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On relaxed and contraction-proximal point algorithms in hilbert spaces

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Abstract

We consider the relaxed and contraction-proximal point algorithms in Hilbert spaces. Some conditions on the parameters for guaranteeing the convergence of the algorithm are relaxed or removed. As a result, we extend some recent results of Ceng-Wu-Yao and Noor-Yao.

Keywords: maximal monotone operator, proximal point algorithm, firmly nonexpansive operator

1. Introduction

Throughout, H denotes a real Hilbert space and A a multi-valued operator with domain $D(A)$. We know that A is called monotone if $\langle u - v, x - y\rangle \geq 0$, for any $u \in Ax$, $v \in Ay$; maximal monotone if its graph $G(A) = \{(x,y): x \in D(A), y \in Ax\}$ is not properly contained in the graph of any other monotone operator. Denote by $S: = \{x \in D(A): 0 \in Ax\}$ the zero set and by $J_c: = (I + cA)^{-1}$ the resolvent of A. It is well known that J_c is single valued and $D(J_c) = H$ for any $c > 0$.

A fundamental problem of monotone operators is that of finding an element x so that $0 \in Ax$. This problem is essential because it includes many concrete examples, such as convex programming and monotone variational inequalities. A successful and powerful algorithm for solving this problem is the well-known proximal point algorithm (PPA), which generates, for any initial guess, $x_0 \in H$, an iterative sequence as

$$x_{n+1} = J_{c_n}(x_n + e_n), \tag{1.1}$$

where (c_n) is a positive real sequence and (e_n) is the error sequence (see [1]). To guarantee the convergence of PPA, there are two kinds of accuracy criterion posed on the error sequence:

$$\text{(I)}\ \|e_n\| \leq \varepsilon_n, \quad \sum_{n=0}^{\infty} \varepsilon_n < \infty \quad \text{or}$$

$$\text{(II)}\ \|e_n\| \leq \eta_n \left\|\tilde{x}_n - x_n\right\|, \quad \sum_{n=0}^{\infty} \eta_n < \infty,$$

where $\tilde{x}_n = J_{c_n}(x_n + e_n)$. In 2001, Han and He [2] proved that in finite dimensional Hilbert space criterion (II) can be replaced by

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$$\text{(II}'\text{)}\ \ \|e_n\| \le \eta_n \left\|\tilde{x}_n - x_n\right\|, \quad \sum_{n=0}^{\infty} \eta_n^2 < \infty.$$

The infinite version was obtained by Marino and Xu [3].

There are various generations or modifications on the PPA. Among them Eckstein and Bertsekas [4] proposed the relaxed proximal point algorithm (RPPA):

$$x_{n+1} = (1-\rho_n)x_n + \rho_n J_{c_n}(x_n) + e_n, \tag{1.2}$$

where $(\rho_n) \subset (0, 2)$ is a relaxation factor. The weak convergence of (1.2) is guaranteed provided that (e_n) satisfies criterion (I),

$$c_n \ge \bar{c} > 0, \quad 0 < \delta \le \rho_n \le 2 - \delta. \tag{1.3}$$

On the other hand, since the PPA does not necessarily converge strongly (see [5]), many authors have conducted worthwhile studies on modifying the PPA so that the strong convergence is guaranteed (see, for instance, [6-8]). In particular, Marino and Xu [3] proposed the contraction-proximal point algorithm (CPPA):

$$x_{n+1} = \lambda_n u + (1-\lambda_n)J_{c_n}(x_n) + e_n, \tag{1.4}$$

where the parameters above satisfy (i) $\lim_n \lambda_n = 0$, $\Sigma_n \lambda_n = \infty$; (ii) either $\Sigma_n|\lambda_n+1- \lambda_n| < \infty$; or $\lim_n \lambda_n/\lambda_n+1 = 1$; (iii) $0 < \underline{c} \le c_n \le \bar{c} < \infty$, $\sum_n |c_{n+1} - c_n| < \infty$; (iv) $\Sigma_n ||e_n|| < \infty$. Under these assumptions, the CPPA converges strongly to $P_S(u)$, the projection of u onto S.

In this article, we shall focus on the RPPA and CPPA. We note that the resolvent is in fact the arithmetic mean of the identity and a nonexpansive operator. By using this fact, we relax or remove some sufficient conditions to guarantee the convergence of the algorithms. As a result, we extend and improve some recent results on the PPA.

