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On a new Hardy-Mulholland-type inequality and its more accurate form

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Abstract

Using weight coefficients and applying the well-known Hermite-Hadamard inequality, a new Hardy-Mulholand-type inequality with a best possible constant factor is given. Furthermore, we also consider the more accurate equivalent forms, the operator expressions and some particular inequalities. The lemmas and theorems provide an extensive account of this type of inequalities.

MSC: 26D15; 47A07

Keywords: Hardy-Mulholand-type inequality; weight coefficient; equivalent form; reverse; operator

1 Introduction

Assuming that $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $a_m, b_n \geq 0$, $a = \{a_m\}_{m=1}^{\infty} \in l^p$, $b = \{b_n\}_{n=1}^{\infty} \in l^q$, $\|a\|_p = (\sum_{m=1}^{\infty} a_m^p)^{\frac{1}{p}} > 0$, $\|b\|_q > 0$, we have the following Hardy-Hilbert inequality with the best possible constant factor $\frac{\pi}{\sin(\pi/p)}$ (*cf.* [1], Theorem 315):

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\pi/p)} \|a\|_p \|b\|_q. \tag{1}$$

The more accurate inequality of (1) is given as follows (*cf.* [2] and Theorem 323 of [1]):

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m b_n}{m+n-\alpha} < \frac{\pi}{\sin(\pi/p)} \|a\|_p \|b\|_q \quad (0 \leq \alpha \leq 1), \tag{2}$$

which is an extension of (1). We still have the following Mulholland inequality similar to (1) with the same best possible constant factor $\frac{\pi}{\sin(\pi/p)}$ (*cf.* [3] or Theorem 343 of [1], replacing $\frac{a_m}{n}$, $\frac{b_n}{n}$ by a_m, b_n):

$$\sum_{m=2}^{\infty} \sum_{n=2}^{\infty} \frac{a_m b_n}{\ln mn} < \frac{\pi}{\sin(\pi/p)} \left(\sum_{m=2}^{\infty} \frac{a_m^p}{m^{1-p}} \right)^{\frac{1}{p}} \left(\sum_{n=2}^{\infty} \frac{b_n^q}{n^{1-q}} \right)^{\frac{1}{q}}. \tag{3}$$

Inequalities (1)-(3) are important in analysis and applications (*cf.* [2, 4–9]).

If $\mu_i, \upsilon_j > 0$ $(i,j \in \mathbf{N} = \{1,2,\ldots\})$,

$$U_m := \sum_{i=1}^{m} \mu_i, \qquad V_n := \sum_{j=1}^{n} \upsilon_j \quad (m,n \in \mathbf{N}), \tag{4}$$

then we have the following Hardy-Hilbert-type inequality (*cf.* Theorem 321 of [1], replacing $\mu_m^{1/q} a_m$ and $\upsilon_n^{1/p} b_n$ by a_m and b_n):

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{a_m b_n}{U_m + V_n} < \frac{\pi}{\sin(\frac{\pi}{p})} \left(\sum_{m=1}^{\infty} \frac{a_m^p}{\mu_m^{p-1}}\right)^{\frac{1}{p}} \left(\sum_{n=1}^{\infty} \frac{b_n^q}{\upsilon_n^{q-1}}\right)^{\frac{1}{q}}. \tag{5}$$

For $\mu_i = \upsilon_j = 1$ $(i,j \in \mathbf{N})$, (5) reduces to (1).

In 2015, Yang [10] gave an extension of (5) as follows: For $0 < \lambda_1, \lambda_2 \leq 1$, $\lambda_1 + \lambda_2 = \lambda$, $\{\mu_m\}_{m=1}^{\infty}$, and $\{\upsilon_n\}_{n=1}^{\infty}$ are decreasing, and $U_\infty = V_\infty = \infty$, we have the following inequality with the best possible constant factor $B(\lambda_1,\lambda_2)$:

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{a_m b_n}{(U_m + V_n)^\lambda} < B(\lambda_1,\lambda_2) \left[\sum_{m=1}^{\infty} \frac{U_m^{p(1-\lambda_1)-1} a_m^p}{\mu_m^{p-1}}\right]^{\frac{1}{p}} \left[\sum_{n=1}^{\infty} \frac{V_n^{q(1-\lambda_2-1)} b_n^q}{\upsilon_n^{q-1}}\right]^{\frac{1}{q}}, \tag{6}$$

where $B(u,v)$ is the beta function defined by (*cf.* [11])

$$B(u,v) := \int_0^\infty \frac{t^{u-1}}{(1+t)^{u+v}}\,dt \quad (u,v > 0). \tag{7}$$

In a similar way, Huang and Yang [12] gave a more accurate inequality of (6) and Yang and Chen [13] obtained a Hardy-Hilbert-type inequality with another kernel and a best possible constant factor.

In this paper, using the way of weight coefficients and applying Hermite-Hadamard's inequality, a Hardy-Mulholland-type inequality with a best possible constant factor similar to (6) is proved, which is an extension of (3). Furthermore, the more accurate Hardy-Mulholland-type inequality is built by introducing a few parameters. We also consider the equivalent forms, the operator expressions and some particular inequalities.

