Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH | Open Access

Some new generalized Gronwall-Bellman-type integral inequalities in two independent variables on time scales

Yulu Wang, Qinghua Feng*, Shixiang Zhou and Feng Xu

*Correspondence: fqhua@sina.com
School of Science, Shandong University of Technology, Zhangzhou Road 12, Zibo, Shandong 255049, China

Abstract
In this paper, some new Gronwall-Bellman-type integral inequalities in two independent variables on time scales are established, which can be used as a handy tool in the research of qualitative and quantitative properties of solutions of dynamic equations on time scales. The inequalities established unify some of the integral inequalities for continuous functions in (Meng and Li in Appl. Math. Comput. 148:381-392, 2004) and their discrete analysis in (Meng and Li in J. Comput. Appl. Math. 158:407-417, 2003).

MSC: 26E70; 26D15; 26D10

Keywords: integral inequality; time scale; dynamic equation; qualitative property; quantitative property

1 Introduction

It is well known that Gronwall-Bellman inequality [1, 2] plays an important role in the research of boundedness, global existence, stability of solutions of differential and integral equations as well as difference equations. During the past decades, a lot of generations of Gronwall-Bellman inequality have been discovered; see, for example, [3–10]. On the other hand, in the 1980s, Hilger created the theory of time scales [11] as a theory capable to contain both difference and differential calculus in a consistent way. Since then many authors have expounded on various aspects of the theory of dynamic equations on time scales. See, for example, [12–14] and the references therein. In these investigations, integral inequalities on time scales have been paid much attention by many authors, and a lot of integral inequalities on time scales have been established, for example, [15–20], which have been designed in order to unify continuous and discrete analysis. But to our knowledge, Gronwall-Bellman-type integral inequalities containing integration on infinite intervals on time scales have been paid little attention in the literature so far.

In this paper, we establish some new Gronwall-Bellman-type integral inequalities in two independent variables containing integration on infinite intervals on time scales, which unify some of the continuous inequalities in [21] and the corresponding discrete analysis in [22].

2 Some preliminaries

Throughout the paper, $\mathbb{R}$ denotes the set of real numbers and $\mathbb{R}_+ = [0, \infty)$. $\mathbb{Z}$ denotes the set of integers. For two given sets G, H, we denote the set of maps from G to H by (G, H).

A time scale is an arbitrary nonempty closed subset of real numbers. In this paper, $\mathbb{T}$ denotes an arbitrary time scale. On $\mathbb{T}$ we define the forward and backward jump operators $\sigma \in (\mathbb{T},\mathbb{T})$ and $\rho \in (\mathbb{T},\mathbb{T})$ such that $\sigma(t) = \inf\{s \in \mathbb{T}, s > t\}$, $\rho(t) = \sup\{s \in \mathbb{T}, s < t\}$.

Definition 2.1 The graininess $\mu \in (\mathbb{T},\mathbb{R}_+)$ is defined by $\mu(t) = \sigma(t) - t$.

Remark 2.1 Obviously, $\mu(t) = 0$ if $\mathbb{T} = \mathbb{R}$, while $\mu(t) = 1$ if $\mathbb{T} = \mathbb{Z}$.

Definition 2.2 A point $t \in \mathbb{T}$ with $t > \inf \mathbb{T}$ is said to be left-dense if $\rho(t) = t$ and right-dense if $\sigma(t) = t$, left-scattered if $\rho(t) < t$ and right-scattered if $\sigma(t) > t$.

Definition 2.3 The set $\mathbb{T}^\kappa$ is defined to be $\mathbb{T}$ if $\mathbb{T}$ does not have a left-scattered maximum; otherwise, it is $\mathbb{T}$ without the left-scattered maximum.

Definition 2.4 A function $f(t) \in (\mathbb{T},\mathbb{R})$ is called rd-continuous if it is continuous in right-dense points and if the left-sided limits exist in left-dense points, while f is called regressive if $1 + \mu(t)f(t) \neq 0$. C_{rd} denotes the set of rd-continuous functions, while $\mathfrak{R}$ denotes the set of all regressive and rd-continuous functions, and $\mathfrak{R}^+ = \{f \mid f \in \mathfrak{R}, 1 + \mu(t)f(t) > 0, \forall t \in \mathbb{T}\}$.

Definition 2.5 For some $t \in \mathbb{T}^\kappa$, and a function $f(t) \in (\mathbb{T},\mathbb{R})$, the *delta derivative* of f is denoted by $f^\Delta(t)$ and satisfies

$$\left|f\big(\sigma(t)\big) - f(s) - f^\Delta(t)\big(\sigma(t) - s\big)\right| \le \varepsilon\left|\sigma(t) - s\right|, \quad \forall \varepsilon > 0,$$

where $s \in \mathfrak{U}$, and $\mathfrak{U}$ is a neighborhood of t. The function $f(t)$ is called *delta differential* on $\mathbb{T}^\kappa$.

Similarly, for some $x \in \mathbb{T}^\kappa$, and a function $f(x,y) \in (\mathbb{T}\times\mathbb{T},\mathbb{R})$, the *partial delta derivative* of $f(x,y)$ with respect to x is denoted by $(f(x,y))^\Delta_x$ and satisfies

$$\left|f\big(\sigma(x),y\big) - f(s,y) - \big(f(x,y)\big)^\Delta_x\big(\sigma(x) - s\big)\right| \le \varepsilon\left|\sigma(x) - s\right|, \quad \forall \varepsilon > 0,$$

where $s \in \mathfrak{U}$, and $\mathfrak{U}$ is a neighborhood of x. The function $f(x,y)$ is called *partial delta differentiable* with respect to x on $\mathbb{T}^\kappa$.

Remark 2.2 If $\mathbb{T} = \mathbb{R}$, then $f^\Delta(t)$ becomes the usual derivative $f'(t)$, while $f^\Delta(t) = f(t+1) - f(t)$ if $\mathbb{T} = \mathbb{Z}$, which represents the forward difference.

Definition 2.6 If $F^\Delta(t) = f(t)$, $t \in \mathbb{T}^\kappa$, then F is called an *antiderivative* of f, and the *Cauchy integral* of f is defined by

$$\int_a^b f(t)\Delta t = F(b) - F(a), \quad \text{where } a,b \in \mathbb{T}.$$

Similarly, for $a,b \in \mathbb{T}$ and a function $f(x,y): \mathbb{T}\times\mathbb{T} \to \mathbb{R}$, the *Cauchy partial integral* of $f(x,y)$ with respect to x is defined by

$$\int_a^b f(x,y)\Delta x = F(b,y) - F(a,y), \quad \text{where } \big(F(x,y)\big)^\Delta_x = f(x,y), x \in \mathbb{T}^\kappa.$$

The following two theorems include some important properties for *partial delta derivative* and *Cauchy partial integral* on time scales.