2. Some lemmas

We know that an operator $T : H \to H$ is called (i) nonexpansive if $||Tx - Ty|| \le || x - y||$ $\forall x,y \in H$; and (ii) firmly nonexpansive if $\langle Tx - Ty, x - y\rangle \ge ||Tx - Ty||^2$ $\forall x,y \in H$. Denote by $\mathrm{Fix}(T) = \{x \in H : x = Tx\}$ the fixed point set of T. It is well known that firmly nonexpansive operators have the following properties.

Lemma 1 (Goebel-Kirk [9]). *Let T be firmly nonexpansive. Then (1) 2T - I is nonexpansive; (2)* $\langle Tx - x, Tx - z\rangle \le 0$ *for all x⌊ H and for all z⌊ H* Fix(T).

It is well known that J_c is firmly nonexpansive and consequently nonexpansive; moreover, $S = \mathrm{Fix}\ (J_c)$. Since the fixed point set of nonexpansive operators is closed convex, the projection P_s onto the solution set S is well defined whenever $S \ne \emptyset$. Hereafter, we assume that S is nonempty. The following lemmas play an important role in our convergence analysis.

Lemma 2 (resolvent identity [3]). *Let c, t >0. Then for any x⌊ H,*

$$J_c x = J_t\left(\frac{t}{c}x + \left(1 - \frac{t}{c}\right)J_c x\right).$$

Lemma 3 ([10]). *Let (ρ_n) be real sequence satisfying*

$$0 < \liminf_{n\to\infty} \rho_n \le \limsup_{n\to\infty}\ \rho_n < 1.$$

Assume that (x_n) *and* (y_n) *are bounded sequences in H satisfying* $x_n+1 = (1 - \rho_n)x_n + \rho_n y_n$. *If*

$$\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0,$$

then $\lim_{n\to\infty}||x_n - y_n|| = 0$.

Lemma 4 *For r, s, > 0, let* $T_r = 2J_r - I$. *Then for any* $x \in H$,

$$\|T_s x - T_r x\| \leq \left|1 - \frac{s}{r}\right| \|x - T_r x\| . \tag{2.1}$$

Proof. Using the resolvent identity, we have

$$\begin{aligned}\|T_s x - T_r x\| &= 2\left\|J_s x - J_s\left(\frac{s}{r}x + \left(1 - \frac{s}{r}\right)J_r x\right)\right\| \\ &\leq 2\left\|x - \left(\frac{s}{r}x + \left(1 - \frac{s}{r}\right)J_r x\right)\right\| \\ &= 2\left|1 - \frac{s}{r}\right| \|x - J_r x\| \\ &= \left|1 - \frac{s}{r}\right| \|x - T_r x\| ,\end{aligned}$$

where the inequality uses the nonexpansive property of the resolvent.

Lemma 5 ([11]). *Let* (ε_n) *and* (s_n) *be positive real sequences. Assume that* $\Sigma_n \varepsilon_n < \infty$. *If either (i)* $s_{n+}1 \leq (1 + \varepsilon_n)s_n$, *or (ii)* $s_{n+}1 \leq \varepsilon_n$, *then the limit of* (s_n) *exists.*

3. The relaxed proximal point algorithm

Under criterion (II'), Ceng et al. [12] considered another type, RPPA:

$$\begin{cases}\tilde{x}_n = J_{c_n}(x_n + e_n), \\ x_{n+1} = (1 - \rho_n)x_n + \rho_n\tilde{x}_n,\end{cases} \tag{3.1}$$

and proved the weak convergence of (3.1) under the assumptions:

$$c_n \geq \bar{c} > 0, \quad 0 < \delta \leq \rho_n \leq 1.$$

We note that the choice of (ρ_n) excludes the case whenever $\rho_n \in (1,2)$, the overrelaxation. The overrelaxation, however, may indeed speed up the convergence of the algorithm (see [13]). Below, we shall improve their conditions on the relaxation factor from $0 < \delta \leq \rho_n \leq 1$ to $0 < \delta \leq \rho_n \leq 2 - \delta$.