2 Some lemmas and an example

In the following of this paper, we assume that $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\mu_i, \upsilon_j > 0$ $(i,j \in \mathbf{N})$, with $\mu_1 = \upsilon_1 = 1$, U_m and V_n are indicated by (4), $\alpha \leq \frac{\mu_2}{2}$, $\beta \leq \frac{\upsilon_2}{2}$, $0 < \lambda_1, \lambda_2 \leq 1$, $\lambda_1 + \lambda_2 = \lambda$, $a_m, b_n \geq 0$, $\|a\|_{p,\Phi_\lambda} := (\sum_{m=2}^{\infty} \Phi_\lambda(m) a_m^p)^{\frac{1}{p}}$, and $\|b\|_{q,\Psi_\lambda} := (\sum_{n=2}^{\infty} \Psi_\lambda(n) b_n^q)^{\frac{1}{q}}$, where

$$\begin{aligned}
\Phi_\lambda(m) &:= \frac{[\ln(U_m - \alpha)]^{p(1-\lambda_1)-1}}{(U_m - \alpha)^{1-p} \mu_{m+1}^{p-1}}, \\
\Psi_\lambda(n) &:= \frac{[\ln(V_n - \beta)]^{q(1-\lambda_2)-1}}{(V_n - \beta)^{1-q} \upsilon_{n+1}^{q-1}} \quad \bigl(m,n \in \mathbf{N}\backslash\{1\}\bigr).
\end{aligned} \tag{8}$$

Lemma 1 *Suppose that $a \in \mathbf{R}$, $f(x)$ in continuous in $[a - \frac{1}{2}, a + \frac{1}{2}]$, $f'(x)$ is strictly increasing in $(a - \frac{1}{2}, a)$ and $(a, a + \frac{1}{2})$, and $f'(a - 0) \leq f'(a + 0)$. We have the following Hermite-*

Hadamard inequality (*cf. Lemma* 1 *of* [14]):

$$f(a) < \int_{a-\frac{1}{2}}^{a+\frac{1}{2}} f(x)\,dx. \tag{9}$$

Example 1 Assuming that $\{\mu_m\}_{m=1}^{\infty}$ and $\{\upsilon_n\}_{n=1}^{\infty}$ are decreasing, we set $\mu(t) := \mu_m$, $t \in (m-1, m]$ $(m \in \mathbf{N})$; $\upsilon(t) := \upsilon_n$, $t \in (n-1, n]$ $(n \in \mathbf{N})$,

$$U(x) := \int_0^x \mu(t)\,dt \quad (x \geq 0), \qquad V(y) := \int_0^y \upsilon(t)\,dt \quad (y \geq 0). \tag{10}$$

Then we have $U(m) = U_m$, $V(n) = V_n$, $U(\infty) = U_\infty$, $V(\infty) = V_\infty$ and

$$U'(x) = \mu(x) = \mu_m, \quad x \in (m-1, m),$$
$$V'(y) = \upsilon(y) = \upsilon_n, \quad y \in (n-1, n)\ (m, n \in \mathbf{N}).$$

For fixed $m, n \in \mathbf{N}\backslash\{1\}$, we define the function $h(x)$ as follows:

$$h(x) := \frac{\ln^{\lambda_2 - 1}(V(x) - \beta)}{(V(x) - \beta)[\ln(U_m - \alpha) + \ln(V(x) - \beta)]^{\lambda}}, \quad x \in \left[n - \frac{1}{2}, n + \frac{1}{2}\right].$$

Then $h(x)$ in continuous in $[n - \frac{1}{2}, n + \frac{1}{2}]$, and, for $x \in (n - \frac{1}{2}, n)$ $(n \in \mathbf{N}\backslash\{1\})$,

$$\begin{aligned} h'(x) = {} & -\left\{ \frac{\ln^{\lambda_2 - 1}(V(x) - \beta)}{(V(x) - \beta)} + \frac{\lambda \ln^{\lambda_2 - 1}(V(x) - \beta)}{\ln[(U_m - \alpha)(V(x) - \beta)]} \right. \\ & \left. + \frac{1 - \lambda_2}{(V(x) - \beta)^{2 - \lambda_2}} \right\} \frac{\upsilon_n}{(V(x) - \beta)\ln^{\lambda}[(U_m - \alpha)(V(x) - \beta)]}. \end{aligned}$$

In view of $1 - \lambda_2 \geq 0$, $h'(x)$ (< 0) is strictly increasing in $(n - \frac{1}{2}, n)$ and

$$\begin{aligned} \lim_{x \to n-} h'(x) = h'(n - 0) = {} & -\left\{ \frac{\ln^{\lambda_2 - 1}(V_n - \beta)}{(V_n - \beta)} + \frac{\lambda \ln^{\lambda_2 - 1}(V_n - \beta)}{\ln[(U_m - \alpha)(V_n - \beta)]} \right. \\ & \left. + \frac{1 - \lambda_2}{(V_n - \beta)^{2 - \lambda_2}} \right\} \frac{\upsilon_n}{(V_n - \beta)\ln^{\lambda}[(U_m - \alpha)(V_n - \beta)]}. \end{aligned}$$

In the same way, for $x \in (n, n + \frac{1}{2})$, we find

$$\begin{aligned} h'(x) = {} & -\left\{ \frac{\ln^{\lambda_2 - 1}(V(x) - \beta)}{(V(x) - \beta)} + \frac{\lambda \ln^{\lambda_2 - 1}(V(x) - \beta)}{\ln[(U_m - \alpha)(V(x) - \beta)]} \right. \\ & \left. + \frac{1 - \lambda_2}{(V(x) - \beta)^{2 - \lambda_2}} \right\} \frac{\upsilon_{n+1}}{(V(x) - \beta)\ln^{\lambda}[(U_m - \alpha)(V(x) - \beta)]}, \end{aligned}$$

$h'(x)$ (< 0) is strictly increasing in $(n, n + \frac{1}{2})$ and

$$\begin{aligned} \lim_{x \to n+} h'(x) = h'(n + 0) = {} & -\left\{ \frac{\ln^{\lambda_2 - 1}(V_n - \beta)}{(V_n - \beta)} + \frac{\lambda \ln^{\lambda_2 - 1}(V_n - \beta)}{\ln[(U_m - \alpha)(V_n - \beta)]} \right. \\ & \left. + \frac{1 - \lambda_2}{(V_n - \beta)^{2 - \lambda_2}} \right\} \frac{\upsilon_{n+1}}{(V_n - \beta)\ln^{\lambda}[(U_m - \alpha)(V_n - \beta)]}. \end{aligned}$$

Since $\upsilon_{n+1} \leq \upsilon_n$, we have $h'(n-0) \leq h'(n+0)$. Then by (9), for $m, n \in \mathbf{N}\backslash\{1\}$, it follows that

$$h(n) < \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} h(x)\,dx = \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} \frac{\ln^{\lambda_2-1}(V(x)-\beta)}{(V(x)-\beta)[\ln(U_m-\alpha)+\ln(V(x)-\beta)]^\lambda}\,dx. \tag{11}$$