Theorem 2.1 *If $f(x,y), g(x,y) \in (\mathbb{T} \times \mathbb{T}, \mathbb{R})$, and $x \in \mathbb{T}^\kappa$, then*
(i)

$$\big(f(x,y)\big)_x^\Delta = \begin{cases} \frac{f(\sigma(x),y)-f(x,y)}{\mu(x)} & \text{if } \mu(x) \neq 0, \\ \lim_{s\to x} \frac{f(x,y)-f(s,y)}{x-s} & \text{if } \mu(x) = 0. \end{cases}$$

(ii) *If f, g are* partial delta differentiable *at x, then fg is also* partial delta differentiable *at x, and*

$$\big(f(x,y)g(x,y)\big)_x^\Delta = \big(f(x,y)\big)_x^\Delta g(x,y) + f\big(\sigma(x),y\big)\big(g(x,y)\big)_x^\Delta.$$

Theorem 2.2 *If $a, b, c \in \mathbb{T}$, $\alpha \in \mathbb{R}$, and $f(x,y), g(x,y) \in C_{\mathrm{rd}}(\mathbb{T} \times \mathbb{T}, \mathbb{R})$, then*
(i) $\int_a^b [f(x,y) + g(x,y)]\Delta x = \int_a^b f(x,y)\Delta x + \int_a^b g(x,y)\Delta x$,
(ii) $\int_a^b (\alpha f)(x,y)\Delta x = \alpha \int_a^b f(x,y)\Delta x$,
(iii) $\int_a^b f(x,y)\Delta x = -\int_b^a f(x,y)\Delta x$,
(iv) $\int_a^b f(x,y)\Delta x = \int_a^c f(x,y)\Delta x + \int_c^b f(x,y)\Delta x$,
(v) $\int_a^a f(x,y)\Delta x = 0$,
(vi) *if $f(x,y) \geq 0$ for all $a \leq x \leq b$, then $\int_a^b f(x,y)\Delta x \geq 0$.*

Remark 2.3 If $b = \infty$, then all the conclusions of Theorem 2.2 still hold.

Definition 2.7 The cylinder transformation ξ_h is defined by

$$\xi_h(z) = \begin{cases} \frac{\mathrm{Log}(1+hz)}{h} & \text{if } h \neq 0 \text{ (for } z \neq -\frac{1}{h}), \\ z & \text{if } h = 0, \end{cases}$$

where Log is the principal logarithm function.

Definition 2.8 For $p(x,y) \in \mathfrak{R}$ with respect to x, the *exponential function* is defined by

$$e_p(x,s) = \exp\left(\int_s^x \xi_{\mu(\tau)}\big(p(\tau,y)\big)\Delta\tau\right), \quad s, x, y \in \mathbb{T}.$$

Definition 2.9 If $\sup_{x\in\mathbb{T}} x = \infty$, $p(x,y) \in \mathfrak{R}$ with respect to x, then we define

$$e_p(\infty,s) = \exp\left(\int_s^\infty \xi_{\mu(\tau)}\big(p(\tau,y)\big)\Delta\tau\right), \quad s, y \in \mathbb{T}.$$

Remark 2.4 If $\mathbb{T} = \mathbb{R}$, then for $x \in \mathbb{R}$

$$\begin{cases} e_p(x,s) = \exp(\int_s^x p(\tau,y)\,d\tau), & s, y \in \mathbb{R}, \\ e_p(\infty,s) = \exp(\int_s^\infty p(\tau,y)\,d\tau), & s, y \in \mathbb{R}. \end{cases}$$

If $\mathbb{T} = \mathbb{Z}$, then for $x \in \mathbb{Z}$

$$\begin{cases} e_p(x,s) = \prod_{\tau=s}^{x-1}[1 + p(\tau,y)], & s, y \in \mathbb{Z} \text{ and } s < x, \\ e_p(\infty,s) = \prod_{\tau=s}^{\infty}[1 + p(\tau,y)], & s, y \in \mathbb{Z}. \end{cases}$$

The following two theorems include some known properties on the *exponential function*.

Theorem 2.3 *If $p(x,y) \in \mathfrak{R}$ with respect to x, then the following conclusions hold*:

(i) $e_p(x,x) \equiv 1$ *and* $e_0(x,s) \equiv 1$,

(ii) $e_p(\sigma(x),s) = (1 + \mu(x)p(x,y))e_p(x,s)$,

(iii) *if* $p \in \mathfrak{R}^+$ *with respect to* x, *then* $e_p(x,s) > 0$, $\forall s, x \in \mathbb{T}$,

(iv) *if* $p \in \mathfrak{R}^+$ *with respect to* x, *then* $\ominus p \in \mathfrak{R}^+$,

(v) $e_p(x,s) = \frac{1}{e_p(s,x)} = e_{\ominus p}(s,x)$,

where $(\ominus p)(x,y) = -\frac{p(x,y)}{1+\mu(x)p(x,y)}$.

Remark 2.5 If $s = \infty$, then Theorem 2.3(v) still holds.

Theorem 2.4 *If $p(x,y) \in \mathfrak{R}$ with respect to x, $x_0 \in \mathbb{T}$ is a fixed number, then the* exponential function $e_p(x,x_0)$ *is the unique solution of the following initial value problem*:

$$\begin{cases} (z(x,y))^{\Delta}_x = p(x,y)z(x,y), \\ z(x_0,y) = 1. \end{cases}$$

Remark 2.6 Theorems 2.1-2.4 are similar to the corresponding theorems in [23, 24].

Remark 2.7 For more details about time scales, we advise the reader to refer to [25].

3 Main results

First we give some important lemmas as follows.

Lemma 3.1 *Suppose* $\sup_{x\in\mathbb{T}^\kappa} x = \infty$. *For every fixed* $y \in \mathbb{T}^\kappa$, $u(x,y), q(x,y) \in C_{\mathrm{rd}}(\mathbb{T}^\kappa \times \mathbb{T}^\kappa, \mathbb{R})$, $p(x,y) \in \mathfrak{R}^+$ *with respect to* x, *and* $u(x,y)$ *is* partial delta differentiable *at* $x \in \mathbb{T}^\kappa$, *then*

$$\big(u(x,y)\big)^{\Delta}_x \ge p(x,y)u(x,y) - q(x,y), \quad x,y \in \mathbb{T}^\kappa, \tag{3.1}$$

implies

$$u(x,y) \le u(\infty,y)e_{\ominus p}(\infty,x) + \int_x^\infty q(s,y)e_p\big(x,\sigma(s)\big)\Delta s, \quad x,y \in \mathbb{T}^\kappa. \tag{3.2}$$

Proof Fix $Y \in \mathbb{T}^\kappa$, since $p(x,Y) \in \mathfrak{R}^+$, then from Theorem 2.3(iv) we have $\ominus p(x,Y) \in \mathfrak{R}^+$; furthermore, from Theorem 2.3(iii) we obtain $e_{\ominus p}(x,\alpha) > 0$, $\forall \alpha \in \mathbb{T}^\kappa$.

