

New Exact Traveling Wave Solutions for the Nonlinear Klein-Gordon Equation

Zaiyun ZHANG

School of Mathematical Science and Computing Technology, Central South University

Changsha, 410075, Hunan province of P.R. CHINA

Department of Mathematics, Hunan Institute of Science and Technology

Yueyang, 414006, Hunan province of P.R. CHINA

e-mail: zhangzaiyun1226@126.com

Received 02.04.2008

Abstract

In this paper we discuss the nonlinear Klein-Gordon equation and we derive the new traveling wave solutions by applying trigonometric function series method. Also, they are complex linear combinations of kink solitary wave solutions and bell solitary wave solutions.

Key Words: Nonlinear Klein-Gordon equation, trig-function series, traveling wave solutions, Jacobi elliptic functions

PACS: 02.30.Jr, 04.20.Jb

1. Introduction

It is well known that traveling wave solutions of Nonlinear Partial Differential Equations (NPDEs) play an important role in the study of nonlinear wave phenomena. The wave phenomena are observed in fluid dynamics, plasma, elastic media, optical fibres, etc. Moreover, the wave phenomena are modeled by kink-shaped tanh solutions and bell-shaped sech solutions in physics. Most important, the NPDE is a Klein-Gordon equation (K-G) which arises in many fields such as nonlinear optics, Josephson array, ferromagnetic materials, charge density waves, and liquid helium and it has periodic solution, soliton solution of the helical wave and the kink form wave. It can be solved by means of inverse scattering method [1]. The K-G equation reads in the form

$$u_{tt} - u_{xx} + \sin u = 0. \quad (1)$$

In past decades, both mathematicians and physicists have made significant progress in seeking traveling wave solutions for NPDE. A comprehensive account of traveling wave solutions to the S-G equation can be found in papers by P. Rosenau and J. M. Hyman [1]; A. C. Scolt, F. F. Chu and D. W. McLaughlin [2]; M. J. Ablowitz, D. J. Kaup, A. C. Newell and H. Segar [3]; R. K. Dodd et al. [4]; and M. J. Ablowitz and P. A. Clarkson [5].

As we all know, there are many methods to study the exact solutions to NPDEs, such as method of symbolic computation [6], the homogeneous balance method [7–9], the tanh and extended methods [10–14], the Lax pairs representation method [15–16], the formal variable separation approach [17], the variation of

parameters approach [18], the special truncated expansion method[19], the Adomian decomposition method [20], Jacobi elliptic function expansion method [21–23], and so on.

Some types of NPDE solutions, such as the K-G equation, have Hamiltonian structure and are completely integrable. Physically, this would give rise to an ideal model, such as for quasi-particle currents.

In this paper, we consider the nonlinear K-G equation as

$$u_{tt} - u_{xx} + \alpha u - \beta u^3 = 0, \tag{2}$$

where α, β are constants. Equation (1.2) describes the propagation of dislocations within crystals, the Bloch wall motion of magnetic crystals, the propagation of a “splay wave” along a lied membrane, the unitary theory for elementary particles and the propagation of magnetic flux on a Josephson line, etc. [1].

In this letter, we apply a new method, using trigonometric series, which is most effective straightforward method to deal with NPDEs. Therefore, it is important to develop a simple and direct method for obtaining traveling wave solutions to the nonlinear K-G equation (1.2).

2. New Explicit Solutions

We introduce the trigonometric series method to seek new explicit solutions and we seek its explicit solution in the form

$$u(x, t) = \phi(\xi), \quad \xi = k(x - ct), \tag{3}$$

where k and c are the wave number and wave speed, respectively. According to the trig-function series method [8, 24], we suppose that equation (1.2) has the traveling wave solution

$$\phi(\xi) = \sum_{i=1}^{i=n} \sin^{i-1} \omega (a_i \sin \omega + a_i \cos \omega) + a_0, \tag{4}$$

and

$$\frac{d\omega}{d\xi} = \sqrt{1 - m^2} \sin \omega. \tag{5}$$

By [25], taking $i = 1$ in the above formula, we have the formal solutions

$$\phi(\xi) = a_0 + a_1 \sin \omega + b_1 \cos \omega, \tag{6}$$

and target equation

$$\frac{d\omega}{d\xi} = \sqrt{1 - m^2} \sin \omega.$$

Substituting (2.1) into (1.2), we obtain the ordinary differential equation (ODE)