Lemma 2 *For $m, n \in \mathbf{N}\backslash\{1\}$, we define the following weight coefficients*:

$$\omega(\lambda_2, m) := \sum_{n=2}^{\infty} \frac{1}{\ln^\lambda[(U_m-\alpha)(V_n-\beta)]} \frac{\upsilon_{n+1}\ln^{\lambda_1}(U_m-\alpha)}{(V_n-\beta)\ln^{1-\lambda_2}(V_n-\beta)}, \tag{12}$$

$$\varpi(\lambda_1, n) := \sum_{m=2}^{\infty} \frac{1}{\ln^\lambda[(U_m-\alpha)(V_n-\beta)]} \frac{\mu_{m+1}\ln^{\lambda_2}(V_n-\beta)}{(U_m-\alpha)\ln^{1-\lambda_1}(U_m-\alpha)}. \tag{13}$$

If $\{\mu_m\}_{m=1}^\infty$ and $\{\upsilon_n\}_{n=1}^\infty$ are decreasing, and $U(\infty) = V(\infty) = \infty$, then

$$\omega(\lambda_2, m) < B(\lambda_1, \lambda_2) \quad \bigl(m \in \mathbf{N}\backslash\{1\}\bigr), \tag{14}$$

$$\varpi(\lambda_1, n) < B(\lambda_1, \lambda_2) \quad \bigl(n \in \mathbf{N}\backslash\{1\}\bigr). \tag{15}$$

Proof For $x \in (n-\frac{1}{2}, n+\frac{1}{2})\backslash\{n\}$, $\upsilon_{n+1} \leq V'(x)$, by (11), we obtain

$$\begin{aligned} \omega(\lambda_2, m) &< \sum_{n=2}^{\infty} \upsilon_{n+1} \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} \frac{\ln^{\lambda_1}(U_m-\alpha)\ln^{\lambda_2-1}(V(x)-\beta)\,dx}{(V(x)-\beta)[\ln(U_m-\alpha)+\ln(V(x)-\beta)]^\lambda} \\ &\leq \sum_{n=2}^{\infty} \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} \frac{\ln^{\lambda_1}(U_m-\alpha)\ln^{\lambda_2-1}(V(x)-\beta)}{[\ln(U_m-\alpha)+\ln(V(x)-\beta)]^\lambda} \frac{V'(x)}{V(x)-\beta}\,dx \\ &= \int_{\frac{3}{2}}^{\infty} \frac{\ln^{\lambda_1}(U_m-\alpha)\ln^{\lambda_2-1}(V(x)-\beta)}{[\ln(U_m-\alpha)+\ln(V(x)-\beta)]^\lambda} \frac{V'(x)}{V(x)-\beta}\,dx. \end{aligned}$$

Setting $t = \frac{\ln(V(x)-\beta)}{\ln(U_m-\alpha)}$, since $V(\frac{3}{2}) - \beta = 1 + \frac{\upsilon_2}{2} - \beta \geq 1$ and $\frac{V'(x)}{V(x)-\beta}\,dx = \ln(U_m-\alpha)\,dt$, we find

$$\omega(\lambda_2, m) < \int_0^\infty \frac{1}{(1+t)^\lambda} t^{\lambda_2-1}\,dt = B(\lambda_1, \lambda_2).$$

Hence, we obtain (14). In the same way, we obtain (15). □

Lemma 3 *Suppose that $\{\mu_m\}_{m=1}^\infty$ and $\{\upsilon_n\}_{n=1}^\infty$ are decreasing, and $U(\infty) = V(\infty) = \infty$.* (i) *For $m, n \in \mathbf{N}\backslash\{1\}$, we have*

$$B(\lambda_1, \lambda_2)\bigl(1 - \theta(\lambda_2, m)\bigr) < \omega(\lambda_2, m), \tag{16}$$

$$B(\lambda_1, \lambda_2)\bigl(1 - \vartheta(\lambda_1, n)\bigr) < \varpi(\lambda_1, n), \tag{17}$$

where

$$\begin{aligned} \theta(\lambda_2, m) &= \frac{1}{B(\lambda_1, \lambda_2)} \frac{\ln^{\lambda_2-1}(1+\upsilon_2-\beta)}{\lambda_2[1+\frac{\ln(1+\upsilon_2\theta(m)-\beta)}{\ln(U_m-\alpha)}]^\lambda} \frac{1}{\ln^{\lambda_2}(U_m-\alpha)} \\ &= O\biggl(\frac{1}{\ln^{\lambda_2}(U_m-\alpha)}\biggr) \in (0,1) \quad \biggl(\theta(m) \in \biggl(\frac{\beta}{\upsilon_2}, 1\biggr)\biggr), \end{aligned} \tag{18}$$

$$\begin{aligned}\vartheta(\lambda_1,n) &= \frac{1}{B(\lambda_1,\lambda_2)}\frac{\ln^{\lambda_1}(1+\mu_2-\alpha)}{\lambda_1[1+\frac{\ln(1+\mu_2\vartheta(n)-\alpha)}{\ln(V_n-\beta)}]^{\lambda}}\frac{1}{\ln^{\lambda_1}(V_n-\beta)}\\ &= O\biggl(\frac{1}{\ln^{\lambda_1}(V_n-\beta)}\biggr)\in(0,1)\quad\biggl(\vartheta(n)\in\biggl(\frac{\alpha}{\mu_2},1\biggr)\biggr);\end{aligned}\tag{19}$$

(ii) *for any* $c>0$, *we have*

$$\sum_{m=2}^{\infty}\frac{\mu_{m+1}}{(U_m-\alpha)\ln^{1+c}(U_m-\alpha)} = \frac{1}{c}\biggl(\frac{1}{\ln^{c}(1+\mu_2-\alpha)}+cO(1)\biggr),\tag{20}$$

$$\sum_{n=2}^{\infty}\frac{\upsilon_{n+1}}{(V_n-\beta)\ln^{1+c}(V_n-\beta)} = \frac{1}{c}\biggl(\frac{1}{\ln^{c}(1+\upsilon_2-\beta)}+c\widetilde{O}(1)\biggr).\tag{21}$$