According to Theorem 2.1(ii),

$$\big[u(x,Y)e_{\ominus p}(x,\alpha)\big]^{\Delta}_x = \big[e_{\ominus p}(x,\alpha)\big]^{\Delta}_x u(x,Y) + e_{\ominus p}\big(\sigma(x),\alpha\big)\big(u(x,Y)\big)^{\Delta}_x. \tag{3.3}$$

On the other hand, from Theorem 2.4 we have

$$\big[e_{\ominus p}(x,\alpha)\big]^{\Delta}_x = (\ominus p)(x,Y)e_{\ominus p}(x,\alpha). \tag{3.4}$$

So, combining (3.3), (3.4) and Theorem 2.3, it follows that

$$\begin{aligned} &\big[u(x,Y)e_{\ominus p}(x,\alpha)\big]^{\Delta}_x \\ &\quad = (\ominus p)(x,Y)e_{\ominus p}(x,\alpha)u(x,Y) + e_{\ominus p}\big(\sigma(x),\alpha\big)\big(u(x,Y)\big)^{\Delta}_x \end{aligned}$$

$$= e_{\ominus p}\big(\sigma(x),\alpha\big)\left[\frac{(\ominus p)(x,Y)}{1+\mu(x)(\ominus p)(x,Y)}u(x,Y) + \big(u(x,Y)\big)_x^{\Delta}\right]$$
$$= e_{\ominus p}\big(\sigma(x),\alpha\big)\big[\big(u(x,Y)\big)_x^{\Delta} - p(x,Y)u(x,Y)\big]. \tag{3.5}$$

Substituting x with s and integration for (3.5) with respect to s from α to ∞ yield

$$u(\infty,Y)e_{\ominus p}(\infty,\alpha) - u(\alpha,Y)e_{\ominus p}(\alpha,\alpha)$$
$$= \int_\alpha^\infty e_{\ominus p}\big(\sigma(s),\alpha\big)\big[\big(u(s,Y)\big)_x^{\Delta} - p(s,Y)u(s,Y)\big]\Delta s. \tag{3.6}$$

Considering $e_{\ominus p}(\alpha,\alpha) = 1$, from (3.1) and (3.6), we have

$$\begin{aligned} u(\infty,Y)e_{\ominus p}(\infty,\alpha) - u(\alpha,Y) &\geq -\int_\alpha^\infty e_{\ominus p}\big(\sigma(s),\alpha\big)q(s,Y)\Delta s \\ &= -\int_\alpha^\infty e_p\big(\alpha,\sigma(s)\big)q(s,Y)\Delta s, \end{aligned}$$

which is followed by

$$u(\alpha,Y) \leq u(\infty,Y)e_{\ominus p}(\infty,\alpha) + \int_\alpha^\infty e_p\big(\alpha,\sigma(s)\big)q(s,Y)\Delta s. \tag{3.7}$$

Since $\alpha \in \mathbb{T}^\kappa$ is arbitrary, then after substituting α with x, we obtain

$$u(x,Y) \leq u(\infty,Y)e_{\ominus p}(\infty,x) + \int_x^\infty q(s,Y)e_p\big(x,\sigma(s)\big)\Delta s, \quad x \in \mathbb{T}^\kappa. \tag{3.8}$$

Considering Y is selected from $\mathbb{T}^\kappa$ arbitrarily, then in fact (3.8) holds for every y in $\mathbb{T}^\kappa$, that is,

$$u(x,y) \leq u(\infty,Y)e_{\ominus p}(\infty,x) + \int_x^\infty q(s,y)e_p\big(x,\sigma(s)\big)\Delta s, \quad x,y \in \mathbb{T}^\kappa,$$

which is the desired inequality. □

Lemma 3.2 [26] *Assume that* $a \geq 0$, $p \geq q \geq 0$, *and* $p \neq 0$, *then for any* $K > 0$

$$a^{\frac{q}{p}} \leq \frac{q}{p}K^{\frac{q-p}{p}}a + \frac{p-q}{p}K^{\frac{q}{p}}.$$

Lemma 3.3 *If* $\sup_{x\in\mathbb{T}} x = \infty$, $p(x,y) \in \mathfrak{R}$ *with respect to* x, *then*

$$\int_x^\infty f(s,y)e_f\big(x,\sigma(s)\big)\Delta s = 1 - e_f(x,\infty) = 1 - e_{\ominus f}(\infty,x). \tag{3.9}$$

Proof According to [25, Theorems 2.39 and 2.36(i)], we have

$$\int_x^{x_0} f(s,y)e_f\big(x,\sigma(s)\big)\Delta s = -\int_{x_0}^x f(s,y)e_f\big(x,\sigma(s)\big)\Delta s = 1 - e_f(x,x_0). \tag{3.10}$$

Then by Theorem 2.3(v) and after letting $x_0 \to \infty$, we obtain the desired result. □

Theorem 3.1 *Suppose* $\sup_{x\in\mathbb{T}^\kappa} x = \infty$, $u, a, f, g \in C_{\mathrm{rd}}(\mathbb{T}\times\mathbb{T}, \mathbb{R}_+)$, p *is a positive number with* $p \geq 1$. *If* $u(x,y)$ *satisfies the following inequality*:

$$u^p(x,y) \leq a(x,y) + b(x,y)\int_x^\infty \int_y^\infty \big[f(s,t)u(s,t) + g(s,t)\big]\Delta t\Delta s, \quad x,y\in\mathbb{T}^\kappa, \tag{3.11}$$

then

$$u(x,y) \leq \left\{a(x,y) + b(x,y)\left[H_1(x,y) + \int_x^\infty H_2(s,y)H_1(s,y)e_{-H_2}\big(x,\sigma(s)\big)\Delta s\right]\right\}^{\frac{1}{p}}, \\ x,y\in\mathbb{T}^\kappa, \tag{3.12}$$

provided that $1-\mu(x)H_2(x,y) > 0$, *where*

$$\begin{cases} H_1(x,y) = \int_x^\infty \int_y^\infty \{f(s,t)[\frac{1}{p}K^{\frac{1-p}{p}}a(s,t) + \frac{p-1}{p}K^{\frac{1}{p}}] + g(s,t)\}\Delta t\Delta s, \\ H_2(x,y) = \int_y^\infty f(x,t)\frac{1}{p}K^{\frac{1-p}{p}}b(x,t)\Delta t, \quad \forall K>0. \end{cases} \tag{3.13}$$

Proof Let

$$v(x,y) = \int_x^\infty \int_y^\infty \big[f(s,t)u(s,t) + g(s,t)\big]\Delta t\Delta s, \quad x,y\in\mathbb{T}^\kappa. \tag{3.14}$$

Then

$$u(x,y) \leq \big[a(x,y) + b(x,y)v(x,y)\big]^{\frac{1}{p}}, \quad x,y\in\mathbb{T}^\kappa. \tag{3.15}$$

On the other hand, from Lemma 3.2 we have

$$\big(a(x,y) + b(x,y)v(x,y)\big)^{\frac{1}{p}} \leq \frac{1}{p}K^{\frac{1-p}{p}}\big(a(x,y) + b(x,y)v(x,y)\big) + \frac{p-1}{p}K^{\frac{1}{p}}. \tag{3.16}$$