$$k^2(c^2 - 1) \frac{d^2\phi}{d\xi^2} + \alpha\phi - \beta\phi^3 = 0. \tag{7}$$

With the aid of Mathematica, from (2.4) and (2.5) we can get

$$\begin{aligned} & k^2(c^2 - 1) \frac{d^2\phi}{d\xi^2} + \alpha\phi - \beta\phi^3 \\ &= [\alpha a_0 - \beta(a_0^3 + 3a_0 b_1^2)] \\ &+ [-k^2(c^2 - 1)(1 + m^2)a_1 + \alpha a_1 - 3\beta(a_0^2 + b_1^2)a_1] \sin \omega \\ &+ [-k^2(c^2 - 1)(1 + m^2)b_1 + \alpha b_1 - \beta(3a_0^2 + b_1^2)b_1] \cos \omega \\ &+ [-6\beta a_0 a_1 b_1] \sin \omega \cos \omega \\ &+ [-3\beta a_0 (a_0^2 - b_1^2)] \sin^2 \omega \\ &+ [2k^2(c^2 - 1)m^2 b_1 - \beta(3a_1^2 - b_1^2)b_1] \sin^2 \omega \cos \omega \\ &+ [2k^2(c^2 - 1)m^2 a_1 - \beta(a_1^2 - 3b_1^2)a_1] \sin^3 \omega \\ &= 0. \end{aligned}$$

Setting the coefficients of $\sin^j \omega \cos^i \omega$ for $i = 0, 1$ and $j = 1, 2, 3$, we have the following equations for constants a_0, a_1, b_1, k to be determined:

$$\begin{aligned} \alpha a_0 - \beta(a_0^3 + 3a_0 b_1^2) &= 0, \\ -k^2(c^2 - 1)(1 + m^2)a_1 + \alpha a_1 - 3\beta(a_0^2 + b_1^2)a_1 &= 0, \\ -k^2(c^2 - 1)(1 + m^2)b_1 + \alpha b_1 - \beta(3a_0^2 + b_1^2)b_1 &= 0, \\ \beta a_0 a_1 b_1 &= 0, \\ \beta a_0(a_0^2 - b_1^2) &= 0, \\ 2k^2(c^2 - 1)m^2 b_1 - \beta(3a_1^2 - b_1^2)b_1 &= 0, \\ 2k^2(c^2 - 1)m^2 a_1 - \beta(a_1^2 - 3b_1^2)a_1 &= 0. \end{aligned}$$

There are three cases for the above equations:

Case 1:

$$a_0 = 0, a_1 = 0, b_1 = \pm \sqrt{-\frac{2\alpha m^2}{2m^2 - 1}}, k = \sqrt{\frac{\alpha}{(2m^2 - 1)(c^2 - 1)}}.$$

So, we obtain the new exact solutions of (1.2):

$$u = b_1 \cos \omega = \pm \sqrt{-\frac{2\alpha m^2}{2m^2 - 1}} \cos \left[\sqrt{\frac{\alpha}{(2m^2 - 1)(c^2 - 1)}} (x - ct) \right].$$

Case 2:

$$a_0 = 0, b_1 = 0, a_1 = \pm \sqrt{\frac{2\alpha m^2}{m^2 + 1}}, k = \sqrt{\frac{\alpha}{(m^2 + 1)(c^2 - 1)}}.$$

from which we obtain the exact solutions of (1.2) as

$$u = a_1 \sin \omega = \pm \sqrt{\frac{2\alpha m^2}{m^2 + 1}} \sin \left[\sqrt{\frac{\alpha}{(m^2 + 1)(c^2 - 1)}} (x - ct) \right].$$

For Jacobi elliptic functions for nonlinear partial differential equations (NPDEs), our interest is in the exact periodic wave solutions. As we all know, there are three basic Jacobi elliptic functions: $sn\xi = sn \frac{\xi}{m}$, $cn\xi = cn \frac{\xi}{m}$ and $dn\xi = dn \frac{\xi}{m}$, where $m(0 < m < 1)$ is the modulus of the elliptic functions, satisfy the relations:

$$\begin{aligned} sn^2 \xi + cn^2 \xi &= 1, dn^2 \xi + m^2 sn^2 \xi = 1, (sn\xi)' = cn\xi dn\xi, \\ (cn\xi)' &= -sn\xi dn\xi, (dn\xi)' = -m^2 sn\xi cn\xi. \end{aligned}$$

So, we derive periodic wave solutions, namely, sn solutions

$$u = \pm \sqrt{-\frac{2\alpha m^2}{2m^2 - 1}} sn \left[\sqrt{\frac{\alpha}{(2m^2 - 1)(c^2 - 1)}} (x - ct) \right],$$

and cn solutions

$$u = a_1 \sin \omega = \pm \sqrt{\frac{2\alpha m^2}{m^2 + 1}} cn \left[\sqrt{\frac{\alpha}{(m^2 + 1)(c^2 - 1)}} (x - ct) \right],$$

respectively.

