q}+\theta)}\right)^q\|f\|_{L^{p,q,\lambda}_{\alpha,\theta,V}(\mathbb{R}^n)}^q\|b\|_{BMO}^q \\
&\lesssim \|f\|_{L^{p,q,\lambda}_{\alpha,\theta,V}(\mathbb{R}^n)}^q\|b\|_{BMO}^q.
\end{aligned}$$

Let $N=[-\frac{\alpha}{q}+1](k_0+1)$. We finally get

$$\big\|[b,T]f\big\|_{L^{p,q,\lambda}_{\alpha,\theta,V}(\mathbb{R}^n)}\lesssim \|f\|_{L^{p,q,\lambda}_{\alpha,\theta,V}(\mathbb{R}^n)}\|b\|_{BMO}.$$

□

4 Schrödinger type operators on $L^{p,q,\lambda}_{\alpha,\theta,V}(\mathbb{R}^n)$

Let $L = -\Delta + V$ be the Schrödinger operator, where $V \in B_s$, $s > n/2$. For $0 < \beta < \frac{n}{2}$, the fractional integral operator associated with L is defined by

$$L^{-\beta}(f)(x) = \int_0^\infty e^{-tL}(f)(x)t^{\beta-1}\,dt.$$

Denote by $K_\beta(x,y)$ the kernel of $L^{-\beta}$. By Lemma 2.12, Bui [20] obtained the following pointwise estimate.

Lemma 4.1 (Proposition 3.3 in [20]) *Let $0 < \beta < \frac{n}{2}$. For $N \in \mathbb{N}$, there is a constant C_N such that*

$$\begin{aligned} K_\beta(x,y) &= \int_0^\infty K_t(x,y)t^{\beta-1}\,dt \\ &\leq \frac{C_N}{(1+|x-y|m_V(x))^N}\frac{1}{|x-y|^{n-2\beta}}, \end{aligned} \tag{4.1}$$

where $K_t(\cdot,\cdot)$ is the kernel of the semigroup e^{-tL}.

Definition 4.2 Let $f \in L^q_{\mathrm{loc}}(\mathbb{R}^n)$. Denote by $|B|$ the Lebesgue measure of the ball $B \subset \mathbb{R}^n$. The fractional Hardy-Littlewood maximal function $M_{\sigma,\gamma}$ is defined by

$$M_{\sigma,\gamma}f(x) = \sup_{x\in B}\left(\frac{1}{|B|^{1-\frac{\sigma\gamma}{n}}}\int_B \left|f(y)\right|^\gamma dy\right)^{\frac{1}{\gamma}}.$$

Lemma 4.3 ([16]) *Suppose $1 < \gamma < p_1 < \frac{n}{\sigma}$ and $\frac{1}{p_2} = \frac{1}{p_1} - \frac{\sigma}{n}$. Then*

$$\|M_{\sigma,\gamma}f\|_{L^{p_2}(\mathbb{R}^n)} \lesssim \|f\|_{L^{p_1}(\mathbb{R}^n)}.$$

As a generalization of the fractional integral associated with L, the operators $V^{\beta_2}(-\Delta + V)^{-\beta_1}$, $0 \leq \beta_2 \leq \beta_1 \leq 1$, have been studied by Sugano [9] systematically. Applying the method of Sugano [9] together with Lemma 4.1, we can obtain the following result for $V^{\beta_2}(-\Delta + V)^{-\beta_1}$, $0 \leq \beta_2 \leq \beta_1 \leq n/2$. We omit the proof.

Theorem 4.4 *Suppose that $V \in B_\infty$. Let $1 < \beta_2 \leq \beta_1 < \frac{n}{2}$. Then*

$$\left|V^{\beta_2}(-\Delta + V)^{-\beta_1}f(x)\right| \lesssim M_{2(\beta_1-\beta_2),1}f(x).$$

In a similar way, by (4.1), we can get the following estimate for the operators $(-\Delta + V)^{-\beta_1}V^{\beta_2}$, $0 \leq \beta_2 \leq \beta_1 < \frac{n}{2}$.

Theorem 4.5 *Suppose that $V \in B_s$ for $s > \frac{n}{2}$. Let $0 \leq \beta_2 \leq \beta_1 < \frac{n}{2}$. Then*

$$\left|(-\Delta + V)^{-\beta_1}\left(V^{\beta_2}f\right)(x)\right| \lesssim M_{2(\beta_1-\beta_2)}(f)(x),$$

where $(\frac{s}{\beta_2})'$ is the conjugate of $(\frac{s}{\beta_2})$.

Proof Let $r = 1/m_V(x)$. By Lemma 4.1 and Hölder's inequality, we have

$$\begin{aligned}&\big|(-\Delta+V)^{-\beta_1}V^{\beta_2}(x)f(x)\big|\\&\quad\lesssim\sum_{k=-\infty}^{\infty}\int_{2^{k-1}r\le|x-y|\le 2^k r}\frac{1}{(1+2^k r m_V(x_0))^N}\frac{1}{(2^k r)^{n-2\beta_1}}V(y)^{\beta_2}\big|f(y)\big|\,dy\\&\quad\lesssim\sum_{k=-\infty}^{\infty}\frac{(2^k r)^{2\beta_2}}{(1+2^k)^N}\bigg(\frac{1}{(2^k r)^n}\int_{B(x,2^k r)}V(y)\,dy\bigg)^{\beta_2}M_{2(\beta_1-\beta_2),(\frac{s}{\beta_2})'}(f)(x).\end{aligned}$$

For $k\ge 1$, because $V(y)\,dy$ is a doubling measure, we have

$$\begin{aligned}\frac{(2^k r)^2}{(2^k r)^n}\int_{B(x,2^k r)}V(y)\,dy&\lesssim C_0^k\cdot 2^{(2-n)k}\frac{r^2}{r^n}\int_{B(x,r)}V(y)\,dy\\&\lesssim\big(2^k\big)^{k_0},\end{aligned}$$

where $k_0 = 2 - n + \log_2 C_0$. For $k \le 0$, Lemma 2.6 implies that

$$\begin{aligned}\frac{(2^k r)^2}{(2^k r)^n}\int_{B(x,2^k r)}V(y)\,dy&\lesssim\bigg(\frac{r}{2^k r}\bigg)^{\frac{n}{s}-2}\frac{r^2}{r^n}\int_{B(x,r)}V(y)\,dy\\&\lesssim\big(2^k\big)^{2-\frac{n}{s}}.\end{aligned}$$

Taking N large enough, we get

$$\big|(-\Delta+V)^{-\beta_1}V^{\beta_2}f(x)\big|\lesssim M_{2(\beta_1-\beta_2),(\frac{s}{\beta_2})'}f(x).\qquad\square$$

By Theorem 4.5 and the duality, we can obtain the following.

Corollary 4.6 *Suppose $V\in B_s$ for $s>\frac{n}{2}$.*

(1) *If $1<(\frac{s}{\beta_2})'<p_1<\frac{n}{2\beta_1-2\beta_2}$ and $\frac{1}{p_2}=\frac{1}{p_1}-\frac{2\beta_1-2\beta_2}{n}$, then*

$$\big\|(-\Delta+V)^{-\beta_1}V^{\beta_2}f\big\|_{L^{p_2}(\mathbb{R}^n)}\lesssim\|f\|_{L^{p_1}(\mathbb{R}^n)},$$

where $\frac{s}{\beta_2}+(\frac{s}{\beta_2})'=1$.