Theorem 6. *Assume that the following conditions hold:*

(a) $c_n \geq \bar{c} > 0$;

(b) $0 < \delta \leq \rho_n \leq 2 - \delta$;

(c) $\sum_n \|e_n\| \leq \eta_n \|\tilde{x}_n - x_n\|, \sum_n \eta_n^2 < \infty$.

Then the sequence generated by (3.1) *converges weakly to a point in S.*

Proof. The key point of our proof is to show $\lim_n s_n = 0$, where $s_n = \|x_n - J_{c_n}(x_n)\|$. To see this, let $z \in S$ be fixed. Since J_{c_n} is firmly nonexpansive and $z \in \mathrm{Fix}(J_{c_n})$, applying Lemma 1 yields $\langle\tilde{x}_n - z, \tilde{x}_n - x_n - e_n\rangle \leq 0$. This together with (3.1) enables us to get

$$\begin{aligned}\|x_{n+1} - z\|^2 - \|x_n - z\|^2 &= \|(x_n - z) + \rho_n(\tilde{x}_n - x_n)\|^2 - \|x_n - z\|^2 \\ &= 2\rho_n\langle x_n - z, \tilde{x}_n - x_n\rangle + \rho_n^2\|\tilde{x}_n - x_n\|^2 \\ &= 2\rho_n\langle \tilde{x}_n - z, \tilde{x}_n - x_n\rangle - \rho_n(2 - \rho_n)\|\tilde{x}_n - x_n\|^2 \\ &\leq 2\rho_n\langle \tilde{x}_n - z, e_n\rangle - \rho_n(2 - \rho_n)\|\tilde{x}_n - x_n\|^2 \\ &= 2\rho_n\langle \tilde{x}_n - x_n, e_n\rangle + 2\rho_n\langle x_n - z, e_n\rangle - \rho_n(2 - \rho_n)\|\tilde{x}_n - x_n\|^2 \\ &\leq 2\rho_n\|e_n\|\,\|\tilde{x}_n - x_n\| + 2\rho_n\|e_n\|\,\|x_n - z\| - \rho_n(2 - \rho_n)\|\tilde{x}_n - x_n\|^2 \\ &\leq 2\rho_n\eta_n\|\tilde{x}_n - x_n\|^2 + 2\rho_n\eta_n\|\tilde{x}_n - x_n\|\,\|x_n - z\| \\ &\quad - \rho_n(2 - \rho_n)\|\tilde{x}_n - x_n\|^2.\end{aligned}$$

Using the basic inequality $2ab \le a^2 / \varepsilon + \varepsilon b^2$ ($a,b \in \mathbb{R}$, $\varepsilon > 0$), we arrive at

$$\begin{aligned} 2\rho_n\eta_n \|x_n - z\| \left\|\tilde{x}_n - x_n\right\| &\le \frac{2\rho_n}{2-\rho_n}(\eta_n \|x_n - z\|)^2 + \frac{2-\rho_n}{2\rho_n}\left(\rho_n \left\|\tilde{x}_n - x_n\right\|\right)^2 \\ &= \frac{2\rho_n\eta_n^2}{2-\rho_n}\|x_n - z\|^2 + \frac{\rho_n(2-\rho_n)}{2}\left\|\tilde{x}_n - x_n\right\|^2 \\ &\le \frac{2(2-\delta)\eta_n^2}{\delta}\|x_n - z\|^2 + \frac{\rho_n(2-\rho_n)}{2}\left\|\tilde{x}_n - x_n\right\|^2 \\ &= \varepsilon_n\|x_n - z\|^2 + \frac{\rho_n(2-\rho_n)}{2}\left\|\tilde{x}_n - x_n\right\|^2, \end{aligned}$$

where $\varepsilon_n = 2(2-\delta)\eta_n^2/\delta$ is a summable sequence. Substituting this into above yields

$$\|x_{n+1} - z\|^2 \le (1+\varepsilon_n)\|x_n - z\|^2 - \frac{\rho_n(2-\rho_n-4\eta_n)}{2}\left\|\tilde{x}_n - x_n\right\|^2.$$

Since by Lemma 5 the limit of $||x_n - z||^2$ exists and $\liminf_n \rho_n (2 - \rho_n - 4\eta_n) \ge \delta (2 - \delta)$, this implies that $\left\|\tilde{x}_n - x_n\right\| \to 0$. On the other hand, we note that for all $n \in \mathbb{N}$

$$s_n \le (1+\eta_n)\left\|x_n - \tilde{x}_n\right\| \to 0;$$

therefore, $\lim_n s_n = 0$. The rest proof is similar to that of [12, Theorem 3.1].

