Gao and Ma *Journal of Inequalities and Applications* (2017) 2017:174
DOI 10.1186/s13660-017-1448-2

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

A new bound on the block restricted isometry constant in compressed sensing

Yi Gao[1*] and Mingde Ma[2]

*Correspondence: gaoyimh@163.com
[1]School of Mathematics and Information Science, Beifang University of Nationalities, Wenchang Road, Yinchuan, 750021, China
Full list of author information is available at the end of the article

Abstract

This paper focuses on the sufficient condition of block sparse recovery with the l_2/l_1-minimization. We show that if the measurement matrix satisfies the block restricted isometry property with $\delta_{2s|\mathcal{I}} < 0.6246$, then every block s-sparse signal can be exactly recovered via the l_2/l_1-minimization approach in the noiseless case and is stably recovered in the noisy measurement case. The result improves the bound on the block restricted isometry constant $\delta_{2s|\mathcal{I}}$ of Lin and Li (Acta Math. Sin. Engl. Ser. 29(7):1401-1412, 2013).

Keywords: compressed sensing; l_2/l_1-minimization; block sparse recovery; block restricted isometry property; null space property

1 Introduction

Compressed sensing [2–4] is a scheme which shows that some signals can be reconstructed from fewer measurements compared to the classical Nyquist-Shannon sampling method. This effective sampling method has a number of potential applications in signal processing, as well as other areas of science and technology. Its essential model is

$$\min_{x\in\mathbb{R}^N} \|x\|_0 \quad s.t. \quad y = Ax, \tag{1}$$

where $\|x\|_0$ denotes the number of non-zero entries of the vector x, an s-sparse vector $x \in \mathbb{R}^N$ is defined by $\|x\|_0 \le s \ll N$. However, the l_0-minimization (1) is a nonconvex and NP-hard optimization problem [5] and thus is computationally infeasible. To overcome this problem, one proposed the l_1-minimization [4, 6–9].

$$\min_{x\in\mathbb{R}^N} \|x\|_1 \quad s.t. \quad y = Ax, \tag{2}$$

where $\|x\|_1 = \sum_{i=1}^N |x_i|$. Candès [10] proved that the solutions to (2) are equivalent to those of (1) provided that the measurement matrices satisfy the restricted isometry property (RIP) [9, 11] with some definite restricted isometry constant (RIC) $\delta_s \in (0,1)$, here δ_s is defined as the smallest constant satisfying

$$(1-\delta_s)\|x\|_2^2 \le \|Ax\|_2^2 \le (1+\delta_s)\|x\|_2^2 \tag{3}$$

for any s-sparse vectors $x \in \mathbb{R}^N$.

However, the standard compressed sensing only considers the sparsity of the recovered signal, but it does not take into account any further structure. In many practical applications, for example, DNA microarrays [12], face recognition [13], color imaging [14], image annotation [15], multi-response linear regression [16], etc., the non-zero entries of sparse signal can be aligned or classified into blocks, which means that they appear in regions in a regular order instead of arbitrarily spread throughout the vector. These signals are called the block sparse signals and has attracted considerable interests; see [17–23] for more information.

Suppose that $x \in \mathbb{R}^N$ is split into m blocks, $x[1], x[2], \ldots, x[m]$, which are of length $d_1, d_2, \ldots, d_m$, respectively, that is,

$$x = [\underbrace{x_1, \ldots, x_{d_1}}_{x[1]}, \underbrace{x_{d_1+1}, \ldots, x_{d_1+d_2}}_{x[2]}, \ldots, \underbrace{x_{N-d_m+1}, \ldots, x_N}_{x[m]}]^T, \tag{4}$$

and $N = \sum_{i=1}^m d_i$. A vector $x \in \mathbb{R}^N$ is called block s-sparse over $\mathcal{I} = \{d_1, d_2, \ldots, d_m\}$ if $x[i]$ is non-zero for at most s indices i [18]. Obviously, $d_i = 1$ for each i, the block sparsity reduces to the conventional definition of a sparse vector. Let

$$\|x\|_{2,0} = \sum_{i=1}^m I\bigl(\|x[i]\|_2\bigr),$$

where $I(x)$ is an indicator function that equals 1 if $x > 0$ and 0 otherwise. So a block s-sparse vector x can be defined by $\|x\|_{2,0} \leq s$, and $\|x\|_0 = \sum_{i=1}^m \|x[i]\|_0$. Also, let Σ_s denote the set of all block s-sparse vectors: $\Sigma_s = \{x \in \mathbb{R}^N : \|x\|_{2,0} \leq s\}$.