(2) *If $1<p_2<\frac{s}{\beta_2}$ and $\frac{1}{p_2}=\frac{1}{p_1}-\frac{2\beta_1-2\beta_2}{n}$, then*

$$\big\|V^{\beta_2}(-\Delta+V)^{-\beta_1}f\big\|_{L^{p_2}(\mathbb{R}^n)}\lesssim\|f\|_{L^{p_1}(\mathbb{R}^n)}.$$

Theorem 4.7 *Suppose that $V\in B_s$, $s\ge\frac{n}{2}$, $\alpha\in(-\infty,0]$, $\lambda\in(0,n)$. Let $1<q<\infty$, $1<\beta_2\le\beta_1<\frac{n}{2}$ and $1<p_2<\frac{s}{\beta_2}$ with $\frac{1}{p_1}-\frac{1}{p_2}=\frac{2\beta_1-2\beta_2}{n}$. If $\frac{\lambda}{q}-\frac{1}{p_1}+\frac{2\beta_1}{n}<\theta<\frac{\lambda}{q}+1-\frac{1}{p_1}$, then*

$$\big\|V^{\beta_2}(-\Delta+V)^{-\beta_1}f\big\|_{L^{p_2,q,\lambda}_{\alpha,\theta,V}}\lesssim\|f\|_{L^{p_1,q,\lambda}_{\alpha,\theta,V}}.$$

Proof For any ball $B(x_0,r)$, write

$$f(y)=\sum_{j=-\infty}^{\infty}f(y)\chi_{E_j}(y)=\sum_{j=-\infty}^{\infty}f_j(y),$$

where $E_j = B(x_0, 2^j r)\backslash B(x_0, 2^{j-1} r)$. Hence, we have

$$\begin{aligned}
&\big(1 + rm_V(x_0)\big)^{\alpha} r^{-\lambda n} \sum_{k=-\infty}^{0} |E_k|^{\theta q} \big\| \chi_k V^{\beta_2}(-\Delta + V)^{-\beta_1} f\big\|^q_{L^{p_2}(\mathbb{R}^n)} \\
&\lesssim \big(1 + rm_V(x_0)\big)^{\alpha} r^{-\lambda n} \sum_{k=-\infty}^{0} |E_k|^{\theta q} \Bigg(\sum_{j=-\infty}^{k-2} \big\| \chi_k V^{\beta_2}(-\Delta + V)^{-\beta_1} f_j\big\|_{L^{p_2}(\mathbb{R}^n)} \Bigg)^q \\
&\quad + \big(1 + rm_V(x_0)\big)^{\alpha} r^{-\lambda n} \sum_{k=-\infty}^{0} |E_k|^{\theta q} \Bigg(\sum_{j=k-1}^{k+1} \big\| \chi_k V^{\beta_2}(-\Delta + V)^{-\beta_1} f_j\big\|_{L^{p_2}(\mathbb{R}^n)} \Bigg)^q \\
&\quad + \big(1 + rm_V(x_0)\big)^{\alpha} r^{-\lambda n} \sum_{k=-\infty}^{0} |E_k|^{\theta q} \Bigg(\sum_{j=k+2}^{\infty} \big\| \chi_k V^{\beta_2}(-\Delta + V)^{-\beta_1} f_j\big\|_{L^{p_2}(\mathbb{R}^n)} \Bigg)^q \\
&= M_1 + M_2 + M_3.
\end{aligned}$$

We first estimate M_2. For $1 < p_2 < \frac{s}{\beta_2}$, by (2) of Corollary 4.6, we can get

$$M_2 \lesssim \frac{(1 + rm_V(x_0))^{\alpha}}{r^{\lambda n}} \sum_{k=-\infty}^{0} |E_k|^{\theta q} \Bigg(\sum_{j=k-1}^{k+1} \|f_j\|_{L^{p_1}(\mathbb{R}^n)} \Bigg)^q \lesssim \|f\|^q_{L^{p_1,q,\lambda}_{\alpha,\theta,V}}.$$

Now we deal with the terms M_1 and M_3. We choose N large enough such that

$$(N/k_0 + 1) - (\log_2 C_0 + 1)\beta_2 + \alpha/q > 0$$

and take positive $N_1 < (N/k_0 + 1) - (\log_2 C_0 + 1)\beta_2$. For M_1, note that if $x \in E_k$, $y \in E_j$ and $j \le k-2$, then $|x - y| \sim 2^k r$. By Lemmas 4.1 and 2.7, we use Hölder's inequality to obtain

$$\begin{aligned}
&\big\| \chi_k V^{\beta_2}(-\Delta + V)^{-\beta_1} f_j\big\|_{L^{p_2}(\mathbb{R}^n)} \\
&\lesssim \Bigg(\int_{E_k} \bigg| V^{\beta_2}(x) \int_{E_j} \frac{1}{(1 + |x - y| m_v(x))^N} \frac{1}{|x - y|^{n-2\beta_1}} f(y)\,dy \bigg|^{p_2} dx \Bigg)^{\frac{1}{p_2}} \\
&\lesssim \frac{1}{(1 + 2^k rm_V(x_0))^{N/k_0+1}} \frac{1}{(2^k r)^{n-2\beta_1}} \int_{E_j} \big|f(y)\big|\,dy \bigg(\int_{E_k} \big|V(x)\big|^{\beta_2 p_2}\,dx \bigg)^{\frac{1}{p_2}} \\
&\lesssim \frac{|E_j|^{1-\frac{1}{p_1}} |E_k|^{\frac{1}{p_2}}}{(1 + 2^k rm_V(x_0))^{N/k_0+1}} \frac{1}{(2^k r)^{n-2\beta_1}} \|f_j\|_{L^{p_1}(\mathbb{R}^n)} \bigg(\frac{1}{|E_k|} \int_{E_k} V(x)^s\,dx \bigg)^{\frac{\beta_2}{s}} \\
&\lesssim \frac{|E_j|^{1-\frac{1}{p_1}} |E_k|^{\frac{1}{p_2}}}{(1 + 2^k rm_V(x_0))^{N/k_0+1}} \frac{1}{(2^k r)^{n-2\beta_1}} \|f_j\|_{L^{p_1}(\mathbb{R}^n)} \bigg(\frac{1}{|B_k|} \int_{B_k} V(x)\,dx \bigg)^{\beta_2} \\
&\lesssim \frac{1}{(1 + 2^k rm_V(x_0))^{N_1}} \frac{1}{(2^k r)^{n-2\beta_1+2\beta_2}} |E_k|^{\frac{1}{p_2}} |E_j|^{1-\frac{1}{p_1}} \|f_j\|_{L^{p_1}(\mathbb{R}^n)},
\end{aligned}$$

where $\frac{1}{p_1} - \frac{1}{p_2} = \frac{2\beta_1 - 2\beta_2}{n}$. Since $\frac{\lambda}{q} - \frac{1}{p_1} + \frac{2\beta_1}{n} < \theta < \frac{\lambda}{q} + 1 - \frac{1}{p_1}$, we obtain

$$\begin{aligned}
M_1 &\lesssim \big(1 + rm_V(x_0)\big)^{\alpha} r^{-\lambda n} \sum_{k=-\infty}^{0} |E_k|^{\theta q} \\
&\quad \times \Bigg(\sum_{j=-\infty}^{k-2} \frac{1}{(1 + 2^k rm_V(x_0))^{N_1}} \frac{1}{(2^k r)^{n-2\beta_1+2\beta_2}} |E_k|^{\frac{1}{p_2}} |E_j|^{1-\frac{1}{p_1}} \|f_j\|_{L^{p_1}(\mathbb{R}^n)} \Bigg)^q
\end{aligned}$$

$$
\begin{aligned}
&\lesssim \left(1+rm_V(x_0)\right)^{\alpha} r^{-\lambda n} \sum_{k=-\infty}^{0} |E_k|^{\theta q} \\
&\quad \times \left(\sum_{j=-\infty}^{k-2} \frac{(1+2^j rm_V(x_0))^{-\frac{\alpha}{q}}}{(1+2^k rm_V(x_0))^{N_1}} \frac{(2^j r)^{\frac{\lambda n}{q}} |E_j|^{-\theta}}{(2^k r)^{n-2\beta_1+2\beta_2}} |E_k|^{\frac{1}{p_2}} |E_j|^{1-\frac{1}{p_1}} \right)^q \|f\|^q_{L^{p_1,\lambda,q}_{\alpha,v,\theta}} \\
&\lesssim \left(1+rm_V(x_0)\right)^{\alpha} r^{-\lambda n} \sum_{k=-\infty}^{0} |E_k|^{\lambda} \left(\sum_{j=-\infty}^{k-2} 2^{(j-k)n(\frac{\lambda}{q}-\theta-\frac{1}{p_1}+1)} \right)^q \|f\|^q_{L^{p_1,\lambda,q}_{\alpha,v,\theta}} \\
&\lesssim \|f\|^q_{L^{p_1,\lambda,q}_{\alpha,v,\theta}}.
\end{aligned}
$$