Combining (3.14), (3.15), and (3.16), we obtain

$$\begin{aligned} v(x,y) &\leq \int_x^\infty \int_y^\infty \big[f(s,t)\big(a(s,t) + b(s,t)v(s,t)\big)^{\frac{1}{p}} + g(s,t)\big]\Delta t\Delta s \\ &\leq \int_x^\infty \int_y^\infty \left\{f(s,t)\left[\frac{1}{p}K^{\frac{1-p}{p}}\big(a(s,t) + b(s,t)v(s,t)\big) + \frac{p-1}{p}K^{\frac{1}{p}}\right] + g(s,t)\right\}\Delta t\Delta s \\ &\leq \int_x^\infty \int_y^\infty \left\{f(s,t)\left[\frac{1}{p}K^{\frac{1-p}{p}}a(s,t) + \frac{p-1}{p}K^{\frac{1}{p}}\right] + g(s,t)\right\}\Delta t\Delta s \\ &\quad + \int_x^\infty \left[\int_y^\infty f(s,t)\frac{1}{p}K^{\frac{1-p}{p}}b(s,t)\Delta t\right]v(s,y)\Delta s \\ &= H_1(x,y) + \int_x^\infty H_2(s,y)v(s,y)\Delta s, \end{aligned} \tag{3.17}$$

where H_1, H_2 are defined in (3.13).

Let $z(x,y) = \int_x^\infty H_2(s,y)v(s,y)\Delta s$, then

$$v(x,y) \leq H_1(x,y) + z(x,y), \quad x,y\in\mathbb{T}^\kappa \tag{3.18}$$

and

$$\begin{aligned}\big(z(x,y)\big)_x^{\Delta} &= -H_2(x,y)v(x,y)\\ &\geq -H_2(x,y)\big[H_1(x,y)+z(x,y)\big]\\ &= -H_2(x,y)z(x,y)-H_2(x,y)H_1(x,y),\end{aligned} \tag{3.19}$$

where H_1, H_2 are defined in (3.13).

Since $1-\mu(x)H_2(x,y)>0$, then in fact $-H_2\in\mathfrak{R}^+$, and a suitable application of Lemma 3.1 to (3.19) yields

$$\begin{aligned}z(x,y) &\leq z(\infty,y)e_{\ominus-H_2}(\infty,x)\\ &\quad+\int_x^{\infty}H_2(s,y)H_1(s,y)e_{-H_2}\big(x,\sigma(s)\big)\Delta s,\quad x,y\in\mathbb{T}^{\kappa}.\end{aligned} \tag{3.20}$$

Since $z(\infty,y)=0$, then combining (3.18) and (3.20) gives

$$v(x,y)\leq H_1(x,y)+\int_x^{\infty}H_2(s,y)H_1(s,y)e_{-H_2}\big(x,\sigma(s)\big)\Delta s,\quad x,y\in\mathbb{T}^{\kappa}. \tag{3.21}$$

Combining (3.15) and (3.21), we can obtain the desired inequality (3.12). □

Since $\mathbb{T}$ is an arbitrary time scale, then if we take $\mathbb{T}$ for some peculiar cases, we can obtain some corollaries immediately. Especially, if we let $\mathbb{T}=\mathbb{R}$ or $\mathbb{T}=\mathbb{Z}$, we obtain the following two corollaries.

Corollary 3.1 *Suppose $\mathbb{T}=\mathbb{R}$, $u,a,f,g\in C(\mathbb{R}\times\mathbb{R},\mathbb{R}_+)$. If $u(x,y)$ satisfies the following inequality*:

$$u^p(x,y)\leq a(x,y)+b(x,y)\int_x^{\infty}\int_y^{\infty}\big[f(s,t)u(s,t)+g(s,t)\big]\,dt\,ds,\quad x,y\in\mathbb{R}, \tag{3.22}$$

then

$$\begin{aligned}u(x,y)\leq{}&\left\{a(x,y)+b(x,y)\left[H_1(x,y)+\int_x^{\infty}H_2(s,y)H_1(s,y)\exp\left(\int_x^s H_2(\tau,y)\,d\tau\right)\Delta s\right]\right\}^{\frac{1}{p}},\\ &x,y\in\mathbb{R},\end{aligned} \tag{3.23}$$

where

$$\begin{cases}H_1(x,y)=\int_x^{\infty}\int_y^{\infty}\{f(s,t)[\frac{1}{p}K^{\frac{1-p}{p}}a(s,t)+\frac{p-1}{p}K^{\frac{1}{p}}]+g(s,t)\}\,dt\,ds,\\ H_2(x,y)=\int_y^{\infty}f(x,t)\frac{1}{p}K^{\frac{1-p}{p}}b(x,t)\,dt,\quad \forall K>0.\end{cases} \tag{3.24}$$

Corollary 3.2 *Suppose $\mathbb{T}=\mathbb{Z}$ and $u,a,f,g\in(\mathbb{Z}\times\mathbb{Z},\mathbb{R}_+)$. If $u(m,n)$ satisfies the following inequality*:

$$u(m,n)\leq a(m,n)+\sum_{s=m}^{\infty}\sum_{t=n}^{\infty}\big[f(s,t)u(s,t)+g(s,t)\big],\quad m,n\in\mathbb{Z}, \tag{3.25}$$

then

$$u(m,n) \le \left\{ a(m,n) + b(m,n)\left[H_1(m,n) + \sum_{s=m}^{\infty} H_2(s,n)H_1(s,n) \prod_{\tau=m}^{s} \frac{1}{1-H_2(\tau,n)} \right] \right\}^{\frac{1}{p}},$$
$$m,n \in \mathbb{Z}, \tag{3.26}$$

provided that $H_2(m,n) < 1$, $m,n \in \mathbb{Z}$, *where*

$$\begin{cases} H_1(m,n) = \sum_{s=m}^{\infty}\sum_{t=n}^{\infty}\{f(s,t)[\frac{1}{p}K^{\frac{1-p}{p}}a(s,t) + \frac{p-1}{p}K^{\frac{1}{p}}] + g(s,t)\}, \\ H_2(m,n) = \sum_{t=n}^{\infty} f(m,t)\frac{1}{p}K^{\frac{1-p}{p}}b(m,t), \quad \forall K > 0. \end{cases} \tag{3.27}$$

Theorem 3.2 *Under the conditions of Theorem* 3.1, *if* $u(x,y)$ *satisfies* (3.11), *then*

$$u(x,y) \le \{a(x,y) + b(x,y)H_1(x,y)e_{\ominus(-H_2)}(\infty,x)\}^{\frac{1}{p}}, \quad x,y \in \mathbb{T}^{\kappa}, \tag{3.28}$$

where H_1, H_2 *are the same as in Theorem* 3.1.