Proof In view of $0\le\beta\le\frac{\upsilon_2}{2}<\upsilon_2$, it follows that $\frac{\beta}{\upsilon_2}+1\ge1$ and $\frac{\beta}{\upsilon_2}+1<2$. By Example 1, $h(x)$ is strictly decreasing in $[n,n+1]$, then for $m\in\mathbf{N}\backslash\{1\}$, we obtain

$$\begin{aligned}\omega(\lambda_2,m) &> \sum_{n=2}^{\infty}\int_n^{n+1}\frac{\ln^{\lambda_1}(U_m-\alpha)\ln^{\lambda_2-1}(V(x)-\beta)\upsilon_{n+1}\,dx}{(V(x)-\beta)[\ln(U_m-\alpha)+\ln(V(x)-\beta)]^{\lambda}}\\ &= \int_2^{\infty}\frac{\ln^{\lambda_1}(U_m-\alpha)\ln^{\lambda_2-1}(V(x)-\beta)}{[\ln(U_m-\alpha)+\ln(V(x)-\beta)]^{\lambda}}\frac{V'(x)}{V(x)-\beta}\,dx\\ &= \int_{\frac{\beta}{\upsilon_2}+1}^{\infty}\frac{\ln^{\lambda_1}(U_m-\alpha)\ln^{\lambda_2-1}(V(x)-\beta)}{[\ln(U_m-\alpha)+\ln(V(x)-\beta)]^{\lambda}}\frac{V'(x)}{V(x)-\beta}\,dx\\ &\quad-\int_{\frac{\beta}{\upsilon_2}+1}^{2}\frac{\ln^{\lambda_1}(U_m-\alpha)\ln^{\lambda_2-1}(V(x)-\beta)}{[\ln(U_m-\alpha)+\ln(V(x)-\beta)]^{\lambda}}\frac{V'(x)\,dx}{V(x)-\beta}.\end{aligned}$$

Setting $t=\frac{\ln(V(x)-\beta)}{\ln(U_m-\alpha)}$, since

$$\ln\biggl(V\biggl(\frac{\beta}{\upsilon_2}+1\biggr)-\beta\biggr)=\ln\biggl(1+\upsilon_2\frac{\beta}{\upsilon_2}-\beta\biggr)=0,$$

we find

$$\begin{aligned}\omega(\lambda_2,m) &> \int_0^{\infty}\frac{1}{(1+t)^{\lambda}}t^{\lambda_2-1}\,dt-\int_{\frac{\beta}{\upsilon_2}+1}^{2}\frac{\ln^{\lambda_1}(U_m-\alpha)\ln^{\lambda_2-1}(V(x)-\beta)}{[\ln(U_m-\alpha)+\ln(V(x)-\beta)]^{\lambda}}\frac{V'(x)}{V(x)-\beta}\,dx\\ &= B(\lambda_1,\lambda_2)\bigl(1-\theta(\lambda_2,m)\bigr),\end{aligned}$$

where

$$\theta(\lambda_2,m):=\frac{\ln^{\lambda_1}(U_m-\alpha)}{B(\lambda_1,\lambda_2)}\int_{\frac{\beta}{\upsilon_2}+1}^{2}\frac{V'(x)\ln^{\lambda_2-1}(V(x)-\beta)\,dx}{(V(x)-\beta)\ln^{\lambda}[(U_m-\alpha)(V(x)-\beta)]}\in(0,1).$$

In view of the integral mid value theorem, there exists a $\theta(m)\in(\frac{\beta}{\upsilon_2},1)$, satisfying

$$\begin{aligned}\theta(\lambda_2,m) &= \frac{1}{B(\lambda_1,\lambda_2)}\frac{\ln^{\lambda_1}(U_m-\alpha)\ln^{\lambda_2-1}(V(x)-\beta)}{[\ln(U_m-\alpha)+\ln(V(1+\theta(m))-\beta)]^{\lambda}}\\ &\quad\times\int_{\frac{\beta}{\upsilon_2}+1}^{2}\ln^{\lambda_2-1}\bigl(V(x)-\beta\bigr)\frac{V'(x)}{V(x)-\beta}\,dx\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{B(\lambda_1,\lambda_2)} \frac{\ln^{\lambda_1}(U_m-\alpha)\ln^{\lambda_2-1}(1+\upsilon_2-\beta)}{[\ln(U_m-\alpha)+\ln(1+\upsilon_2\theta(m)-\beta)]^{\lambda}} \\
&= \frac{1}{B(\lambda_1,\lambda_2)} \frac{\ln^{\lambda_2-1}(1+\upsilon_2-\beta)}{\lambda_2[1+\frac{\ln(1+\upsilon_2\theta(m)-\beta)}{\ln(U_m-\alpha)}]^{\lambda}} \frac{1}{\ln^{\lambda_2}(U_m-\alpha)}.
\end{aligned}
$$

Since we find

$$
0 < \theta(\lambda_2, m) \le \frac{\ln^{\lambda_2-1}(1+\upsilon_2-\beta)}{\lambda_2} \frac{1}{\ln^{\lambda_2}(U_m-\alpha)},
$$

namely, $\theta(\lambda_2, m) = O(\frac{1}{\ln^{\lambda_2}(U_m-\alpha)})$, we have (16) and (18). In the same way, we obtain (17) and (19).