For M_3, note that when $x \in E_k$, $y \in E_j$ and $j \geq k+2$, then $|x-y| \sim 2^j r$. Similar to E_1, we have

$$
\begin{aligned}
&\left\| \chi_k V^{\beta_2}(-\Delta+V)^{-\beta_1} f_j \right\|_{L^{p_2}(\mathbb{R}^n)} \\
&\quad \lesssim \frac{1}{(1+2^j rm_V(x_0))^{N/k_0+1}} \frac{1}{(2^j r)^{n-2\beta_1}} \int_{E_j} \left|f(y)\right| dy \left(\int_{E_k} \left|V(x)\right|^{\beta_2 p_2} dx \right)^{\frac{1}{p_2}} \\
&\quad \lesssim \frac{1}{(1+2^j rm_V(x_0))^{N_1}} |E_j|^{\frac{2\beta_1}{n}-\frac{1}{p_1}} |E_k|^{\frac{1}{p_2}-\frac{2\beta_2}{n}} \|f_j\|_{L^{p_1}(\mathbb{R}^n)},
\end{aligned}
$$

where $\frac{1}{p_1}-\frac{1}{p_2}=\frac{2\beta_1-2\beta_2}{n}$. Since $\frac{\lambda}{q}-\frac{1}{p_1}+\frac{2\beta_1}{n}<\theta<\frac{\lambda}{q}+1-\frac{1}{p_1}$, we obtain

$$
\begin{aligned}
M_3 &\lesssim \left(1+rm_V(x_0)\right)^{\alpha} r^{-\lambda n} \sum_{k=-\infty}^{0} |E_k|^{\theta q} \\
&\quad \times \left(\sum_{j=k+2}^{\infty} \frac{1}{(1+2^j rm_V(x_0))^{N_1}} |E_j|^{\frac{2\beta_1}{n}-\frac{1}{p_1}} |E_k|^{\frac{1}{p_2}-\frac{2\beta_2}{n}} \|f_j\|_{L^{p_1}(\mathbb{R}^n)} \right)^q \\
&\lesssim \left(1+rm_V(x_0)\right)^{\alpha} r^{-\lambda n} \sum_{k=-\infty}^{0} |E_k|^{\theta q} \\
&\quad \times \left(\sum_{j=k+2}^{\infty} \frac{(1+2^j rm_V(x_0))^{-\frac{\alpha}{q}} (2^j r)^{\frac{\lambda n}{q}} |E_j|^{-\theta}}{(1+2^j rm_V(x_0))^{N_1}} \frac{|E_k|^{\frac{1}{p_2}-\frac{2\beta_2}{n}}}{|E_j|^{\frac{2\beta_1}{n}-\frac{1}{p_1}}} \right)^q \|f\|^q_{L^{p_1,q,\lambda}_{\alpha,\theta,V}} \\
&\lesssim \left(1+rm_V(x_0)\right)^{\alpha} r^{-\lambda n} \sum_{k=-\infty}^{0} |E_k|^{\lambda} \left(\sum_{j=k+2}^{\infty} 2^{(k-j)n(\theta-\frac{\lambda}{q}+\frac{1}{p_1}+\frac{2\beta_1}{n})} \right)^q \|f\|^q_{L^{p_1,q,\lambda}_{\alpha,\theta,V}} \\
&\lesssim \|f\|^q_{L^{p_1,q,\lambda}_{\alpha,\theta,V}}.
\end{aligned}
$$

Choosing N large enough, we obtain

$$
\left\| V^{\beta_2}(-\Delta+V)^{-\beta_1} f \right\|_{L^{p_2,q,\lambda}_{\alpha,\theta,V}} \lesssim \|f\|_{L^{p_1,q,\lambda}_{\alpha,\theta,V}}. \qquad \square
$$

Theorem 4.8 *Suppose that* $V \in B_s$, $s \geq \frac{n}{2}$, $\alpha \in (-\infty,0]$, $\lambda \in (0,n)$ *and* $1<q<\infty$. *Let* $0<\beta_2 \leq \beta_1 < \frac{n}{2}$, $\frac{s}{s-\beta_2}<p_1<\frac{n}{2\beta_1-2\beta_2}$ *with* $\frac{1}{p_2}=\frac{1}{p_1}-\frac{2\beta_1-2\beta_2}{n}$. *If* $\frac{\lambda}{q}-\frac{1}{p_2}<\theta<\frac{\lambda}{q}-\frac{1}{p_2}+1-\frac{2\beta_1}{n}$, *then*

$$
\left\| (-\Delta+V)^{-\beta_1} V^{\beta_2} f \right\|_{L^{p_2,q,\lambda}_{\alpha,\theta,V}} \lesssim \|f\|_{L^{p_1,q,\lambda}_{\alpha,\theta,V}}.
$$

Proof For any ball $B(x_0,r)$, let $E_j = B(x_0,2^j r)\backslash B(x_0,2^{j-1}r)$. We can decompose f as follows:

$$f(y) = \sum_{j=-\infty}^{\infty} f(y)\chi_{E_j}(y) = \sum_{j=-\infty}^{\infty} f_j(y).$$

Similar to the proof of Theorem 4.7, we have

$$\begin{aligned}
&\big(1+rm_V(x_0)\big)^{\alpha} r^{-\lambda n} \sum_{k=-\infty}^{0} |E_k|^{\theta q} \big\| \chi_k(-\Delta+V)^{-\beta_1}V^{\beta_2}f\big\|_{L^{p_2}(\mathbb{R}^n)}^q \\
&\quad\lesssim \big(1+rm_V(x_0)\big)^{\alpha} r^{-\lambda n} \sum_{k=-\infty}^{0} |E_k|^{\theta q} \left(\sum_{j=-\infty}^{k-2} \big\| \chi_k(-\Delta+V)^{-\beta_1}V^{\beta_2}f_j\big\|_{L^{p_2}(\mathbb{R}^n)}\right)^q \\
&\qquad + C\big(1+rm_V(x_0)\big)^{\alpha} r^{-\lambda n} \sum_{k=-\infty}^{0} |E_k|^{\theta q} \left(\sum_{j=k-1}^{k+1} \big\| \chi_k(-\Delta+V)^{-\beta_1}V^{\beta_2}f_j\big\|_{L^{p_2}(\mathbb{R}^n)}\right)^q \\
&\qquad + C\big(1+rm_V(x_0)\big)^{\alpha} r^{-\lambda n} \sum_{k=-\infty}^{0} |E_k|^{\theta q} \left(\sum_{j=k+2}^{\infty} \big\| \chi_k(-\Delta+V)^{-\beta_1}V^{\beta_2}f_j\big\|_{L^{p_2}(\mathbb{R}^n)}\right)^q \\
&\quad= L_1 + L_2 + L_3.
\end{aligned}$$