Proof By Lemma 3.3 we have

$$\int_x^{\infty} H_2(s,y)e_{-H_2}(x,\sigma(s))\Delta s = e_{-H_2}(x,\infty) - 1 = e_{\ominus(-H_2)}(\infty,x) - 1. \tag{3.29}$$

On the other hand, considering $H_1(x,y)$ is decreasing in x, from (3.12) we have

$$\begin{aligned} u(x,y) &\le \left\{ a(x,y) + b(x,y)\left[H_1(x,y) + \int_x^{\infty} H_2(s,y)H_1(s,y)e_{-H_2}(x,\sigma(s))\Delta s \right] \right\}^{\frac{1}{p}} \\ &\le \left\{ a(x,y) + b(x,y)H_1(x,y)\left[1 + \int_x^{\infty} H_2(s,y)e_{-H_2}(x,\sigma(s))\Delta s \right] \right\}^{\frac{1}{p}} \\ &= \{a(x,y) + b(x,y)H_1(x,y)e_{\ominus(-H_2)}(\infty,x)\}^{\frac{1}{p}}, \end{aligned}$$

which is the desired result. □

If we let $\mathbb{T} = \mathbb{R}$ or $\mathbb{T} = \mathbb{Z}$ in Theorem 3.2, then we obtain the following two corollaries.

Corollary 3.3 *Under the conditions of Corollary* 3.1, *if* $u(x,y)$ *satisfies* (3.22), *then*

$$u(x,y) \le \left\{ a(x,y) + b(x,y)H_1(x,y)\exp\left(\int_x^{\infty} H_2(s,y)\,ds \right) \right\}^{\frac{1}{p}}, \quad x,y \in \mathbb{R}, \tag{3.30}$$

where H_1, H_2 *are defined the same as in* (3.22).

Corollary 3.4 *Under the conditions of Corollary* 3.2, *if* $u(m,n)$ *satisfies* (3.25), *then*

$$u(m,n) \le \left\{ a(m,n) + b(m,n)H_1(m,n)\prod_{s=m}^{\infty}\frac{1}{1-H_2(s,n)} \right\}^{\frac{1}{p}}, \quad m,n \in \mathbb{Z}, \tag{3.31}$$

provided that $H_2(m,n) < 1$, $m,n \in \mathbb{Z}$, *where* H_1, H_2 *are defined the same as in* (3.27).

Remark 3.1 Corollary 3.3 is equivalent to [21, Theorem 2], while Corollary 3.4 is equivalent to [22, Theorem 2] with a slight difference.

Theorem 3.3 *Suppose* $\sup_{x\in\mathbb{T}^\kappa} x = \infty$, $u, a, f, g, h \in C_{\mathrm{rd}}(\mathbb{T}\times\mathbb{T}, \mathbb{R}_+)$, *$p$ is a positive number with* $p \geq 1$. *If* $u(x,y)$ *satisfies the following inequality*:

$$\begin{aligned} u^p(x,y) \leq{}& a(x,y) + \int_x^\infty f(s,y)u^p(s,y)\Delta s \\ &+ \int_x^\infty \int_y^\infty \bigl[g(s,t)u(s,t) + h(s,t)\bigr]\Delta t\Delta s, \quad x,y\in\mathbb{T}^\kappa, \end{aligned} \tag{3.32}$$

then

$$\begin{aligned} u(x,y) \leq{}& \Bigl\{\Bigl[a(x,y) + \widetilde{H}_1(x,y) + \int_x^\infty \widetilde{H}_2(s,y)\widetilde{H}_1(s,y)e_{-\widetilde{H}_2}\bigl(x,\sigma(s)\bigr)\Delta s\Bigr] \\ &\times e_{\ominus(-f)}(\infty,x)\Bigr\}^{\frac{1}{p}}, \end{aligned} \tag{3.33}$$

provided that $-f \in \mathfrak{R}^+$, *where*

$$\begin{cases} \widetilde{H}_1(x,y) = \int_x^\infty \int_y^\infty \{g(s,t)[e_{\ominus(-f)}(\infty,s)]^{\frac{1}{p}} \\ \qquad \times [\frac{1}{p}K^{\frac{1-p}{p}}a(s,t) + \frac{p-1}{p}K^{\frac{1}{p}}] + h(s,t)\}\Delta t\Delta s, \\ \widetilde{H}_2(x,y) = \int_y^\infty g(x,t)[e_{\ominus(-f)}(\infty,x)]^{\frac{1}{p}}\frac{1}{p}K^{\frac{1-p}{p}}\Delta t, \quad \forall K > 0. \end{cases} \tag{3.34}$$

Proof Let

$$v(x,y) = a(x,y) + \int_x^\infty \int_y^\infty \bigl[g(s,t)u(s,t) + h(s,t)\bigr]\Delta t\Delta s \tag{3.35}$$

and

$$z(x,y) = \int_x^\infty f(s,y)u^p(s,y)\Delta s. \tag{3.36}$$

Then

$$u^p(x,y) \leq v(x,y) + z(x,y), \quad x,y\in\mathbb{T}^\kappa. \tag{3.37}$$

Furthermore,

$$\begin{aligned} \bigl(z(x,y)\bigr)_x^\Delta &= -f(x,y)u^p(x,y) \\ &\geq -f(x,y)\bigl[v(x,y) + z(x,y)\bigr] \\ &= -f(x,y)z(x,y) - f(x,y)v(x,y). \end{aligned}$$

Since $1 - \mu(x)f(x,y) > 0$, then $-f \in \mathfrak{R}^+$, and an application of Lemma 3.1 yields

$$z(x,y) \leq z(\infty,y)e_{\ominus -f}(\infty,x) + \int_x^\infty f(s,y)v(s,y)e_{-f}\bigl(x,\sigma(s)\bigr)\Delta s, \quad x,y\in\mathbb{T}^\kappa. \tag{3.38}$$

Considering $z(\infty, y) = 0$, and $v(x,y)$ is decreasing in x, combining (3.37) and (3.38), we obtain

$$\begin{aligned}
u^p(x,y) &\le v(x,y) + z(x,y) \\
&\le v(x,y) + z(\infty,y)e_{\ominus -f}(\infty,x) + \int_x^\infty f(s,y)v(s,y)e_{-f}\big(x,\sigma(s)\big)\Delta s \\
&= v(x,y) + \int_x^\infty f(s,y)v(s,y)e_{-f}\big(x,\sigma(s)\big)\Delta s \\
&\le v(x,y)\left[1 + \int_x^\infty f(s,y)e_{-f}\big(x,\sigma(s)\big)\Delta s\right] \\
&= v(x,y)e_{\ominus(-f)}(\infty,x), \quad x,y \in \mathbb{T}^\kappa, \qquad (3.39)
\end{aligned}$$

where Lemma 3.3 is used in the last step.