To recover a block sparse signal, similar to the standard l_0-minimization, one seeks the sparsest block sparse vector via the following l_2/l_0-minimization [13, 17, 18]:

$$\min_{x \in \mathbb{R}^N} \|x\|_{2,0} \quad s.t. \quad y = Ax. \tag{5}$$

But the l_2/l_0-minimization problem is also NP-hard. It is natural to use the l_2/l_1-minimization to replace the l_2/l_0-minimization [13, 17, 18, 24].

$$\min_{x \in \mathbb{R}^N} \|x\|_{2,1} \quad s.t. \quad y = Ax, \tag{6}$$

where

$$\|x\|_{2,1} = \sum_{i=1}^m \|x[i]\|_2. \tag{7}$$

To characterize the performance of this method, Eldar and Mishali [18] proposed the block restricted isometry property (block RIP).

Definition 1 (Block RIP) Given a matrix $A \in \mathbb{R}^{n \times N}$, for every block s-sparse $x \in \mathbb{R}^N$ over $\mathcal{I} = \{d_1, d_2, \ldots, d_m\}$, there exists a positive constant $0 < \delta_{s|\mathcal{I}} < 1$, such that

$$(1 - \delta_{s|\mathcal{I}})\|x\|_2^2 \leq \|Ax\|_2^2 \leq (1 + \delta_{s|\mathcal{I}})\|x\|_2^2, \tag{8}$$

then the matrix A satisfies the s-order block RIP over $\mathcal{I}$, and the smallest constant $\delta_{s|\mathcal{I}}$ satisfying the above inequality (8) is called the block RIC of A.

Obviously, the block RIP is an extension of the standard RIP, but it is a less stringent requirement comparing to the standard RIP [18, 25]. Eldar *et al.* [18] proved that the l_2/l_1-minimization can exactly recover any block s-sparse signal when the measurement matrices A satisfy the block RIP with $\delta_{2s|\mathcal{I}} < 0.414$. The block RIC can be improved, for example, Lin and Li [1] improved the bound to $\delta_{2s|\mathcal{I}} < 0.4931$, and established another sufficient condition $\delta_{s|\mathcal{I}} < 0.307$ for exact recovery. So far, to the best of our knowledge, there is no paper that further focuses on improvement of the block RIC. As mentioned in [1, 26, 27], like RIC, there are several benefits for improving the bound on $\delta_{2s|\mathcal{I}}$. First, it allows more measurement matrices to be used in compressed sensing. Secondly, for the same matrix A, it allows for recovering a block sparse signal with more non-zero entries. Furthermore, it gives better error estimation in a general problem to recover noisy compressible signals. Therefore, this paper addresses improvement of the block RIC, we consider the following minimization for the inaccurate measurement, $y = Ax + e$ with $\|e\|_2 \le \epsilon$:

$$\min_{x\in\mathbb{R}^N} \|x\|_{2,1} \quad s.t. \quad \|y - Ax\|_2 \le \epsilon. \tag{9}$$

Our main result is stated in the following theorem.

Theorem 1 *Suppose that the* $2s$ *block RIC of the matrix* $A \in \mathbb{R}^{n\times N}$ *satisfies*

$$\delta_{2s|\mathcal{I}} < \frac{4}{\sqrt{41}} \approx 0.6246. \tag{10}$$

If x^* *is a solution to* (9), *then there exist positive constants* C_1, D_1 *and* C_2, D_2, *and we have*

$$\left\|x - x^*\right\|_{2,1} \le C_1\sigma_s(x)_{2,1} + D_1\sqrt{s}\epsilon, \tag{11}$$

$$\left\|x - x^*\right\|_2 \le \frac{C_2}{\sqrt{s}}\sigma_s(x)_{2,1} + D_2\epsilon, \tag{12}$$