We now turn to the RPPA (1.2). Under the criterion (I), the assumptions on relaxation factors can be relaxed to $\Sigma\rho_n(2 - \rho_n) = \infty$ (see [3, Theorem 3.3]). Since the proof there is very technical, we wang to restate this result with a simple proof.

Theorem 7. *Assume that the following conditions hold:*

(a) $\Sigma_n ||e_n|| < \infty$;

(b) $\Sigma_n \rho_n(2 - \rho_n) = \infty$;

(c) $0 < \bar{c} \le c_n \le \tilde{c} < \infty$;

(d) $\Sigma_n |c_n+1- c_n| < \infty$.

Then the sequence generated by (1.2) *converges weakly to a point in S.*

Proof. The key step is to show $\lim_n s_n = 0$, where $s_n = \left\|x_n - J_{c_n}(x_n)\right\|$. It has been shown that $\Sigma_n \rho_n (2 - \rho_n)s_n < \infty$ (see [3, Lemma 3.2]). Therefore, it remains to show that $\lim_n s_n$ exists. By letting $T_n = 2J_n - I$, we rewrite (2) as

$$x_{n+1} = \left(1 - \frac{\rho_n}{2}\right)x_n + \frac{\rho_n}{2}T_n x_n + e_n.$$

In view of Lemma 4 and condition (c),

$$\begin{aligned} \|T_{n+1}x_{n+1} - T_n x_n\| &\le \|T_{n+1}x_{n+1} - T_{n+1}x_n\| + \|T_{n+1}x_n - T_n x_n\| \\ &\le \|x_n - x_{n+1}\| + \|T_{n+1}x_n - T_n x_n\| \\ &\le \|x_n - x_{n+1}\| + \left|1 - \frac{c_{n+1}}{c_n}\right| \|T_n x_n - x_n\| \\ &\le \|x_n - x_{n+1}\| + \frac{|c_{n+1} - c_n|}{\bar{c}}\|T_n x_n - x_n\| \\ &\le \|x_n - x_{n+1}\| + M|c_{n+1} - c_n|, \end{aligned}$$

where $M > 0$ is a suitable number. Consequently,

$$\begin{aligned}
\|x_{n+1} - T_{n+1}x_{n+1}\| &= \left\| \left(1 - \frac{\rho_n}{2}\right) x_n + \frac{\rho_n}{2} T_n x_n + e_n - T_{n+1}x_{n+1} \right\| \\
&= \left\| \left(1 - \frac{\rho_n}{2}\right) (x_n - T_n x_n) + (T_n x_n - T_{n+1}x_{n+1}) + e_n \right\| \\
&\leq \left(1 - \frac{\rho_n}{2}\right) \|x_n - T_n x_n\| + \|T_n x_n - T_{n+1}x_{n+1}\| + \|e_n\| \\
&\leq \left(1 - \frac{\rho_n}{2}\right) \|x_n - T_n x_n\| + \|x_n - x_{n+1}\| \\
&\quad + M|c_{n+1} - c_n| + \|e_n\| \\
&= \left(1 - \frac{\rho_n}{2}\right) \|x_n - T_n x_n\| + \left\| \frac{\rho_n}{2}(x_n - T_n x_n) + e_n \right\| \\
&\quad + M|c_{n+1} - c_n| + \|e_n\| \\
&\leq \|x_n - T_n x_n\| + M|c_{n+1} - c_n| + 2\,\|e_n\|\,.
\end{aligned}$$

Using $s_n = || x_n - T_n x_n||/2$, we therefore arrive at

$$s_{n+1} \leq s_n + \sigma_n,$$

where $\sigma_n = 2M\ |c_n+1- c_n| + 4||e_n||$ satisfying $\Sigma_n\ \sigma_n < \infty$ (due to (a) and (d)). By Lemma 5, we finally conclude that $\lim_n s_n = 0$.