By (3.35), (3.39) and Lemma 3.2, we have

$$\begin{aligned}
v(x,y) &\le a(x,y) + \int_x^\infty \int_y^\infty \big\{g(s,t)\big[v(s,t)e_{\ominus(-f)}(\infty,s)\big]^{\frac{1}{p}} + h(s,t)\big\}\Delta t \Delta s \\
&\le a(x,y) + \int_x^\infty \int_y^\infty \bigg\{g(s,t)\big[e_{\ominus(-f)}(\infty,s)\big]^{\frac{1}{p}} \\
&\quad \times \left[\frac{1}{p}K^{\frac{1-p}{p}}v(s,t) + \frac{p-1}{p}K^{\frac{1}{p}}\right] + h(s,t)\bigg\}\Delta t \Delta s \\
&= a(x,y) + w(x,y), \qquad (3.40)
\end{aligned}$$

where

$$\begin{aligned}
w(x,y) = \int_x^\infty \int_y^\infty &\bigg\{g(s,t)\big[e_{\ominus(-f)}(\infty,s)\big]^{\frac{1}{p}} \\
&\times \left[\frac{1}{p}K^{\frac{1-p}{p}}v(s,t) + \frac{p-1}{p}K^{\frac{1}{p}}\right] + h(s,t)\bigg\}\Delta t \Delta s. \qquad (3.41)
\end{aligned}$$

Considering $w(x,y)$ is decreasing in y, it follows that

$$\begin{aligned}
w(x,y) &\le \int_x^\infty \int_y^\infty \bigg\{g(s,t)\big[e_{\ominus(-f)}(\infty,s)\big]^{\frac{1}{p}} \\
&\quad \times \left[\frac{1}{p}K^{\frac{1-p}{p}}\big(a(s,t) + w(s,t)\big) + \frac{p-1}{p}K^{\frac{1}{p}}\right] + h(s,t)\bigg\}\Delta t \Delta s \\
&\le \int_x^\infty \int_y^\infty \bigg\{g(s,t)\big[e_{\ominus(-f)}(\infty,s)\big]^{\frac{1}{p}}\left[\frac{1}{p}K^{\frac{1-p}{p}}a(s,t) + \frac{p-1}{p}K^{\frac{1}{p}}\right] + h(s,t)\bigg\}\Delta t \Delta s \\
&\quad + \int_x^\infty \bigg\{\int_y^\infty g(s,t)\big[e_{\ominus(-f)}(\infty,s)\big]^{\frac{1}{p}}\frac{1}{p}K^{\frac{1-p}{p}}\Delta t\bigg\}w(s,y)\Delta s \\
&= \widetilde{H}_1(x,y) + \int_x^\infty \widetilde{H}_2(s,y)w(s,y)\Delta s, \qquad (3.42)
\end{aligned}$$

where $\widetilde{H}_1$, $\widetilde{H}_2$ are defined in (3.34).

We notice that the structure of (3.42) is similar to that of (3.17). So, following in a similar manner to the process of (3.17)-(3.21), we obtain

$$w(x,y) \le \widetilde{H}_1(x,y) + \int_x^\infty \widetilde{H}_2(s,y)\widetilde{H}_1(s,y)e_{-\widetilde{H}_2}\big(x,\sigma(s)\big)\Delta s, \quad x,y \in \mathbb{T}^\kappa. \tag{3.43}$$

Combining (3.39), (3.40), and (3.43), we get the desired inequality (3.33). □

Theorem 3.4 *Under the conditions of Theorem* 3.3, *if* $u(x,y)$ *satisfies* (3.32), *then*

$$u(x,y) \le \big\{\big[a(x,y) + \widetilde{H}_1(x,y)e_{\ominus(-\widetilde{H}_2)}(\infty,x)\big]e_{\ominus(-f)}(\infty,x)\big\}^{\frac{1}{p}}, \quad x,y \in \mathbb{T}^\kappa, \tag{3.44}$$

where $\widetilde{H}_1$, $\widetilde{H}_2$ *are defined the same as in* (3.34).

The proof of Theorem 3.4 is similar to that of Theorem 3.2, and we omit it here.

Corollary 3.5 *Suppose* $\mathbb{T} = \mathbb{R}$, $u, a, f, g \in C(\mathbb{R} \times \mathbb{R}, \mathbb{R}_+)$. *If* $u(x,y)$ *satisfies the following inequality*:

$$\begin{aligned} u^p(x,y) \le{}& a(x,y) + \int_x^\infty f(s,y)u^p(s,y)\,ds \\ &+ \int_x^\infty \int_y^\infty \big[g(s,t)u(s,t) + h(s,t)\big]\,dt\,ds, \quad x,y \in \mathbb{R}, \end{aligned} \tag{3.45}$$

then

$$u(x,y) \le \left\{\left[a(x,y) + \widetilde{H}_1(x,y)\exp\left(\int_x^\infty \widetilde{H}_2(s,y)\,ds\right)\right]\exp\left(\int_x^\infty f(s,y)\,ds\right)\right\}^{\frac{1}{p}}, \tag{3.46}$$

where

$$\begin{cases} \widetilde{H}_1(x,y) = \int_x^\infty \int_y^\infty \{g(s,t)[\exp(\int_s^\infty f(\tau,y)\,d\tau)]^{\frac{1}{p}} \\ \qquad \times [\frac{1}{p}K^{\frac{1-p}{p}}a(s,t) + \frac{p-1}{p}K^{\frac{1}{p}}] + h(s,t)\}\,dt\,ds, \\ \widetilde{H}_2(x,y) = \int_y^\infty g(x,t)[\exp(\int_x^\infty f(s,y)\,ds)]^{\frac{1}{p}}\frac{1}{p}K^{\frac{1-p}{p}}\,dt, \quad \forall K > 0. \end{cases} \tag{3.47}$$

The proof for Corollary 3.5 is similar to that for Corollary 3.1.

Remark 3.2 Corollary 3.5 is equivalent to [21, Theorem 4].

Remark 3.3 If we take $\mathbb{T} = \mathbb{R}$ in Theorem 3.3, or $\mathbb{T} = \mathbb{Z}$ in Theorems 3.3 and 3.4, then we can obtain another three corollaries, which are omitted here. Especially, if we take $\mathbb{T} = \mathbb{Z}$ in Theorem 3.4, then Theorem 3.4 reduces to [22, Theorem 4] with a slight difference, which is one case of discrete inequality.

Remark 3.4 The inequalities with two independent variables established in (3.11) and (3.32) are essentially different from the cases with a single variable. As these inequalities can be used in deriving bounds for solutions of certain dynamic equations in two independent variables, which are shown in Section 4, while inequalities with a single variable can only be used to the boundedness analysis for solutions of certain dynamic equations in a single variable.

Remark 3.5 The inequalities established above are related to infinite intervals. The main difference between the results here and those of finite intervals lies in their applications. The results above are valid in the qualitative analysis of certain dynamic equations including integrals on infinite intervals, which are shown in the two examples in Section 4, while results of finite intervals are invalid in such cases.

4 Some applications

In this section, we present some applications for the results established above.

Example 1 Consider the following dynamic equation:

$$\bigl(u^p(x,y)\bigr)_x^{\Delta} = -\int_y^{\infty} F\bigl(s,t,u(s,t)\bigr)\Delta t, \quad x,y \in \mathbb{T}^{\kappa}, \tag{4.1}$$

with the condition $u(\infty,y) = \varphi(y)$, where $u \in C_{\mathrm{rd}}(\mathbb{T} \times \mathbb{T}, \mathbb{R})$, $\varphi \in C_{\mathrm{rd}}(\mathbb{T}, \mathbb{R})$, and p is a constant with $p \geq 1$.