where the constants C_1, D_1 *and* C_2, D_2 *depend only on* $\delta_{2s|\mathcal{I}}$, *written as*

$$C_1 = \frac{2(4\sqrt{1-\delta_{2s|\mathcal{I}}^2} + 3\delta_{2s|\mathcal{I}})}{4\sqrt{1-\delta_{2s|\mathcal{I}}^2} - 5\delta_{2s|\mathcal{I}}}, \tag{13}$$

$$D_1 = \frac{16\sqrt{1+\delta_{2s|\mathcal{I}}}}{4\sqrt{1-\delta_{2s|\mathcal{I}}^2} - 5\delta_{2s|\mathcal{I}}}, \tag{14}$$

$$C_2 = \frac{2(4\sqrt{1-\delta_{2s|\mathcal{I}}^2} + 3\delta_{2s|\mathcal{I}})^2}{(4\sqrt{1-\delta_{2s|\mathcal{I}}^2} - \delta_{2s|\mathcal{I}})(4\sqrt{1-\delta_{2s|\mathcal{I}}^2} - 5\delta_{2s|\mathcal{I}})}, \tag{15}$$

$$D_2 = \frac{8\sqrt{1+\delta_{2s|\mathcal{I}}}(12\sqrt{1-\delta_{2s|\mathcal{I}}^2} + \delta_{2s|\mathcal{I}})}{(4\sqrt{1-\delta_{2s|\mathcal{I}}^2} - \delta_{2s|\mathcal{I}})(4\sqrt{1-\delta_{2s|\mathcal{I}}^2} - 5\delta_{2s|\mathcal{I}})}, \tag{16}$$

and $\sigma_s(x)_{2,1}$ denotes the best block s-term approximation error of $x \in \mathbb{R}^N$ in l_2/l_1 norm, i.e.,

$$\sigma_s(x)_{2,1} := \inf_{z\in\Sigma_s} \|x - z\|_{2,1}. \tag{17}$$

Corollary 1 *Under the same assumptions as in Theorem* 1, *suppose that $e = 0$ and x is block s-sparse, then x can be exactly recovered via the l_2/l_1-minimization* (6).

The remainder of the paper is organized as follows. In Section 2, we introduce the $l_{2,1}$ robust block NSP that can characterize the stability and robustness of the l_1-minimization with noisy measurement (9). In Section 3, we show that the condition (10) can conclude the $l_{2,1}$ robust block NSP, which means to implement the proof of our main result. Section 4 is for our conclusions. The last section is an appendix including an important lemma.

2 Block null space property

Although null space property (NSP) is a very important concept in approximation theory [28, 29], it provides a necessary and sufficient condition of the existence and uniqueness of the solution to the l_1-minimization (2), so NSP has drawn extensive attention for studying the characterization of measurement matrix in compressed sensing [30]. It is natural to extend the classic NSP to the block sparse case. For this purpose, we introduce some notations. Suppose that $x \in \mathbb{R}^N$ is an m-block signal, whose structure is like (4), we set $S \subset \{1, 2, \dots, m\}$ and by S^C we mean the complement of the set S with respect to $\{1, 2, \dots, m\}$, *i.e.*, $S^C = \{1, 2, \dots, m\} \setminus S$. Let x_S denote the vector equal to x on a block index set S and zero elsewhere, then $x = x_S + x_{S^C}$. Here, to investigate the solution to the model (9), we introduce the $l_{2,1}$ robust block NSP, for more information on other forms of block NSP, we refer the reader to [23, 31].

Definition 2 ($l_{2,1}$ robust block NSP) Given a matrix $A \in \mathbb{R}^{n\times N}$, for any set $S \subset \{1, 2, \dots, m\}$ with $\mathrm{card}(S) \le s$ and for all $v \in \mathbb{R}^N$, if there exist constants $0 < \tau < 1$ and $\gamma > 0$, such that

$$\|v_S\|_2 \le \frac{\tau}{\sqrt{s}} \|v_{S^C}\|_{2,1} + \gamma \|Av\|_2, \tag{18}$$

then the matrix A is said to satisfy the $l_{2,1}$ robust block NSP of order s with τ and γ.