For L_2, because $1 < \frac{s}{s-\beta_2} < p_1 < \frac{n}{2\beta_1-\beta_2}$, we use Corollary 4.6 to obtain

$$L_2 \lesssim \frac{(1+rm_V(x_0))^{\alpha}}{r^{\lambda n}} \sum_{k=-\infty}^{0} |E_k|^{\theta q} \left(\sum_{j=k-1}^{k+1} \|f_j\|_{L^{p_1}(\mathbb{R}^n)}\right)^q \lesssim \|f\|_{L^{p_1,q,\lambda}_{\alpha,\theta,V}}^q.$$

For L_1, we can see that if $x \in E_k$ and $y \in E_j$, then $|x-y| \sim 2^k r$ for $j \le k-2$. By Hölder's inequality and the fact that $V \in B_s$, we deduce from Lemmas 4.1 and 2.7 that

$$\begin{aligned}
&\big\| \chi_k(-\Delta+V)^{-\beta_1}V^{\beta_2}f_j\big\|_{L^{p_2}(\mathbb{R}^n)}^q \\
&\quad\lesssim \frac{1}{(1+2^k rm_V(x_0))^{N/k_0+1}} \frac{|E_k|^{\frac{1}{p_2}}}{(2^k r)^{n-2\beta_1}} \int_{E_j} V(x)^{\beta_2}\big|f(y)\big|\,dy \\
&\quad\lesssim \frac{1}{(1+2^k rm_V(x_0))^{N/k_0+1}} \frac{|E_k|^{\frac{1}{p_2}}}{(2^k r)^{n-2\beta_1}} |E_j|^{1-\frac{1}{p_1}} \left(\frac{1}{|B_j|}\int_{B_j} V(x)\,dx\right)^{\beta_2} \|f_j\|_{L^{p_1}(\mathbb{R}^n)} \\
&\quad\lesssim \frac{1}{(1+2^k rm_V(x_0))^{N_2}} \frac{|E_k|^{\frac{1}{p_2}}}{(2^k r)^{n-2\beta_1}} |E_j|^{1-\frac{1}{p_1}} \big(2^j r\big)^{-2\beta_2} \|f_j\|_{L^{p_1}(\mathbb{R}^n)},
\end{aligned}$$

where $\frac{1}{p_2} = \frac{1}{p_1} - \frac{2\beta_1-2\beta_2}{n}$ and $N_2 < (N/k_0+1) - (\log_2 C_0 + 1)\beta_2$. Since $\frac{\lambda}{q} - \frac{1}{p_2} < \theta < \frac{\lambda}{q} - \frac{1}{p_2} + 1 - \frac{2\beta_1}{n}$, we obtain

$$\begin{aligned}
L_1 &\lesssim \big(1+rm_V(x_0)\big)^{\alpha} r^{-\lambda n} \sum_{k=-\infty}^{0} |E_k|^{\theta q} \\
&\quad\times \left(\sum_{j=-\infty}^{k-2} \frac{1}{(1+2^k rm_V(x_0))^{N_2}} \frac{|E_k|^{\frac{1}{p_2}}}{(2^k r)^{n-2\beta_1}} |E_j|^{1-\frac{1}{p_1}} \big(2^j r\big)^{-2\beta_2} \|f_j\|_{L^{p_1}(\mathbb{R}^n)}\right)^q
\end{aligned}$$

$$\begin{aligned}
&\lesssim \big(1+rm_V(x_0)\big)^{\alpha} r^{-\lambda n} \sum_{k=-\infty}^{0} |E_k|^{\theta q} \\
&\quad \times \left(\sum_{j=-\infty}^{k-2} \frac{(1+2^j rm_V(x_0))^{-\frac{\alpha}{q}}}{(1+2^k rm_V(x_0))^{N_2}} \frac{(2^j r)^{\frac{\lambda n}{q}} |E_j|^{-\theta}}{(2^k r)^{n-2\beta_1}} \frac{|E_k|^{\frac{1}{p_2}} |E_j|^{1-\frac{1}{p_1}}}{(2^j r)^{2\beta_2}} \right)^q \|f\|^q_{L^{p_1,\lambda,q}_{\alpha,V,\theta}} \\
&\lesssim \big(1+rm_V(x_0)\big)^{\alpha} r^{-\lambda n} \sum_{k=-\infty}^{0} |E_k|^{\lambda} \left(\sum_{j=-\infty}^{k-2} 2^{(k-j)n(\theta-\frac{\lambda}{q}+\frac{1}{p_2}-1+\frac{2\beta_1}{n})} \right)^q \|f\|^q_{L^{p_1,\lambda,q}_{\alpha,\theta,V}} \\
&\lesssim \|f\|^q_{L^{p_1,q,\lambda}_{\alpha,V,\theta}}.
\end{aligned}$$

For L_3, note that when $x \in E_k$, $y \in E_j$ and $j \geq k+2$, then $|x-y| \sim 2^j r$. Similar to E_1, we have

$$\begin{aligned}
&\big\| \chi_k(-\Delta+V)^{-\beta_1} V^{\beta_2} f_j \big\|^q_{L^{p_2}(\mathbb{R}^n)} \\
&\quad \lesssim \frac{1}{(1+2^j rm_V(x_0))^{N/k_0+1}} \frac{|E_k|^{\frac{1}{p_2}}}{(2^j r)^{n-2\beta_1}} \int_{E_j} V(x)^{\beta_2} \big|f(y)\big|\,dy \\
&\quad \lesssim \frac{1}{(1+2^j rm_V(x_0))^{N_2}} \frac{|E_k|^{\frac{1}{p_2}}}{(2^j r)^{n-2\beta_1}} |E_j|^{1-\frac{1}{p_1}} \big(2^j r\big)^{-2\beta_2} \|f_j\|_{L^{p_1}(\mathbb{R}^n)},
\end{aligned}$$

where $\frac{1}{p_2} = \frac{1}{p_1} - \frac{2\beta_1-2\beta_2}{n}$ and $N_2 < (N/k_0+1) - (\log_2 C_0+1)\beta_2$. Since $\frac{\lambda}{q} - \frac{1}{p_2} < \theta < \frac{\lambda}{q} - \frac{1}{p_2} + 1 - \frac{2\beta_1}{n}$, we obtain

$$\begin{aligned}
L_3 &\lesssim \big(1+rm_V(x_0)\big)^{\alpha} r^{-\lambda n} \sum_{k=-\infty}^{0} |E_k|^{\theta q} \\
&\quad \times \left(\sum_{j=k+2}^{\infty} \frac{1}{(1+2^j rm_V(x_0))^{N_2}} \frac{|E_k|^{\frac{1}{p_2}}}{(2^j r)^{n-2\beta_1}} |E_j|^{1-\frac{1}{p_1}} \big(2^j r\big)^{-2\beta_2} \|f_j\|_{L^{p_1}(\mathbb{R}^n)} \right)^q \\
&\lesssim \big(1+rm_V(x_0)\big)^{\alpha} r^{-\lambda n} \sum_{k=-\infty}^{0} |E_k|^{\lambda} \left(\sum_{j=k+2}^{\infty} 2^{(k-j)n(\theta-\frac{\lambda}{q}+\frac{1}{p_2})} \right)^q \|f\|^q_{L^{p_1,q,\lambda}_{\alpha,\theta,V}} \\
&\lesssim \|f\|^q_{L^{p_1,q,\lambda}_{\alpha,\theta,V}}.
\end{aligned}$$

Let N be large enough. We finally get $\|(-\Delta+V)^{-\beta_1} V^{\beta_2} f\|_{L^{p_2,q,\lambda}_{\alpha,\theta,V}} \lesssim \|f\|_{L^{p_1,q,\lambda}_{\alpha,\theta,V}}$. □

5 Boundedness of the commutators on $L^{p,q,\lambda}_{\alpha,\theta,V}(\mathbb{R}^n)$

In this section, let $b \in BMO(\mathbb{R}^n)$. We consider the boundedness of commutators $[b,(-\Delta+V)^{-\beta_1} V^{\beta_2}]$ and its duality on the generalized Morrey spaces $L^{p,q,\lambda}_{\alpha,\theta,V}(\mathbb{R}^n)$. For this purpose, we prove the commutator $[b,(-\Delta+V)^{-\beta_1} V^{\beta_2}]$ is bounded from $L^{p_1}(\mathbb{R}^n)$ to $L^{p_2}(\mathbb{R}^n)$. For the sake of simplicity, we denote by $b_{2^k r}$ the mean value of b on the ball $B(x_0, 2^k r)$.