Theorem 4.1 *If $u(x,y)$ is a solution of* (4.1), *and suppose $|F(s,t,u)| \leq f(s,t)|u| + g(s,t)$, where $f,g \in C_{\mathrm{rd}}(\mathbb{T} \times \mathbb{T}, \mathbb{R}_+)$, then we have*

$$\begin{aligned} u(x,y) \leq {}& \biggl\{ \bigl|\varphi(y)\bigr| + \biggl[H_1(x,y) + \int_x^{\infty} H_2(s,y)H_1(s,y)e_{-H_2}\bigl(x,\sigma(s)\bigr)\Delta s \biggr] \biggr\}^{\frac{1}{p}}, \\ & x,y \in \mathbb{T}^{\kappa}, \end{aligned} \tag{4.2}$$

provided that $1 - \mu(x)H_2(x,y) > 0$, *where*

$$\begin{cases} H_1(x,y) = \int_x^{\infty}\int_y^{\infty} \{ f(s,t)[\frac{1}{p}K^{\frac{1-p}{p}}|\varphi(t)| + \frac{p-1}{p}K^{\frac{1}{p}}] + g(s,t)\}\Delta t\Delta s, \\ H_2(x,y) = \int_y^{\infty} f(x,t)\frac{1}{p}K^{\frac{1-p}{p}}\Delta t, \quad \forall K > 0. \end{cases} \tag{4.3}$$

Proof Considering $u(\infty,y) = \varphi(y)$, then the equivalent integral equation of (4.1) can be denoted by

$$u^p(x,y) = \varphi(y) + \int_x^{\infty}\int_y^{\infty} F\bigl(s,t,u(s,t)\bigr)\Delta t\Delta s, \quad x,y \in \mathbb{T}^{\kappa}. \tag{4.4}$$

So, we have

$$\begin{aligned} \bigl|u(x,y)\bigr|^p &\leq \bigl|\varphi(y)\bigr| + \int_x^{\infty}\int_y^{\infty} \bigl|F\bigl(s,t,u(s,t)\bigr)\bigr|\Delta t\Delta s \\ &\leq \bigl|\varphi(y)\bigr| + \int_x^{\infty}\int_y^{\infty} \bigl[f(s,t)\bigl|u(s,t)\bigr| + g(s,t)\bigr]\Delta t\Delta s. \end{aligned} \tag{4.5}$$

Then a suitable application of Theorem 3.1 yields the desired inequality (4.2). □

In the last step of Theorem 4.1, if we apply Theorem 3.2 to (4.5), then we can obtain the following theorem.

Theorem 4.2 *Under the conditions of Theorem* 4.1, *if* $u(x,y)$ *is a solution of* (4.1), *then*

$$u(x,y) \le \left\{ \left|\varphi(y)\right| + H_1(x,y) e_{\ominus(-H_2)}(\infty, x) \right\}^{\frac{1}{p}}, \quad x, y \in \mathbb{T}^\kappa, \tag{4.6}$$

where H_1, H_2 *are defined in* (4.3).

Theorem 4.3 *Suppose* $p = 1$ *in* (4.1) *and* $|F(s,t,u_1)| - |F(s,t,u_2)| \le f(s,t)|u_1 - u_2| + g(s,t)$, *where* f, g *are defined the same as in Theorem* 4.1, *then under the condition* $u(\infty, y) = \varphi(y)$, *Eq*. (4.1) *has at most one solution.*

Proof Let $u_1(x,y)$ and $u_2(x,y)$ be two solutions of Eq. (4.1). Then we have

$$\bigl(u_1(x,y)\bigr)_x^\Delta = -\int_y^\infty F\bigl(s,t,u_1(s,t)\bigr)\Delta t, \quad x, y \in \mathbb{T}^\kappa \tag{4.7}$$

and

$$\bigl(u_2(x,y)\bigr)_x^\Delta = -\int_y^\infty F\bigl(s,t,u_2(s,t)\bigr)\Delta t, \quad x, y \in \mathbb{T}^\kappa. \tag{4.8}$$

From (4.7) and (4.8), we obtain

$$\bigl(u_1(x,y) - u_2(x,y)\bigr)_x^\Delta = -\int_y^\infty \bigl[F\bigl(s,t,u_1(s,t)\bigr) - F\bigl(s,t,u_2(s,t)\bigr)\bigr]\Delta t, \quad x, y \in \mathbb{T}^\kappa. \tag{4.9}$$

On the other hand, since $u_1(\infty, y) = u_2(\infty, y) = \varphi(y)$, then an integration for (4.9) with respect to x from x to ∞ yields

$$u_1(x,y) - u_2(x,y) = \int_x^\infty \int_y^\infty \bigl[F\bigl(s,t,u_1(s,t)\bigr) - F\bigl(s,t,u_2(s,t)\bigr)\bigr]\Delta t \Delta s, \quad x, y \in \mathbb{T}^\kappa. \tag{4.10}$$

Then

$$\begin{aligned} \left|u_1(x,y) - u_2(x,y)\right| &\le \int_x^\infty \int_y^\infty \left|F\bigl(s,t,u_1(s,t)\bigr) - F\bigl(s,t,u_2(s,t)\bigr)\right|\Delta t \Delta s \\ &\le \int_x^\infty \int_y^\infty \bigl[f(s,t)\left|u_1(s,t) - u_2(s,t)\right| + g(s,t)\bigr]\Delta t \Delta s. \end{aligned} \tag{4.11}$$

A suitable application of Theorem 3.1 yields

$$\left|u_1(x,y) - u_2(x,y)\right| \le 0,$$

that is, $u_1(x,y) \equiv u_2(x,y)$, and the proof is complete. □

Example 2 Consider the following dynamic equation:

$$\begin{aligned} u^p(x,y) = \varphi(y) &+ \int_x^\infty F_1\bigl(s,y,u(s,y)\bigr)\Delta s \\ &+ \int_x^\infty \int_y^\infty F_2\bigl(s,t,u(s,t)\bigr)\Delta t \Delta s, \quad x, y \in \mathbb{T}^\kappa, \end{aligned} \tag{4.12}$$

where $u \in C_{\mathrm{rd}}(\mathbb{T} \times \mathbb{T}, \mathbb{R})$, $\varphi \in C_{\mathrm{rd}}(\mathbb{T}, \mathbb{R})$, and $p \ge 1$ is a constant.