For any $c > 0$, it follows that

$$
\begin{aligned}
&\sum_{m=2}^{\infty} \frac{\mu_{m+1}}{(U_m-\alpha)\ln^{1+c}(U_m-\alpha)} \\
&\quad\le \sum_{m=2}^{\infty} \frac{\mu_m}{(U_m-\alpha)\ln^{1+c}(U_m-\alpha)} \\
&\quad= \frac{\mu_2}{(U_2-\alpha)\ln^{1+c}(U_2-\alpha)} + \sum_{m=3}^{\infty} \frac{\mu_m}{(U_m-\alpha)\ln^{1+c}(U_m-\alpha)} \\
&\quad= \frac{\mu_2}{(U_2-\alpha)\ln^{1+c}(U_2-\alpha)} + \sum_{m=3}^{\infty} \int_{m-1}^{m} \frac{U'(x)\,dx}{(U_m-\alpha)\ln^{1+c}(U_m-\alpha)} \\
&\quad< \frac{\mu_2}{(U_2-\alpha)\ln^{1+c}(U_2-\alpha)} + \sum_{m=3}^{\infty} \int_{m-1}^{m} \frac{U'(x)\,dx}{(U(x)-\alpha)\ln^{1+c}(U(x)-\alpha)} \\
&\quad= \frac{\mu_2}{(U_2-\alpha)\ln^{1+c}(U_2-\alpha)} + \int_{2}^{\infty} \frac{U'(x)\,dx}{(U(x)-\alpha)\ln^{1+c}(U(x)-\alpha)} \\
&\quad= \frac{\mu_2}{(U_2-\alpha)\ln^{1+c}(U_2-\alpha)} + \frac{1}{c\ln^{c}(1+\mu_2-\alpha)} \\
&\quad= \frac{1}{c}\left[\frac{1}{\ln^{c}(1+\mu_2-\alpha)} + \frac{c\mu_2}{(U_2-\alpha)\ln^{1+c}(U_2-\alpha)}\right], \\
&\sum_{m=2}^{\infty} \frac{\mu_{m+1}}{(U_m-\alpha)\ln^{1+c}(U_m-\alpha)} \\
&\quad= \sum_{m=2}^{\infty} \int_{m}^{m+1} \frac{U'(x)\,dx}{(U_m-\alpha)\ln^{1+c}(U_m-\alpha)} \\
&\quad> \sum_{m=2}^{\infty} \int_{m}^{m+1} \frac{U'(x)}{(U(x)-\alpha)\ln^{1+c}(U(x)-\alpha)}\,dx \\
&\quad= \int_{2}^{\infty} \frac{U'(x)\,dx}{(U(x)-\alpha)\ln^{1+c}(U(x)-\alpha)} = \frac{1}{c\ln^{c}(1+\mu_2-\alpha)}.
\end{aligned}
$$

Hence we obtain (20). In the same way, we obtain (21). □

3 Main results

We define the following functions:

$$\widetilde{\Phi}_\lambda(m) := \omega(\lambda_2, m)\frac{[\ln(U_m-\alpha)]^{p(1-\lambda_1)-1}}{(U_m-\alpha)^{1-p}\mu_{m+1}^{p-1}},$$
$$\widetilde{\Psi}_\lambda(n) := \varpi(\lambda_1, n)\frac{[\ln(V_n-\beta)]^{q(1-\lambda_2)-1}}{(V_n-\beta)^{1-q}\upsilon_{n+1}^{q-1}} \quad \bigl(m,n\in\mathbf{N}\backslash\{1\}\bigr). \tag{22}$$

Theorem 1 *We have the following equivalent inequalities*:

$$I := \sum_{n=2}^{\infty}\sum_{m=2}^{\infty}\frac{a_m b_n}{\ln^\lambda[(U_m-\alpha)(V_n-\beta)]} \le \|a\|_{p,\widetilde{\Phi}_\lambda}\|b\|_{q,\widetilde{\Psi}_\lambda}, \tag{23}$$

$$J := \left\{\sum_{n=2}^{\infty}\frac{\upsilon_{n+1}\ln^{p\lambda_2-1}(V_n-\beta)}{(\varpi(\lambda_1,n))^{p-1}(V_n-\beta)}\left[\sum_{m=2}^{\infty}\frac{a_m}{\ln^\lambda[(U_m-\alpha)(V_n-\beta)]}\right]^p\right\}^{\frac{1}{p}} \le \|a\|_{p,\widetilde{\Phi}_\lambda}. \tag{24}$$

Proof By Hölder's inequality (*cf.* [15]) and (13), we find

$$\begin{aligned}
&\left[\sum_{m=2}^{\infty}\frac{a_m}{\ln^\lambda[(U_m-\alpha)(V_n-\beta)]}\right]^p \\
&\quad= \left[\sum_{m=2}^{\infty}\frac{1}{\ln^\lambda[(U_m-\alpha)(V_n-\beta)]}\right. \\
&\qquad\left.\times\left(\frac{(U_m-\alpha)^{1/q}\ln^{(1-\lambda_1)/q}(U_m-\alpha)}{\mu_{m+1}^{1/q}\ln^{(1-\lambda_2)/p}(V_n-\beta)}a_m\right)\left(\frac{\mu_{m+1}^{1/q}\ln^{(1-\lambda_2)/p}(V_n-\beta)}{(U_m-\alpha)^{1/q}\ln^{(1-\lambda_1)/q}(U_m-\alpha)}\right)\right]^p \\
&\quad\le \sum_{m=2}^{\infty}\frac{1}{\ln^\lambda[(U_m-\alpha)(V_n-\beta)]}\frac{(U_m-\alpha)^{p-1}\ln^{(1-\lambda_1)p/q}(U_m-\alpha)}{\mu_{m+1}^{p/q}\ln^{1-\lambda_2}(V_n-\beta)}a_m^p \\
&\qquad\times\left[\sum_{m=2}^{\infty}\frac{\mu_{m+1}}{\ln^\lambda[(U_m-\alpha)(V_n-\beta)]}\frac{\ln^{(1-\lambda_2)(q-1)}(V_n-\beta)}{(U_m-\alpha)\ln^{1-\lambda_1}(U_m-\alpha)}\right]^{p-1} \\
&\quad= \frac{(\varpi(\lambda_1,n))^{p-1}(V_n-\beta)}{\upsilon_{n+1}\ln^{p\lambda_2-1}(V_n-\beta)} \\
&\qquad\times\sum_{m=2}^{\infty}\frac{\upsilon_{n+1}(U_m-\alpha)^{p-1}\ln^{(1-\lambda_1)(p-1)}(U_m-\alpha)}{\mu_{m+1}^{p-1}(V_n-\beta)\ln^\lambda[(U_m-\alpha)(V_n-\beta)]\ln^{1-\lambda_2}(V_n-\beta)}a_m^p.
\end{aligned} \tag{25}$$