Our main result relies heavily on this definition. A natural question is what relationship between this robust block NSP and the block RIP. Indeed, from the next section, we shall see that the block RIP with condition (10) can lead to the $l_{2,1}$ robust block NSP, that is, the $l_{2,1}$ robust block NSP is weaker than the block RIP to some extent. The spirit of this definition is first to imply the following theorem.

Theorem 2 *For any set $S \subset \{1, 2, \dots, m\}$ with* $\mathrm{card}(S) \le s$, *the matrix $A \in \mathbb{R}^{n\times N}$ satisfies the $l_{2,1}$ robust block NSP of order s with constants $0 < \tau < 1$ and $\gamma > 0$, then, for all vectors $x, z \in \mathbb{R}^N$,*

$$\|x - z\|_{2,1} \le \frac{1+\tau}{1-\tau} \big(\|z\|_{2,1} - \|x\|_{2,1} + 2\|x_{S^C}\|_{2,1}\big) + \frac{2\gamma\sqrt{s}}{1-\tau} \big\|A(x - z)\big\|_2. \tag{19}$$

Proof For $x, z \in \mathbb{R}^N$, setting $v = x - z$, we have

$$\|v_{S^C}\|_{2,1} \leq \|x_{S^C}\|_{2,1} + \|z_{S^C}\|_{2,1},$$
$$\|x\|_{2,1} = \|x_{S^C}\|_{2,1} + \|x_S\|_{2,1} \leq \|x_{S^C}\|_{2,1} + \|v_S\|_{2,1} + \|z_S\|_{2,1},$$

which yield

$$\|v_{S^C}\|_{2,1} \leq 2\|x_{S^C}\|_{2,1} + \|z\|_{2,1} - \|x\|_{2,1} + \|v_S\|_{2,1}. \tag{20}$$

Clearly, for an m-block vector $x \in \mathbb{R}^N$ is like (4), l_2-norm $\|x\|_2$ can be rewritten as

$$\|x\|_2 = \|x\|_{2,2} = \sum_{i=1}^{m} \left(\|x[i]\|_2^2\right)^{1/2}. \tag{21}$$

Thus, we have $\|v_S\|_{2,2} = \|v_S\|_2$ and $\|v_S\|_{2,1} \leq \sqrt{s}\|v_S\|_{2,2}$. So the $l_{2,1}$ robust block NSP implies

$$\|v_S\|_{2,1} \leq \tau\|v_{S^C}\|_{2,1} + \gamma\sqrt{s}\|Av\|_2. \tag{22}$$

Combining (20) with (22), we can get

$$\|v_{S^C}\|_{2,1} \leq \frac{1}{1-\tau}\left(2\|x_{S^C}\|_{2,1} + \|z\|_{2,1} - \|x\|_{2,1}\right) + \frac{\gamma\sqrt{s}}{1-\tau}\|Av\|_2.$$

Using (22) once again, we derive

$$\begin{aligned}\|v\|_{2,1} &= \|v_{S^C}\|_{2,1} + \|v_S\|_{2,1} \leq (1+\tau)\|v_{S^C}\|_{2,1} + \gamma\sqrt{s}\|Av\|_2 \\ &\leq \frac{1+\tau}{1-\tau}\left(2\|x_{S^C}\|_{2,1} + \|z\|_{2,1} - \|x\|_{2,1}\right) + \frac{2\gamma\sqrt{s}}{1-\tau}\|Av\|_2,\end{aligned}$$

which is the desired inequality. □

The $l_{2,1}$ robust block NSP is vital to characterize the stability and robustness of the l_2/l_1-minimization with noisy measurement (9), which is the following result.