Theorem 4.4 *Let $u(x,y)$ be a solution of* (4.12). *If* $|F_1(x,y,u)| \le f(x,y)|u|^p$, $|F_2(x,y,z)| \le g(x,y)|z| + h(x,y)$, *where* $f,g,h \in C_{\mathrm{rd}}(\mathbb{T} \times \widetilde{\mathbb{T}}, \mathbb{R}_+)$, *then the following estimate holds*:

$$u(x,y) \le \left\{ \left[\left|\varphi(y)\right| + \widetilde{H}_1(x,y) + \int_x^\infty \widetilde{H}_2(s,y)\widetilde{H}_1(s,y)e_{-\widetilde{H}_2}\left(x,\sigma(s)\right)\Delta s \right] e_{\ominus(-f)}(\infty,x) \right\}^{\frac{1}{p}},$$
$$x,y \in \mathbb{T}^\kappa, \tag{4.13}$$

where

$$\begin{cases} \widetilde{H}_1(x,y) = \int_x^\infty \int_y^\infty \{g(s,t)[e_{\ominus(-f)}(\infty,s)]^{\frac{1}{p}} \\ \qquad \times [\frac{1}{p}K^{\frac{1-p}{p}}|\varphi(t)| + \frac{p-1}{p}K^{\frac{1}{p}}] + h(s,t)\}\Delta t\Delta s, \\ \widetilde{H}_2(x,y) = \int_y^\infty g(x,t)[e_{\ominus(-f)}(\infty,x)]^{\frac{1}{p}} \frac{1}{p}K^{\frac{1-p}{p}}\Delta t, \quad \forall K > 0. \end{cases} \tag{4.14}$$

Proof From (4.12) we have

$$\begin{aligned} \left|u^p(x,y)\right| &\le \left|\varphi(y)\right| + \int_x^\infty \left|F_1\left(s,y,u(s,y)\right)\right|\Delta s + \int_x^\infty \int_y^\infty \left|F_2\left(s,t,u(s,t)\right)\right|\Delta t\Delta s \\ &\le \left|\varphi(y)\right| + \int_x^\infty f(s,y)\left|u(s,y)\right|^p \Delta s \\ &\quad + \int_x^\infty \int_y^\infty \left[g(s,t)\left|u(s,t)\right| + h(s,t)\right]\Delta t\Delta s, \quad x,y \in \mathbb{T}^\kappa. \end{aligned} \tag{4.15}$$

Then using Theorem 3.3 in (4.15), we can obtain the desired result. □

5 Conclusions

We have established several new Gronwall-Bellman-type integral inequalities containing integration on infinite intervals on time scales, which unify some known continuous and discrete results in the literature. As one can see from the present applications, the inequalities established are useful in the investigation of qualitative and quantitative properties of solutions of certain dynamic equations on time scales.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
QF and FX carried out the main part of this article. All authors read and approved the final manuscript.

Acknowledgements
The authors would like to thank the referees very much for their valuable suggestions on improving this paper.

Received: 30 January 2013 Accepted: 23 April 2013 Published: 8 May 2013

References
1. Gronwall, TH: Note on the derivatives with respect to a parameter of solutions of a system of differential equations. Ann. Math. **20**, 292-296 (1919)
2. Bellman, R: The stability of solutions of linear differential equations. Duke Math. J. **10**, 643-647 (1943)
3. Ou-lang, L: The boundedness of solutions of linear differential equations $y'' + A(t)y' = 0$. Shuxue Jinzhan **3**, 409-418 (1957)
4. Li, WN, Han, MA, Meng, FW: Some new delay integral inequalities and their applications. J. Comput. Appl. Math. **180**, 191-200 (2005)
5. Lipovan, O: A retarded integral inequality and its applications. J. Math. Anal. Appl. **285**, 436-443 (2003)
6. Ma, QH, Yang, EH: Some new Gronwall-Bellman-Bihari type integral inequalities with delay. Period. Math. Hung. **44**(2), 225-238 (2002)

7. Cheung, WS, Ren, JL: Discrete non-linear inequalities and applications to boundary value problems. J. Math. Anal. Appl. **319**, 708-724 (2006)
8. Ma, QH: Estimates on some power nonlinear Volterra-Fredholm type discrete inequalities and their applications. J. Comput. Appl. Math. **233**, 2170-2180 (2010)
9. Li, LZ, Meng, FW, He, LL: Some generalized integral inequalities and their applications. J. Math. Anal. Appl. **372**, 339-349 (2010)
10. Lipovan, O: Integral inequalities for retarded Volterra equations. J. Math. Anal. Appl. **322**, 349-358 (2006)
11. Hilger, S: Analysis on measure chains - a unified approach to continuous and discrete calculus. Results Math. **18**, 18-56 (1990)
12. Bohner, M, Erbe, L, Peterson, A: Oscillation for nonlinear second order dynamic equations on a time scale. J. Math. Anal. Appl. **301**, 491-507 (2005)
13. Xing, Y, Han, M, Zheng, G: Initial value problem for first-order integro-differential equation of Volterra type on time scales. Nonlinear Anal. TMA **60**(3), 429-442 (2005)
14. Agarwal, RP, Bohner, M, O'Regan, D, Peterson, A: Dynamic equations on time scales: a survey. J. Comput. Appl. Math. **141**(1-2), 1-26 (2006)
15. Liu, WJ, Li, CC, Hao, YM: Further generalization of some double integral inequalities and applications. Acta Math. Univ. Comen. **77**(1), 147-154 (2008)
16. Cheng, XL: Improvement of some Ostrowski-Grüss type inequalities. Comput. Math. Appl. **42**, 109-114 (2001)
17. Bohner, M, Matthews, T: The Grüss inequality on time scales. Commun. Math. Anal. **3**(1), 1-8 (2007)
18. Ngô, QA: Some mean value theorems for integrals on time scales. Appl. Math. Comput. **213**, 322-328 (2009)
19. Liu, WJ, Ngô, QA: Some Iyengar-type inequalities on time scales for functions whose second derivatives are bounded. Appl. Math. Comput. **216**, 3244-3251 (2010)
20. Liu, WJ, Ngô, QA: A generalization of Ostrowski inequality on time scales for *k* points. Appl. Math. Comput. **203**, 754-760 (2008)
21. Meng, FW, Li, WN: On some new integral inequalities and their applications. Appl. Math. Comput. **148**, 381-392 (2004)
22. Meng, FW, Li, WN: On some new nonlinear discrete inequalities and their applications. J. Comput. Appl. Math. **158**, 407-417 (2003)
23. Agarwal, PR, Bohner, M, Peterson, A: Inequalities on time scales: a survey. Math. Inequal. Appl. **4**(4), 535-557 (2001)
24. Li, WN: Some new dynamic inequalities on time scales. J. Math. Anal. Appl. **319**, 802-814 (2006)
25. Bohner, M, Peterson, A: Dynamic Equations on Time Scales: An Introduction with Applications. Birkhäuser, Boston (2001)
26. Jiang, FC, Meng, FW: Explicit bounds on some new nonlinear integral inequality with delay. J. Comput. Appl. Math. **205**, 479-486 (2007)

doi:10.1186/1029-242X-2013-234
Cite this article as: Wang et al.: **Some new generalized Gronwall-Bellman-type integral inequalities in two independent variables on time scales.** *Journal of Inequalities and Applications* 2013 **2013**:234.