Then by (12) we obtain

$$\begin{aligned}
J &\le \left[\sum_{n=2}^{\infty}\sum_{m=2}^{\infty}\frac{\upsilon_{n+1}(U_m-\alpha)^{p-1}a_m^p}{\ln^\lambda[(U_m-\alpha)(V_n-\beta)]}\frac{\ln^{(1-\lambda_1)(p-1)}(U_m-\alpha)}{\mu_{m+1}^{p-1}(V_n-\beta)\ln^{1-\lambda_2}(V_n-\beta)}\right]^{\frac{1}{p}} \\
&= \left[\sum_{m=2}^{\infty}\sum_{n=2}^{\infty}\frac{\upsilon_{n+1}(U_m-\alpha)^{p-1}a_m^p}{\ln^\lambda[(U_m-\alpha)(V_n-\beta)]}\frac{\ln^{(1-\lambda_1)(p-1)}(U_m-\alpha)}{\mu_{m+1}^{p-1}(V_n-\beta)\ln^{1-\lambda_2}(V_n-\beta)}\right]^{\frac{1}{p}} \\
&= \left[\sum_{m=2}^{\infty}\omega(\lambda_2,m)\frac{\ln^{p(1-\lambda_1)-1}(U_m-\alpha)}{(U_m-\alpha)^{1-p}\mu_{m+1}^{p-1}}a_m^p\right]^{\frac{1}{p}},
\end{aligned} \tag{26}$$

namely, (24) follows. By Hölder's inequality (*cf.* [15]), we find

$$I = \sum_{n=2}^{\infty}\left[\frac{\upsilon_{n+1}^{1/p}\ln^{\lambda_2-\frac{1}{p}}(V_n-\beta)}{(\varpi(\lambda_1,n))^{\frac{1}{q}}(V_n-\beta)^{1/p}}\sum_{m=1}^{\infty}\frac{a_m}{\ln^{\lambda}[(U_m-\alpha)(V_n-\beta)]}\right] \times\left[\left(\varpi(\lambda_1,n)\right)^{\frac{1}{q}}\frac{\ln^{\frac{1}{p}-\lambda_2}(V_n-\beta)}{(V_n-\beta)^{-1/p}\upsilon_{n+1}^{1/p}}b_n\right]\le J\|b\|_{q,\widetilde{\Psi}_\lambda}. \tag{27}$$

Then by (24), (23) follows.

On the other hand, suppose that (23) is valid. We set

$$b_n := \frac{\upsilon_{n+1}\ln^{p\lambda_2-1}(V_n-\beta)}{(\varpi(\lambda_1,n))^{p-1}(V_n-\beta)}\left[\sum_{m=2}^{\infty}\frac{a_m}{\ln^{\lambda}[(U_m-\alpha)(V_n-\beta)]}\right]^{p-1}, \quad n\in\mathbf{N}\backslash\{1\}. \tag{28}$$

Then we have $J^p = \|b\|_{q,\widetilde{\Psi}_\lambda}^q$. If $J = 0$, then (24) is trivially valid; if $J = \infty$, then in view of (26), (24) takes the form of an equality. Suppose that $0 < J < \infty$. By (23), we obtain

$$\|b\|_{q,\widetilde{\Psi}_\lambda}^q = J^p = I \le \|a\|_{p,\widetilde{\Phi}_\lambda}\|b\|_{q,\widetilde{\Psi}_\lambda}, \tag{29}$$

$$\|b\|_{q,\widetilde{\Psi}_\lambda}^{q-1} = J \le \|a\|_{p,\widetilde{\Phi}_\lambda}, \tag{30}$$

namely, (24) follows, which is equivalent to (23). □

Theorem 2 *Assuming that* $\{\mu_m\}_{m=1}^{\infty}$ *and* $\{\upsilon_n\}_{n=1}^{\infty}$ *are decreasing,* $U(\infty) = V(\infty) = \infty$, $0 < \|a\|_{p,\Phi_\lambda}, \|b\|_{q,\Psi_\lambda} < \infty$, *we have the following equivalent inequalities*:

$$\sum_{n=2}^{\infty}\sum_{m=2}^{\infty}\frac{a_m b_n}{\ln^{\lambda}[(U_m-\alpha)(V_n-\beta)]} < B(\lambda_1,\lambda_2)\|a\|_{p,\Phi_\lambda}\|b\|_{q,\Psi_\lambda}, \tag{31}$$

$$J_1 := \left\{\sum_{n=2}^{\infty}\frac{\upsilon_{n+1}\ln^{p\lambda_2-1}(V_n-\beta)}{V_n-\beta} \times\left[\sum_{m=2}^{\infty}\frac{a_m}{\ln^{\lambda}[(U_m-\alpha)(V_n-\beta)]}\right]^p\right\}^{\frac{1}{p}} < B(\lambda_1,\lambda_2)\|a\|_{p,\Phi_\lambda}, \tag{32}$$

where the constant factor $B(\lambda_1,\lambda_2)$ *is the best possible.*

Proof Applying (14) and (15) in (23) and (24), we have the equivalent inequalities (31) and (32).