Theorem 5.1 *Suppose that $V \in B_s$, $s \geq \frac{n}{2}$ and $b \in BMO(\mathbb{R}^n)$.*

(i) *If $0 < \beta_2 \leq \beta_1 < \frac{n}{2}$, $\frac{s}{s-\beta_2} < p_1 < \frac{n}{2\beta_1-2\beta_2}$, $\frac{1}{p_2} = \frac{1}{p_1} - \frac{2\beta_1-2\beta_2}{n}$, then*

$$\big\| \big[b,(-\Delta+V)^{-\beta_1} V^{\beta_2}\big] f \big\|_{L^{p_2}(\mathbb{R}^n)} \lesssim \|f\|_{L^{p_1}(\mathbb{R}^n)} \|b\|_{BMO}.$$

(ii) *If* $1 < p_2 < \frac{s}{\beta_2}$ *and* $\frac{1}{p_2} = \frac{1}{p_1} - \frac{2\beta_1 - 2\beta_2}{n}$, *then*

$$\left\| \left[b, V^{\beta_2}(-\Delta + V)^{-\beta_1}\right] f \right\|_{L^{p_2}(\mathbb{R}^n)} \lesssim \|f\|_{L^{p_1}(\mathbb{R}^n)} \|b\|_{BMO}.$$

Proof We only prove (i). (ii) can be obtained by duality. Because $\beta_2 \leq \beta_1$, we can decompose the operator $(-\Delta + V)^{-\beta_1} V^{\beta_2}$ as

$$(-\Delta + V)^{-\beta_1} V^{\beta_2} = (-\Delta + V)^{\beta_2 - \beta_1} (-\Delta + V)^{-\beta_2} V^{\beta_2}.$$

Denote by $L^{\beta_2 - \beta_1}$ and T_{β_2} the operators $(-\Delta + V)^{\beta_2 - \beta_1}$ and $(-\Delta + V)^{-\beta_2} V^{\beta_2}$, respectively. Then we can get

$$\begin{aligned}
&\left[b, (-\Delta + V)^{-\beta_1} V^{\beta_2}\right] f(x) \\
&\quad = \left[b, (-\Delta + V)^{\beta_2 - \beta_1} (-\Delta + V)^{-\beta_2} V^{\beta_2}\right] f(x) \\
&\quad = b L^{\beta_2 - \beta_1} T_{\beta_2} f(x) - L^{\beta_2 - \beta_1} T_{\beta_2}(bf)(x) \\
&\quad = b L^{\beta_2 - \beta_1} T_{\beta_2} f(x) - L^{\beta_2 - \beta_1}\left(b T_{\beta_2} f(x)\right) \\
&\qquad + L^{\beta_2 - \beta_1}\left(b T_{\beta_2} f(x)\right) - L^{\beta_2 - \beta_1} T_{\beta_2}(bf)(x) \\
&\quad = \left[b, L^{\beta_2 - \beta_1}\right] T_{\beta_2} f(x) + L^{\beta_2 - \beta_1} [b, T_{\beta_2}] f(x).
\end{aligned}$$

By (1) of Corollary 4.6, we can get

$$\begin{aligned}
&\left\| \left[b, (-\Delta + V)^{-\beta_1} V^{\beta_2}\right] f \right\|_{L^{p_2}(\mathbb{R}^n)} \\
&\quad \lesssim \left\| \left[b, L^{\beta_2 - \beta_1}\right] T_{\beta_2} f \right\|_{L^{p_2}(\mathbb{R}^n)} + \left\| L^{\beta_2 - \beta_1} [b, T_{\beta_2}] f \right\|_{L^{p_2}(\mathbb{R}^n)} \\
&\quad \lesssim \|T_{\beta_2} f\|_{L^{p_1}(\mathbb{R}^n)} + \left\| [b, T_{\beta_2}] f \right\|_{L^{p_1}(\mathbb{R}^n)} \\
&\quad \lesssim \|f\|_{L^{p_1}(\mathbb{R}^n)}.
\end{aligned}$$

This completes the proof. □

In the rest of this section, we prove the boundedness of the commutators $[b, V^{\beta_2}(-\Delta + V)^{-\beta_1}]$ and $[b, (-\Delta + V)^{-\beta_1} V^{\beta_2}]$ on $L^{p_2,q,\lambda}_{\alpha,\theta,V}(\mathbb{R}^n)$, respectively.

Theorem 5.2 *Suppose that* $V \in B_s$, $s \geq \frac{n}{2}$, $\alpha \in (-\infty, 0]$ *and* $\lambda \in (0, n)$. *Let* $1 < q < \infty$, $1 < \beta_2 \leq \beta_1 < \frac{n}{2}$ *and* $1 < p_2 < \frac{s}{\beta_2}$ *with* $\frac{1}{p_1} - \frac{1}{p_2} = \frac{2\beta_1 - 2\beta_2}{n}$. *If* $\frac{\lambda}{q} - \frac{1}{p_1} + \frac{2\beta_1}{n} < \theta < \frac{\lambda}{q} + 1 - \frac{1}{p_1}$, *then for* $b \in BMO(\mathbb{R}^n)$,

$$\left\| \left[b, V^{\beta_2}(-\Delta + V)^{-\beta_1}\right] f \right\|_{L^{p_2,q,\lambda}_{\alpha,\theta,V}} \lesssim \|f\|_{L^{p_1,q,\lambda}_{\alpha,\theta,V}} \|b\|_{BMO}.$$

Proof For any ball $B(x_0, r)$, we have

$$f(y) = \sum_{j=-\infty}^{\infty} f(y) \chi_{E_j}(y) = \sum_{j=-\infty}^{\infty} f_j(y),$$

where $E_j = B(x_0, 2^j r)\backslash B(x_0, 2^{j-1}r)$. Hence, we have

$$\begin{aligned}
&\big(1 + rm_V(x_0)\big)^{\alpha} r^{-\lambda n} \sum_{k=-\infty}^{0} |E_k|^{\theta q} \big\| \chi_k\big[b, V^{\beta_2}(-\Delta + V)^{-\beta_1}\big]f \big\|_{L^{p_2}(\mathbb{R}^n)}^q \\
&\quad\lesssim \big(1 + rm_V(x_0)\big)^{\alpha} r^{-\lambda n} \sum_{k=-\infty}^{0} |E_k|^{\theta q} \left(\sum_{j=-\infty}^{k-2} \big\| \chi_k\big[b, V^{\beta_2}(-\Delta + V)^{-\beta_1}\big]f_j \big\|_{L^{p_2}(\mathbb{R}^n)} \right)^q \\
&\qquad+ \big(1 + rm_V(x_0)\big)^{\alpha} r^{-\lambda n} \sum_{k=-\infty}^{0} |E_k|^{\theta q} \left(\sum_{j=k-1}^{k+1} \big\| \chi_k\big[b, V^{\beta_2}(-\Delta + V)^{-\beta_1}\big]f_j \big\|_{L^{p_2}(\mathbb{R}^n)} \right)^q \\
&\qquad+ \big(1 + rm_V(x_0)\big)^{\alpha} r^{-\lambda n} \sum_{k=-\infty}^{0} |E_k|^{\theta q} \left(\sum_{j=k+2}^{\infty} \big\| \chi_k\big[b, V^{\beta_2}(-\Delta + V)^{-\beta_1}\big]f_j \big\|_{L^{p_2}(\mathbb{R}^n)} \right)^q \\
&\quad=: D_1 + D_2 + D_3.
\end{aligned}$$