Especially, when $m \rightarrow 1$, the Jacobi elliptic functions degenerate to the functions

$$sn\xi \rightarrow \tanh \xi, \quad cn\xi \rightarrow \operatorname{sech} \xi, \quad dn\xi \rightarrow \operatorname{sech} \xi.$$

So, we derive kink solitary wave solutions

$$u = \pm \sqrt{-2\alpha} \tanh \left[\sqrt{\frac{\alpha}{c^2 - 1}} (x - ct) \right],$$

and bell solitary wave solutions

$$u = \pm\sqrt{\alpha} \operatorname{sech} \left[\sqrt{\frac{\alpha}{2(c^2-1)}}(x-ct) \right]$$

respectively.

Case 3:

$$a_0 = 0, a_1^2 + b_1^2 = 0, k = \sqrt{\frac{2\alpha}{(2-m^2)(c^2-1)}},$$

$$a_1 = \pm\sqrt{\frac{\alpha m^2}{\beta(2-m^2)}}, b_1 = \pm\sqrt{\frac{\alpha m^2}{\beta(2-m^2)}}i,$$

where $i^2 = -1$.

So, we get the exact solutions of (1.2) as

$$u = \pm\sqrt{\frac{\alpha m^2}{\beta(2-m^2)}} \left\{ \sin \left[\sqrt{\frac{2\alpha}{(2-m^2)(c^2-1)}}(x-ct) \right] + i \cos \left[\sqrt{\frac{2\alpha}{(2-m^2)(c^2-1)}}(x-ct) \right] \right\}.$$

Now integrating $\frac{d\omega}{d\xi} = \sin \omega$, and taking the integration constant zero, we obtain

$$\sin \omega = \frac{2 \exp(\pm\xi)}{\exp(\pm 2\xi) + 1} = \operatorname{sech} \xi. \tag{8}$$

At the same time, we get

$$\cos \omega = \pm \tanh \xi. \tag{9}$$

3. Traveling Wave Solutions

According to (2.6), (2.7) and the solutions in Cases 1–3 above, we have the following solitary wave solutions of equation (1.2).

I: If a_0, a_1, b_1, k, c satisfy Case 1, then

$$u_1(x, t) = \pm\sqrt{-2\alpha} \tanh \left[\sqrt{\frac{\alpha}{c^2-1}}(x-ct) \right].$$

II: If a_0, a_1, b_1, k satisfy Case 2, then

$$u_2(x, t) = \pm\sqrt{\alpha} \operatorname{sech} \left[\sqrt{\frac{\alpha}{2(c^2-1)}}(x-ct) \right].$$

III: If a_0, a_1, b_1, k satisfy Case 3, then we get new exact traveling solutions as

$$\begin{aligned} u_3(x, t) &= \pm\sqrt{\frac{\alpha}{\beta}}(\tanh \xi + \operatorname{sech} \xi) \\ &= \pm\sqrt{\frac{\alpha}{\beta}} \exp(\pm i\omega) \\ &= \pm\sqrt{\frac{\alpha}{\beta}} \exp \left\{ \pm i \arcsin \left[\operatorname{sech} \sqrt{\frac{2\alpha}{c^2-1}}(x-ct) \right] \right\}, \end{aligned}$$

and

$$u_4(x, t) = \pm\sqrt{\frac{\alpha}{\beta}} \exp \left\{ \pm i \arccos \left[\tanh \sqrt{\frac{2\alpha}{c^2-1}}(x-ct) \right] \right\},$$

where $i^2 = -1$.

Note that as $|\xi| \rightarrow \infty$, $u_3(x, t) = u_4(x, t) = \pm\sqrt{\frac{\alpha}{\beta}}$ and $u_3(x, t), u_4(x, t)$ are complex linear combinations of kink solitary wave solutions $u_1(x, t)$ and bell solitary wave solutions $u_2(x, t)$. Also, equation (1.2) does not have exact traveling wave solutions if $a_1, b_1 \in R$ in Case 3.

4. Conclusion and Discussion

The trig-function series method is used to construct wide classes of periodic traveling wave solutions of the NPDEs arising in nonlinear physics, such as Klein-Gordon equation [26], and the Landou-Ginzburg-Higgs equation [26–28]. The results revealed remarkable relations of solitary pattern, periodic solutions or solitons. The availability of computer systems like Mathematica or Maple facilitates the tedious algebraic calculations. The methods which we have proposed in this paper is also a standard, direct and effective method, which allow us to do complicate and tedious calculation.

It is worth noting that the proposed method is simple and effective and gives more solutions. The applied method will be used in further work to establish more entirely new solutions for other kinds of NPDEs.

Acknowledgement

The author would like to express sincerely thanks to the referees careful readings and useful comments on this note.

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