For $\varepsilon\in(0,p\lambda_1)$, we set $\widetilde{\lambda}_1 = \lambda_1 - \frac{\varepsilon}{p}$ $(\in(0,1))$, $\widetilde{\lambda}_2 = \lambda_2 + \frac{\varepsilon}{p}$ (>0), and

$$\widetilde{a}_m := \frac{\mu_{m+1}}{U_m-\alpha}\ln^{\widetilde{\lambda}_1-1}(U_m-\alpha), \qquad \widetilde{b}_n := \frac{\upsilon_{n+1}}{V_n-\beta}\ln^{\widetilde{\lambda}_2-\varepsilon-1}(V_n-\beta). \tag{33}$$

Then by (20), (21), and (17), we obtain

$$\begin{aligned}
&\|\widetilde{a}\|_{p,\Phi_\lambda}\|\widetilde{b}\|_{q,\Psi_\lambda} \\
&\quad= \left[\sum_{m=2}^{\infty}\frac{\mu_{m+1}}{(U_m-\alpha)\ln^{1+\varepsilon}(U_m-\alpha)}\right]^{\frac{1}{p}}\left[\sum_{n=2}^{\infty}\frac{\upsilon_{n+1}}{(V_n-\beta)\ln^{1+\varepsilon}(V_n-\beta)}\right]^{\frac{1}{q}} \\
&\quad= \frac{1}{\varepsilon}\left[\frac{1}{\ln^{\varepsilon}(1+\mu_2-\alpha)}+\varepsilon O(1)\right]^{\frac{1}{p}}\left[\frac{1}{\ln^{\varepsilon}(1+\upsilon_2-\beta)}+\varepsilon\widetilde{O}(1)\right]^{\frac{1}{q}}, \\
&\widetilde{I} := \sum_{n=2}^{\infty}\sum_{m=2}^{\infty}\frac{\widetilde{a}_m\widetilde{b}_n}{\ln^{\lambda}[(U_m-\alpha)(V_n-\beta)]} \\
&\quad= \sum_{n=2}^{\infty}\left[\sum_{m=2}^{\infty}\frac{1}{\ln^{\lambda}[(U_m-\alpha)(V_n-\beta)]}\frac{\mu_{m+1}\ln^{\widetilde{\lambda}_2}(V_n-\beta)}{(U_m-\alpha)\ln^{1-\widetilde{\lambda}_1}(U_m-\alpha)}\right] \\
&\qquad\times\frac{\upsilon_{n+1}}{(V_n-\beta)\ln^{\varepsilon+1}(V_n-\beta)} = \sum_{n=2}^{\infty}\varpi(\widetilde{\lambda}_1,n)\frac{\upsilon_{n+1}}{(V_n-\beta)\ln^{\varepsilon+1}(V_n-\beta)} \\
&\quad\geq B(\widetilde{\lambda}_1,\widetilde{\lambda}_2)\left[\sum_{n=2}^{\infty}\frac{\upsilon_{n+1}}{(V_n-\beta)\ln^{\varepsilon+1}(V_n-\beta)}\right. \\
&\qquad\left.-\sum_{n=2}^{\infty}O\left(\frac{\upsilon_{n+1}}{(V_n-\beta)\ln^{\lambda_1+\frac{\varepsilon}{q}+1}(V_n-\beta)}\right)\right] \\
&\quad= \frac{1}{\varepsilon}B(\widetilde{\lambda}_1,\widetilde{\lambda}_2)\left[\frac{1}{\ln^{\varepsilon}(1+\upsilon_2-\beta)}+\varepsilon\bigl(\widetilde{O}(1)-O(1)\bigr)\right].
\end{aligned}$$

If there exists a positive constant $K \leq B(\lambda_1,\lambda_2)$, such that (31) is valid when replacing $B(\lambda_1,\lambda_2)$ by K, then in particular, we have $\varepsilon\widetilde{I} < \varepsilon K\|\widetilde{a}\|_{p,\Phi_\lambda}\|\widetilde{b}\|_{q,\Psi_\lambda}$, namely

$$\begin{aligned}
&B\left(\lambda_1-\frac{\varepsilon}{p},\lambda_2+\frac{\varepsilon}{p}\right)\left[\frac{1}{\ln^{\varepsilon}(1+\upsilon_2-\beta)}+\varepsilon\bigl(\widetilde{O}(1)-O(1)\bigr)\right] \\
&\quad< K\left[\frac{1}{\ln^{\varepsilon}(1+\mu_2-\alpha)}+\varepsilon O(1)\right]^{\frac{1}{p}}\left[\frac{1}{\ln^{\varepsilon}(1+\upsilon_2-\beta)}+\varepsilon\widetilde{O}(1)\right]^{\frac{1}{q}}.
\end{aligned}$$

It follows that $B(\lambda_1,\lambda_2) \leq K(\varepsilon \to 0^+)$. Hence, $K = B(\lambda_1,\lambda_2)$ is the best possible constant factor of (31).

Similarly, we can obtain

$$I \leq J_1\|b\|_{q,\Psi_\lambda}. \tag{34}$$

Hence, we can prove that the constant factor $B(\lambda_1,\lambda_2)$ in (32) is the best possible. Otherwise, we would reach a contradiction by (34) that the constant factor in (31) is not the best possible. □

We find $\Psi_\lambda^{1-p}(n) = \frac{\upsilon_{n+1}}{V_n-\beta}\ln^{p\lambda_2-1}(V_n-\beta)$, and we define the following weighted normed spaces:

$$l_{p,\Phi_\lambda} := \bigl\{a = \{a_m\}_{m=2}^{\infty}; \|a\|_{p,\Phi_\lambda} < \infty\bigr\},$$

$$l_{q,\Psi_\lambda} := \left\{b = \{b_n\}_{n=2}^\infty; \|b\|_{q,\Psi_\lambda} < \infty\right\},$$
$$l_{p,\Psi_\lambda^{1-p}} := \left\{c = \{c_n\}_{n=2}^\infty; \|c\|_{p,\Psi_\lambda^{1-p}} < \infty\right\}.$$

Assuming that $a = \{a_m\}_{m=2}^\infty \in l_{p,\Phi_\lambda}$, setting

$$c = \{c_n\}_{n=2}^\infty, \qquad c_n := \sum_{m=2}^\infty \frac{a_m}{\ln^\lambda[(U_m - \alpha)(V_n - \beta)]}, \quad n \in \mathbf{N}\backslash\{1\},$$

we can rewrite (32) as follows:

$$\|c\|_{p,\Psi_\lambda^{1-p}} < B(\lambda_1, \lambda_2)\|a\|_{p,\Phi_\lambda} < \infty,$$

namely, $c \in l_{p,\Psi_\lambda^{1-p}}$.