4. The contraction-proximal point algorithm

Recently, Yao and Noor [14] extended the CPPA to the following form:

$$x_{n+1} = \lambda_n u + r_n x_n + \delta_n J_{c_n}(x_n) + e_n, \tag{4.1}$$

where $(\lambda_n),(r_n),(\delta_n) \subseteq (0,1)$ and $\lambda_n + r_n + \delta_n = 1$. They proved the strong convergence of the algorithm provided that (i) $c_n \geq \bar{c} > 0, \lim_n |c_{n+1} - c_n| = 0$; (ii) $0 < \liminf_n r_n \leq \limsup_n r_n < 1$; and (iii) $\Sigma_n ||e_n|| < \infty$. Also, they claimed that their algorithm includes the CPPA as a special case. This is, however, not the case, because condition (ii) excludes the special case $r_n \equiv 0$. To overcome this drawback, we shall show the same result by replacing condition (ii) with the weak condition:

$$\limsup_{n\to\infty} r_n < 1 \Leftrightarrow \liminf_{n\to\infty} \delta_n > 0.$$

In this situation, the CPPA is evidently a special case of algorithm (4.1). The idea of the following proof is followed by the second author [15].

Theorem 8. *Let be* (λ_n), (r_n) *and* (δ_n) *be parameters in* (4.1). *Assume that the following conditions hold:*

(a) $\lim_n \lambda_n = 0$, $\Sigma_n \lambda_n = \infty$;

(b) $\limsup_n r_n < 1 \Leftrightarrow \liminf_n \delta_n > 0$;

(c) $c_n \geq \bar{c} > 0, |c_{n+1} - c_n| \to 0$;

(d) $\Sigma_n ||e_n|| < \infty$.

Then the sequence generated by (4.1) *converges strongly to* $P_S(u)$.

Proof. All we need to do is to prove $||x_n+1- x_n|| \to 0$, since the rest proof is similar to that of [14, Theorem 3.3]. To this end, set $J_n = J_{c_n}$ and $T_n = 2J_n - I$. It then follows from (4.1) that

$$\begin{aligned}
x_{n+1} &= \lambda_n u + r_n x_n + \frac{\delta_n}{2}(I + T_n)x_n + e_n \\
&= \left(r_n + \frac{\delta_n}{2}\right) x_n + \lambda_n u + \frac{\delta_n}{2} T_n x_n + e_n.
\end{aligned}$$

Let $\rho_n = \lambda_n + (\delta_n/2)$. Then the algorithm has the form:

$$x_{n+1} = (1-\rho_n)x_n + \rho_n y_n, \tag{4.2}$$

where $y_n = (2\lambda_n u + \delta_n T_n x_n + 2e_n)/2\rho_n$. Using nonexpansiveness of T_n and Lemma 4, we have

$$\begin{aligned} \|T_{n+1}x_{n+1} - T_n x_n\| &\le \|T_{n+1}x_{n+1} - T_{n+1}x_n\| + \|T_{n+1}x_n - T_n x_n\| \\ &\le \|x_{n+1} - x_n\| + \left|1 - \frac{c_{n+1}}{c_n}\right| \|T_n x_n - x_n\| \\ &\le \|x_{n+1} - x_n\| + \frac{|c_n - c_{n+1}|}{\bar{c}} \|T_n x_n - x_n\|. \end{aligned} \tag{4.3}$$

On the other hand, it follows from the definition of y_n that

$$\begin{aligned} \|y_{n+1} - y_n\| &= \Big\| \frac{1}{2\rho_{n+1}}(2\lambda_{n+1}u + \delta_{n+1}T_{n+1}x_{n+1} + 2e_{n+1}) \\ &\quad - \frac{1}{2\rho_n}(2\lambda_n u + \delta_n T_n x_n + 2e_n)\Big\| \\ &\le \left|\frac{\lambda_{n+1}}{\rho_{n+1}} - \frac{\lambda_n}{\rho_n}\right| \|u\| + \frac{\|e_{n+1}\|}{\rho_{n+1}} + \frac{\|e_n\|}{\rho_n} \\ &\quad + \left\|\frac{\delta_{n+1}}{2\rho_{n+1}}T_{n+1}x_{n+1} - \frac{\delta_n}{2\rho_n}T_n x_n\right\| \\ &\le \left|\frac{\lambda_{n+1}}{\rho_{n+1}} - \frac{\lambda_n}{\rho_n}\right| \|u\| + \frac{\|e_{n+1}\|}{\rho_{n+1}} + \frac{\|e_n\|}{\rho_n} \\ &\quad + \left|\frac{\delta_{n+1}}{2\rho_{n+1}} - \frac{\delta_n}{2\rho_n}\right| \|T_{n+1}x_{n+1}\| \\ &\quad + \frac{\delta_n}{2\rho_n} \|T_{n+1}x_{n+1} - T_n x_n\|. \end{aligned} \tag{4.4}$$