For D_2, by (ii) of Theorem 5.1, we have

$$\begin{aligned}
D_2 &\lesssim \big(1 + rm_V(x_0)\big)^{\alpha} r^{-\lambda n} \sum_{k=-\infty}^{0} |E_k|^{\theta q} \left(\sum_{j=k-1}^{k+1} \|f_j\|_{L^{p_1}(\mathbb{R}^n)} \right)^q \|b\|_{BMO}^q \\
&\lesssim \|f\|_{L^{p_1,q,\lambda}_{\alpha,\theta,V}}^q \|b\|_{BMO}^q.
\end{aligned}$$

For D_1, by Lemmas 2.7 and 4.1, we obtain

$$\begin{aligned}
&\big\| \chi_k\big[b, V^{\beta_2}(-\Delta + V)^{-\beta_1}\big]f_j \big\|_{L^{p_2}(\mathbb{R}^n)} \\
&\quad\lesssim \frac{1}{(1 + 2^k rm_V(x_0))^{N/k_0+1}} \frac{1}{(2^k r)^{n-2\beta_1}} \left(\int_{E_k} \left| \int_{E_j} V^{\beta_2}(x)\big(b(x) - b(y)\big)f(y)\,dy \right|^{p_2} dx \right)^{\frac{1}{p_2}} \\
&\quad\lesssim \frac{1}{(1 + 2^k rm_V(x_0))^{N/k_0+1}} \frac{1}{(2^k r)^{n-2\beta_1}} \left[\left(\int_{E_k} V^{\beta_2 p_2}(x)\big|b(x) - b_{2^k r}\big|^{p_2}\,dx \right)^{\frac{1}{p_2}} \int_{E_j} \big|f(y)\big|\,dy \right. \\
&\qquad\left. + \left(\int_{E_k} V^{\beta_2 p_2}(x)\,dx \right)^{\frac{1}{p_2}} \int_{E_j} \big|b(y) - b_{2^k r}\big| \big|f(y)\big|\,dy \right] \\
&\quad\lesssim \frac{\|b\|_{BMO}}{(1 + 2^k rm_V(x_0))^{N/k_0+1}} \frac{1}{(2^k r)^{n-2\beta_1}} \left[\left(\int_{E_k} V(x)\,dx \right)^{\beta_2} |E_k|^{\frac{1}{p_2}-\beta_2} \int_{E_j} \big|f(y)\big|\,dy \right. \\
&\qquad\left. + \left(\int_{E_k} V(x)\,dx \right)^{\beta_2} |E_k|^{\frac{1}{p_2}-\beta_2} |E_j|^{1-\frac{1}{p_1}} (k-j)\|f_j\|_{L^{p_1}(\mathbb{R}^n)} \right] \\
&\quad\lesssim \frac{\|b\|_{BMO}}{(1 + 2^k rm_V(x_0))^{N_1}} \frac{k-j}{(2^k r)^{n-2\beta_1}} |E_k|^{\frac{1}{p_2}-\frac{2\beta_2}{n}} |E_j|^{1-\frac{1}{p_1}} \|f_j\|_{L^{p_1}(\mathbb{R}^n)},
\end{aligned}$$

where $\frac{1}{p_1} - \frac{1}{p_2} = \frac{2\beta_1-2\beta_2}{n}$ and $N_1 < (N/k_0 + 1) - (\log_2 C_0 + 1)\beta_2$. Since $\frac{\lambda}{q} - \frac{1}{p_1} + \frac{2\beta_1}{n} < \theta < \frac{\lambda}{q} + 1 - \frac{1}{p_1}$, we obtain

$$\begin{aligned}
D_1 &\lesssim \|b\|_{BMO}^q \big(1 + rm_V(x_0)\big)^{\alpha} r^{-\lambda n} \sum_{k=-\infty}^{0} |E_k|^{\theta q} \\
&\quad\times \left(\sum_{j=-\infty}^{k-2} \frac{1}{(1 + 2^k rm_V(x_0))^{N_1}} \frac{k-j}{(2^k r)^{n-2\beta_1}} |E_k|^{\frac{1}{p_2}-\frac{2\beta_2}{n}} |E_j|^{1-\frac{1}{p_1}} \|f_j\|_{L^{p_1}(\mathbb{R}^n)} \right)^q
\end{aligned}$$

$$
\begin{aligned}
&\lesssim \|b\|_{BMO}^q \big(1+rm_V(x_0)\big)^{\alpha} r^{-\lambda n} \sum_{k=-\infty}^{0} |E_k|^{\theta q} \\
&\quad \times \left(\sum_{j=-\infty}^{k-2} \frac{(1+2^j rm_V(x_0))^{-\frac{\alpha}{q}}}{(1+2^k rm_V(x_0))^{N_1}} \frac{(2^j r)^{\frac{\lambda n}{q}} |E_j|^{-\theta}}{(2^k r)^{n-2\beta_1}} \frac{|E_k|^{\frac{1}{p_2}-\frac{2\beta_2}{n}}}{|E_j|^{\frac{1}{p_1}-1}} (k-j) \right)^q \|f\|^q_{L^{p_1,q,\lambda}_{\alpha,\theta,V}} \\
&\lesssim \|b\|_{BMO}^q \frac{(1+rm_V(x_0))^{\alpha}}{r^{\lambda n}} \sum_{k=-\infty}^{0} |E_k|^{\lambda} \left(\sum_{j=-\infty}^{k-2} (k-j) 2^{(j-k)n(\frac{\lambda}{q}-\theta-\frac{1}{p_1}+1)} \right)^q \|f\|^q_{L^{p_1,q,\lambda}_{\alpha,\theta,V}} \\
&\lesssim \|f\|^q_{L^{p_1,q,\lambda}_{\alpha,\theta,V}} \|b\|^q_{BMO}.
\end{aligned}
$$