Definition 1 Define a Hardy-Mulholland-type operator $T : l_{p,\Phi_\lambda} \to l_{p,\Psi_\lambda^{1-p}}$ as follows: For any $a = \{a_m\}_{m=2}^\infty \in l_{p,\Phi_\lambda}$, there exists a unique representation $Ta = c \in l_{p,\Psi_\lambda^{1-p}}$. We set the formal inner product of Ta and $b = \{b_n\}_{n=2}^\infty \in l_{q,\Psi_\lambda}$ as follows:

$$(Ta, b) := \sum_{n=2}^\infty \left[\sum_{m=2}^\infty \frac{a_m}{\ln^\lambda[(U_m - \alpha)(V_n - \beta)]}\right] b_n. \tag{35}$$

Then we can rewrite (31) and (32) as follows:

$$(Ta, b) < B(\lambda_1, \lambda_2)\|a\|_{p,\Phi_\lambda}\|b\|_{q,\Psi_\lambda}, \tag{36}$$
$$\|Ta\|_{p,\Psi_\lambda^{1-p}} < B(\lambda_1, \lambda_2)\|a\|_{p,\Phi_\lambda}. \tag{37}$$

We set the norm of operator T as follows:

$$\|T\| := \sup_{a(\neq\theta)\in l_{p,\Phi_\lambda}} \frac{\|Ta\|_{p,\Psi_\lambda^{1-p}}}{\|a\|_{p,\Phi_\lambda}}.$$

By (37), we find $\|T\| \leq B(\lambda_1, \lambda_2)$. Since the constant factor in (37) is the best possible, it follows that $\|T\| = B(\lambda_1, \lambda_2)$.

Remark 1 (i) For $\alpha = \beta = 0$ in (31) and (32), setting

$$\varphi_\lambda(m) := \frac{(\ln U_m)^{p(1-\lambda_1)-1}}{U_m^{1-p}\mu_{m+1}^{p-1}}, \qquad \psi_\lambda(n) := \frac{(\ln V_n)^{q(1-\lambda_2)-1}}{V_n^{1-q}\upsilon_{n+1}^{q-1}} \quad \left(m, n \in \mathbf{N}\backslash\{1\}\right),$$

we have the following equivalent Hardy-Mulholland-type inequalities:

$$\sum_{n=2}^\infty \sum_{m=2}^\infty \frac{a_m b_n}{\ln^\lambda(U_m V_n)} < B(\lambda_1, \lambda_2)\|a\|_{p,\varphi_\lambda}\|b\|_{q,\psi_\lambda}, \tag{38}$$

$$\left\{\sum_{n=2}^\infty \frac{\upsilon_{n+1}}{V_n} \ln^{p\lambda_2 - 1} V_n \left[\sum_{m=2}^\infty \frac{a_m}{\ln^\lambda(U_m V_n)}\right]^p\right\}^{\frac{1}{p}} < B(\lambda_1, \lambda_2)\|a\|_{p,\varphi_\lambda}. \tag{39}$$

For $\lambda = 1$, $\lambda_1 = \frac{1}{q}$, $\lambda_2 = \frac{1}{p}$ in (38) and (39), we have the following equivalent inequality:

$$\sum_{n=2}^{\infty}\sum_{m=2}^{\infty}\frac{a_m b_n}{\ln(U_m V_n)} < \frac{\pi}{\sin(\frac{\pi}{p})}\left[\sum_{m=2}^{\infty}\left(\frac{U_m}{\mu_{m+1}}\right)^{p-1} a_m^p\right]^{\frac{1}{p}}\left[\sum_{n=2}^{\infty}\left(\frac{V_n}{\upsilon_{n+1}}\right)^{q-1} b_n^q\right]^{\frac{1}{q}}, \tag{40}$$

$$\left\{\sum_{n=2}^{\infty}\frac{\upsilon_{n+1}}{V_n}\left[\sum_{m=2}^{\infty}\frac{a_m}{\ln(U_m V_n)}\right]^p\right\}^{\frac{1}{p}} < \frac{\pi}{\sin(\frac{\pi}{p})}\left[\sum_{m=2}^{\infty}\left(\frac{U_m}{\mu_{m+1}}\right)^{p-1} a_m^p\right]^{\frac{1}{p}}. \tag{41}$$

Hence, (38) is an extension of (40), and (31) is a more accurate inequality of (38) (for $0 < \alpha \leq \frac{\mu_2}{2}$, $0 < \beta \leq \frac{\upsilon_2}{2}$).

(ii) For $\mu_i = \upsilon_j = 1$ $(i, j \in \mathbf{N})$, $\lambda = 1$, $\lambda_1 = \frac{1}{q}$, $\lambda_2 = \frac{1}{p}$ in (31), we reduce our case to the following inequality: For $\alpha, \beta \leq \frac{1}{2}$,

$$\begin{aligned}&\sum_{m=2}^{\infty}\sum_{n=2}^{\infty}\frac{a_m b_n}{\ln[(m-\alpha)(n-\beta)]}\\&\quad < \frac{\pi}{\sin(\pi/p)}\left[\sum_{m=2}^{\infty}\frac{a_m^p}{(m-\alpha)^{1-p}}\right]^{\frac{1}{p}}\left[\sum_{n=2}^{\infty}\frac{b_n^q}{(n-\beta)^{1-q}}\right]^{\frac{1}{q}},\end{aligned} \tag{42}$$

Hence, (42) is a more accurate inequality of (3) (for $0 < \alpha, \beta \leq \frac{1}{2}$).

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
BY carried out the mathematical studies, participated in the sequence alignment and drafted the manuscript. AL and LH participated in the design of the study and performed the numerical analysis. All authors read and approved the final manuscript.

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