Since (x_n) is bounded and T_n is nonexpansive, we can find $M > 0$ so that $(||T_n x_n|| + ||x_n|| + ||u||) \le M$ for all $n \in \mathbb{N}$ Adding (4.3) and (4.4) and noting $\delta_n \le 2\rho_n$ yield

$$\begin{aligned} \|y_{n+1} - y_n\| &\le \left|\frac{\lambda_{n+1}}{\rho_{n+1}} - \frac{\lambda_n}{\rho_n}\right| \|u\| + \frac{\|e_{n+1}\|}{\rho_{n+1}} + \frac{\|e_n\|}{\rho_n} \\ &\quad + \left|\frac{\delta_{n+1}}{2\rho_{n+1}} - \frac{\delta_n}{2\rho_n}\right| \|T_{n+1}x_{n+1}\| \\ &\quad + \|x_{n+1} - x_n\| + \frac{|c_n - c_{n+1}|}{\bar{c}} \|T_n x_n - x_n\| \\ &\le \|x_{n+1} - x_n\| + M\left(\left|\frac{\lambda_{n+1}}{\rho_{n+1}} - \frac{\lambda_n}{\rho_n}\right| + \frac{\|e_{n+1}\|}{\rho_{n+1}}\right. \\ &\quad \left. + \frac{\|e_n\|}{\rho_n} + \left|\frac{\delta_{n+1}}{2\rho_{n+1}} - \frac{\delta_n}{2\rho_n}\right| + \frac{|c_n - c_{n+1}|}{\bar{c}}\right). \end{aligned}$$

With the knowledge that $||e_n|| \to 0$ and

$$\frac{\lambda_n}{\rho_n} = \frac{2\lambda_n}{2\lambda_n + \delta_n} \to 0, \quad \frac{\delta_n}{2\rho_n} = \frac{\delta_n}{2\lambda_n + \delta_n} \to 1,$$

we therefore deduce from (b) and (c) that

$$\begin{aligned} &\limsup_{n\to\infty}(\left\|y_{n+1}-y_n\right\| - \|x_{n+1}-x_n\|) \\ &\quad\le \limsup_{n\to\infty}\ M\left(\left|\frac{\lambda_{n+1}}{\rho_{n+1}}-\frac{\lambda_n}{\rho_n}\right| + \frac{\|e_{n+1}\|}{\rho_{n+1}} + \frac{\|e_n\|}{\rho_n}\right. \\ &\qquad \left. + \left|\frac{\delta_{n+1}}{2\rho_{n+1}}-\frac{\delta_n}{2\rho_n}\right| + \frac{|c_n-c_{n+1}|}{\bar{c}}\right) \to 0. \end{aligned}$$

Note that $\liminf_n \rho_n = \liminf_n(\delta_n/2) > 0$ and $\limsup_n \rho_n = \limsup_n (\delta_n/2) \le 1/2 < 1$. On the other hand, it is easy to check that (x_n) is bounded and so is (y_n) We therefore apply Lemma 3 to yield $\lim_n ||x_n - y_n|| = 0$. By means of (4.2), we finally have

$$\|x_{n+1}-x_n\| = \rho_n \left\|x_n - y_n\right\| \to,$$

and thus the required result at once follows.

As a corollary, we improve [3, Theorem 4.1] as follows.

Theorem 9. *Assume that the following conditions hold:*

(a) $\lim_n \lambda_n = 0$, $\Sigma_n \lambda_n = \infty$;

(b) $c_n \ge \bar{c} > 0$, $|c_{n+1} - c_n| \to 0$;

(c) $\Sigma_n ||e_n|| < \infty$.

Then the sequence generated by (1.4) *converges strongly to* $P_S(u)$.

Abbreviations
CPPA: contraction-proximal point algorithm; PPA: proximal point algorithm; RPPA: relaxed proximal point algorithm.

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Authors' contributions
Both authors contributed equally to this work. All authors read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

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