For D_3, because $\frac{1}{p_1} - \frac{1}{p_2} = \frac{2\beta_1 - 2\beta_2}{n}$ and $N_1 < (N/k_0 + 1) - (\log_2 C_0 + 1)\beta_2$, we have

$$
\begin{aligned}
&\big\| \chi_k \big[b, V^{\beta_2}(-\Delta + V)^{-\beta_1}\big] f_j \big\|_{L^{p_2}(\mathbb{R}^n)} \\
&\quad \lesssim \frac{1}{(1+2^j rm_V(x_0))^{N/k_0+1}} \frac{1}{(2^j r)^{n-2\beta_1}} \left(\int_{E_k} \left| \int_{E_j} V(x)^{\beta_2} \big(b(x)-b(y)\big) f(y)\,dy \right|^{p_2} dx \right)^{\frac{1}{p_2}} \\
&\quad \lesssim \frac{j-k}{(1+2^j rm_V(x_0))^{N_1}} |E_j|^{\frac{2\beta_1}{n}-\frac{1}{p_1}} |E_k|^{\frac{1}{p_2}-\frac{2\beta_2}{n}} \|b\|_{BMO} \|f_j\|_{L^{p_1}(\mathbb{R}^n)},
\end{aligned}
$$

where we have used the fact that $|x-y| \sim 2^j r$ for $x \in E_k$, $y \in E_j$ and $j \geq k+2$. Since $\frac{\lambda}{q} - \frac{1}{p_1} + \frac{2\beta_1}{n} < \theta < \frac{\lambda}{q} + 1 - \frac{1}{p_1}$, we obtain

$$
\begin{aligned}
D_3 &\lesssim \big(1+rm_V(x_0)\big)^{\alpha} r^{-\lambda n} \sum_{k=-\infty}^{0} |E_k|^{\theta q} \\
&\quad \times \left(\sum_{j=k+2}^{\infty} \frac{j-k}{(1+2^j rm_V(x_0))^{N_1}} |E_j|^{\frac{2\beta_1}{n}-\frac{1}{p_1}} |E_k|^{\frac{1}{p_2}-\frac{2\beta_2}{n}} \|b\|_{BMO} \|f_j\|_{L^{p_1}(\mathbb{R}^n)} \right)^q \\
&\lesssim \|b\|_{BMO}^q \big(1+rm_V(x_0)\big)^{\alpha} r^{-\lambda n} \sum_{k=-\infty}^{0} |E_k|^{\theta q} \\
&\quad \times \left(\sum_{j=k+2}^{\infty} \frac{(1+2^j rm_V(x_0))^{-\frac{\alpha}{q}} (2^j r)^{\frac{\lambda n}{q}} |E_j|^{-\theta}}{(1+2^j rm_V(x_0))^{N_1}} \frac{|E_k|^{\frac{1}{p_2}-\frac{2\beta_2}{n}}}{|E_j|^{\frac{2\beta_1}{n}-\frac{1}{p_1}}} (j-k) \right)^q \|f\|^q_{L^{p_1,q,\lambda}_{\alpha,\theta,V}} \\
&\lesssim \|b\|_{BMO}^q \big(1+rm_V(x_0)\big)^{\alpha} r^{-\lambda n} \sum_{k=-\infty}^{0} |E_k|^{\lambda} \left(\sum_{j=k+2}^{\infty} (j-k) 2^{(k-j)n(\theta-\frac{\lambda}{q}+\frac{1}{p_1}+\frac{2\beta_1}{n})} \right)^q \|f\|^q_{L^{p_1,q,\lambda}_{\alpha,\theta,V}} \\
&\lesssim \|f\|^q_{L^{p_1,q,\lambda}_{\alpha,\theta,V}} \|b\|^q_{BMO}.
\end{aligned}
$$

Let N be large enough. Finally, we get

$$
\big\| \big[b, V^{\beta_2}(-\Delta + V)^{-\beta_1}\big] f \big\|_{L^{p_2,q,\lambda}_{\alpha,\theta,V}} \lesssim \|f\|_{L^{p_1,q,\lambda}_{\alpha,\theta,V}} \|b\|_{BMO}. \qquad \square
$$

Theorem 5.3 *Suppose that* $V \in B_s$, $s \geq \frac{n}{2}$ *and* $b \in BMO(\mathbb{R}^n)$. *Let* $\alpha \in (-\infty, 0]$, $\lambda \in (0, n)$ *and* $1 < q < \infty$. *If* $0 < \beta_2 \leq \beta_1 < \frac{n}{2}$, $\frac{s}{s-\beta_2} < p_1 < \frac{n}{2\beta_1 - 2\beta_2}$, $\frac{1}{p_2} = \frac{1}{p_1} - \frac{2\beta_1 - 2\beta_2}{n}$, $\frac{\lambda}{q} - \frac{1}{p_2} < \theta < \frac{\lambda}{q} - \frac{1}{p_2} + 1 - \frac{2\beta_1}{n}$, *then*

$$
\big\| \big[b, (-\Delta + V)^{-\beta_1} V^{\beta_2}\big] f \big\|_{L^{p_2,q,\lambda}_{\alpha,\theta,V}} \lesssim \|f\|_{L^{p_1,q,\lambda}_{\alpha,\theta,V}} \|b\|_{BMO}.
$$

Proof Similarly, we can decompose f based on an arbitrary ball $B(x_0,r)$ as follows:

$$f(y)=\sum_{j=-\infty}^{\infty}f(y)\chi_{E_j}(y)=\sum_{j=-\infty}^{\infty}f_j(y),$$

where $E_j=B(x_0,2^jr)\backslash B(x_0,2^{j-1}r)$. Hence, we have

$$\begin{aligned}
&\bigl(1+rm_V(x_0)\bigr)^{\alpha}r^{-\lambda n}\sum_{k=-\infty}^{0}|E_k|^{\theta q}\bigl\|\chi_k\bigl[b,(-\Delta+V)^{-\beta_1}V^{\beta_2}\bigr]f\bigr\|_{L^{p_2}(\mathbb{R}^n)}^q\\
&\quad\lesssim\bigl(1+rm_V(x_0)\bigr)^{\alpha}r^{-\lambda n}\sum_{k=-\infty}^{0}|E_k|^{\theta q}\Biggl(\sum_{j=-\infty}^{k-2}\bigl\|\chi_k\bigl[b,(-\Delta+V)^{-\beta_1}V^{\beta_2}\bigr]f_j\bigr\|_{L^{p_2}(\mathbb{R}^n)}\Biggr)^q\\
&\qquad+\bigl(1+rm_V(x_0)\bigr)^{\alpha}r^{-\lambda n}\sum_{k=-\infty}^{0}|E_k|^{\theta q}\Biggl(\sum_{j=k-1}^{k+1}\bigl\|\chi_k\bigl[b,(-\Delta+V)^{-\beta_1}V^{\beta_2}\bigr]f_j\bigr\|_{L^{p_2}(\mathbb{R}^n)}\Biggr)^q\\
&\qquad+\bigl(1+rm_V(x_0)\bigr)^{\alpha}r^{-\lambda n}\sum_{k=-\infty}^{0}|E_k|^{\theta q}\Biggl(\sum_{j=k+2}^{\infty}\bigl\|\chi_k\bigl[b,(-\Delta+V)^{-\beta_1}V^{\beta_2}\bigr]f_j\bigr\|_{L^{p_2}(\mathbb{R}^n)}\Biggr)^q\\
&\quad=F_1+F_2+F_3.
\end{aligned}$$

Applying Theorem 5.1, we can get

$$\begin{aligned}
F_2&\lesssim\frac{(1+rm_V(x_0))^{\alpha}}{r^{\lambda n}}\sum_{k=-\infty}^{0}|E_k|^{\theta q}\Biggl(\sum_{j=k-1}^{k+1}\|f_j\|_{L^{p_1}(\mathbb{R}^n)}\Biggr)^q\|b\|_{BMO}^q\\
&\lesssim\|f\|_{L_{\alpha,\theta,V}^{p_1,q,\lambda}}^q\|b\|_{BMO}^q.
\end{aligned}$$

For F_1, by Hölder's inequality and the fact that $V\in B_s$, we apply Lemmas 4.1 and 2.7 to deduce that

$$\begin{aligned}
&\bigl\|\chi_k\bigl[b,(-\Delta+V)^{-\beta_1}V^{\beta_2}\bigr]f_j\bigr\|_{L^{p_2}(\mathbb{R}^n)}\\
&\quad\lesssim\frac{1}{(1+2^krm_V(x_0))^{N/k_0+1}}\frac{1}{(2^kr)^{n-2\beta_1}}\biggl(\int_{E_k}\biggl|\int_{E_j}\bigl(b(x)-b(y)\bigr)V^{\beta_2}(y)f(y)\,dy\biggr|^{p_2}dx\biggr)^{\frac{1}{p_2}}\\
&\quad\lesssim\frac{1}{(1+2^krm_V(x_0))^{N/k_0+1}}\frac{1}{(2^kr)^{n-2\beta_1}}\\
&\qquad\times\biggl[\biggl(\int_{E_k}\bigl|b(x)-b_{2^kr}\bigr|^{p_2}dx\biggr)^{\frac{1}{p_2}}\int_{E_j}\bigl|V^{\beta_2}(y)f(y)\bigr|\,dy\\
&\qquad+|E_k|^{\frac{1}{p_2}}\int_{E_j}\bigl|b(y)-b_{2^kr}\bigr|\bigl|V^{\beta_2}(y)f(y)\bigr|\,dy\biggr]\\
&\quad\lesssim\frac{(\int_{E_j}V(y)\,dy)^{\beta_2}}{(1+2^krm_V(x_0))^{N/k_0+1}}\frac{k-j}{(2^kr)^{n-2\beta_1}}|E_k|^{\frac{1}{p_2}}|E_j|^{1-\frac{1}{p_1}}\|b\|_{BMO}\|f_j\|_{L^{p_1}(\mathbb{R}^n)}\\
&\quad\lesssim\|b\|_{BMO}\frac{k-j}{(1+2^krm_V(x_0))^{N_2}}|E_k|^{\frac{1}{p_2}+\frac{2\beta_1}{n}-1}|E_j|^{1-\frac{1}{p_1}-\frac{2\beta_2}{n}}\|f_j\|_{L^{p_1}(\mathbb{R}^n)},
\end{aligned}$$

where $\frac{1}{p_2} = \frac{1}{p_1} - \frac{2\beta_1 - 2\beta_2}{n}$ and $N_2 < (N/k_0 + 1) - (\log_2 C_0 + 1)\beta_2$. Since $\frac{\lambda}{q} - \frac{1}{p_2} < \theta < \frac{\lambda}{q} - \frac{1}{p_2} + 1 - \frac{2\beta_1}{n}$, we obtain

$$\begin{aligned}
F_1 &\lesssim \|b\|_{BMO}^q \big(1 + rm_V(x_0)\big)^\alpha r^{-\lambda n} \sum_{k=-\infty}^{0} |E_k|^{\theta q} \\
&\quad \times \left(\sum_{j=-\infty}^{k-2} \frac{k-j}{(1 + 2^k rm_V(x_0))^{N_2}} |E_k|^{\frac{1}{p_2} + \frac{2\beta_1}{n} - 1} |E_j|^{1 - \frac{1}{p_1} - \frac{2\beta_2}{n}} \|f_j\|_{L^{p_1}(\mathbb{R}^n)} \right)^q \\
&\lesssim \|b\|_{BMO}^q \big(1 + rm_V(x_0)\big)^\alpha r^{-\lambda n} \sum_{k=-\infty}^{0} |E_k|^{\theta q} \\
&\quad \times \left(\sum_{j=-\infty}^{k-2} \frac{(1 + 2^j rm_V(x_0))^{-\frac{\alpha}{q}}}{(1 + 2^k rm_V(x_0))^{N_2}} \big(2^j r\big)^{\frac{\lambda n}{q}} |E_j|^{-\theta} \frac{|E_j|^{1 - \frac{1}{p_2} - \frac{2\beta_1}{n}}}{|E_k|^{1 - \frac{1}{p_2} - \frac{2\beta_1}{n}}} (k-j) \right)^q \|f\|_{L^{p_1,q,\lambda}_{\alpha,\theta,V}}^q \\
&\lesssim \|b\|_{BMO}^q \frac{(1 + rm_V(x_0))^\alpha}{r^{\lambda n}} \sum_{k=-\infty}^{0} |E_k|^\lambda \left(\sum_{j=-\infty}^{k-2} (k-j) 2^{(k-j)n(\theta - \frac{\lambda}{q} + \frac{1}{p_2} - 1 + \frac{2\beta_1}{n})} \right)^q \|f\|_{L^{p_1,q,\lambda}_{\alpha,\theta,V}}^q \\
&\lesssim \|f\|_{L^{p_1,q,\lambda}_{\alpha,\theta,V}}^q \|b\|_{BMO}^q.
\end{aligned}$$

For F_3, note that when $x \in E_k$, $y \in E_j$ and $j \geq k + 2$, then $|x - y| \sim 2^j r$. Similar to F_1, we have

$$\begin{aligned}
&\big\| \chi_k \big[b, (-\Delta + V)^{-\beta_1} V^{\beta_2}\big] f_j \big\|_{L^{p_2}(\mathbb{R}^n)} \\
&\quad \lesssim \frac{1}{(1 + 2^j rm_V(x_0))^{N/k_0 + 1}} \frac{1}{(2^j r)^{n - 2\beta_1}} \left(\int_{E_k} \left| \int_{E_j} \big(b(x) - b(y)\big) V(y)^{\beta_2} f(y)\, dy \right|^{p_2} dx \right)^{\frac{1}{p_2}} \\
&\quad \lesssim \frac{j-k}{(1 + 2^j rm_V(x_0))^{N_2}} |E_k|^{\frac{1}{p_2}} |E_j|^{-\frac{1}{p_2}} \|f_j\|_{L^{p_1}(\mathbb{R}^n)} \|b\|_{BMO},
\end{aligned}$$

where $\frac{1}{p_2} = \frac{1}{p_1} - \frac{2\beta_1 - 2\beta_2}{n}$ and $N_2 < (N/k_0 + 1) - (\log_2 C_0 + 1)\beta_2$. Since $\frac{\lambda}{q} - \frac{1}{p_2} < \theta < \frac{\lambda}{q} - \frac{1}{p_2} + 1 - \frac{2\beta_1}{n}$, we obtain

$$\begin{aligned}
F_3 &\lesssim \|b\|_{BMO}^q \big(1 + rm_V(x_0)\big)^\alpha r^{-\lambda n} \sum_{k=-\infty}^{0} |E_k|^{\theta q} \\
&\quad \times \left(\sum_{j=k+2}^{\infty} \frac{(1 + 2^j rm_V(x_0))^{-\frac{\alpha}{q}}}{(1 + 2^j rm_V(x_0))^{N_2}} \big(2^j r\big)^{\frac{\lambda n}{q}} |E_j|^{-\theta} \frac{|E_k|^{\frac{1}{p_2}}}{|E_j|^{\frac{1}{p_2}}} (j-k) \|f_j\|_{L^{p_1}(\mathbb{R}^n)} \right)^q \\
&\lesssim \|b\|_{BMO}^q \big(1 + rm_V(x_0)\big)^\alpha r^{-\lambda n} \sum_{k=-\infty}^{0} |E_k|^\lambda \left(\sum_{j=k+2}^{\infty} (j-k) 2^{(k-j)n(\theta - \frac{\lambda}{q} + \frac{1}{p_2})} \right)^q \|f\|_{L^{p_1,q,\lambda}_{\alpha,\theta,V}}^q \\
&\lesssim \|f\|_{L^{p_1,q,\lambda}_{\alpha,\theta,V}}^q \|b\|_{BMO}^q.
\end{aligned}$$

Let N be large enough. We finally get

$$\big\| \big[b, (-\Delta + V)^{-\beta_1} V^{\beta_2}\big] f \big\|_{L^{p_2,q,\lambda}_{\alpha,\theta,V}} \lesssim \|f\|_{L^{p_1,q,\lambda}_{\alpha,\theta,V}} \|b\|_{BMO}. \qquad \square$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

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