Theorem 3 *Suppose that the matrix $A \in \mathbb{R}^{n\times N}$ satisfies the $l_{2,1}$ robust block NSP of order s with constants $0 < \tau < 1$ and $\gamma > 0$, if x^* is a solution to the l_2/l_1-minimization with $y = Ax + e$ and $\|e\|_2 \leq \epsilon$, then there exist positive constants C_3, D_3 and C_4, D_4, and we have*

$$\|x - x^*\|_{2,1} \leq C_3\sigma_s(x)_{2,1} + D_3\sqrt{s}\epsilon, \tag{23}$$
$$\|x - x^*\|_2 \leq \frac{C_4}{\sqrt{s}}\sigma_s(x)_{2,1} + D_4\epsilon, \tag{24}$$

where

$$C_3 = \frac{2(1+\tau)}{1-\tau}; \qquad D_3 = \frac{4\gamma}{1-\tau}. \tag{25}$$
$$C_4 = \frac{2(1+\tau)^2}{1-\tau}; \qquad D_4 = \frac{2\gamma(3+\tau)}{1-\tau}. \tag{26}$$

Proof In Theorem 2, by S denote an index set of s largest l_2-norm terms out of m blocks in x, (23) is a direct corollary of Theorem 2 if we notice that $\|x_{S^C}\|_{2,1} = \sigma_s(x)_{2,1}$ and $\|A(x-x^*)\|_2 \le 2\epsilon$. Equation (24) is a result of Theorem 7 for $q = 1$ in [23]. □

3 Proof of the main result

From Theorem 3, we see that the inequalities (23) and (24) are the same as in (11) and (12) up to constants, respectively. This means that we shall only show that the condition (10) implies the $l_{2,1}$ robust block NSP for implementing the proof of our main result.

Theorem 4 *Suppose that the* $2s$ *block RIC of the matrix* $A \in \mathbb{R}^{n\times N}$ *obeys* (10), *then the matrix A satisfies the* $l_{2,1}$ *robust block NSP of order s with constants* $0 < \tau < 1$ *and* $\gamma > 0$, *where*

$$\tau = \frac{4\delta_{2s|\mathcal{I}}}{4\sqrt{1-\delta_{2s|\mathcal{I}}^2} - \delta_{2s|\mathcal{I}}}, \qquad \gamma = \frac{4\sqrt{1+\delta_{2s|\mathcal{I}}}}{4\sqrt{1-\delta_{2s|\mathcal{I}}^2} - \delta_{2s|\mathcal{I}}}. \tag{27}$$

Proof The proof relies on a technique introduced in [30]. Suppose that the matrix A has the block RIP with $\delta_{2s|\mathcal{I}}$. Let v be divided into m blocks whose structure is like (38). Let $S =: S_0$ be an index set of s largest l_2-norm terms out of m blocks in v. We begin by dividing S^C into subsets of size s, S_1 is the first s largest l_2-norm terms in S^C, S_2 is the next s largest l_2-norm terms in S^C, etc. Since the vector v_S is block s-sparse, according to the block RIP, for $|t| \le \delta_{2s|\mathcal{I}}$, we can write

$$\|Av_S\|_2^2 = (1+t)\|v_S\|_2^2. \tag{28}$$

We are going to establish that, for any $j \ge 1$,

$$\bigl|\langle Av_S, Av_{S_j}\rangle\bigr| \le \sqrt{\delta_{2s|\mathcal{I}}^2 - t^2}\,\|v_S\|_2\|v_{S_j}\|_2. \tag{29}$$

To do so, we normalize the vectors v_S and v_{S_j} by setting $u =: v_S/\|v_S\|_2$ and $w =: v_{S_j}/\|v_{S_j}\|_2$. Then, for $\alpha, \beta > 0$, we write

$$2\langle Au, Aw\rangle = \frac{1}{\alpha+\beta}\bigl[\bigl\|A(\alpha u + w)\bigr\|_2^2 - \bigl\|A(\beta u - w)\bigr\|_2^2 - \bigl(\alpha^2 - \beta^2\bigr)\|Au\|_2^2\bigr]. \tag{30}$$

By the block RIP, on the one hand, we have

$$\begin{aligned} 2\langle Au, Aw\rangle &\le \frac{1}{\alpha+\beta}\bigl[(1+\delta_{2s|\mathcal{I}})\|\alpha u + w\|_2^2 - (1-\delta_{2s|\mathcal{I}})\|\beta u - w\|_2^2\bigr] \\ &\quad - \frac{1}{\alpha+\beta}\bigl(\alpha^2 - \beta^2\bigr)(1+t)\|u\|_2^2 \\ &= \frac{1}{\alpha+\beta}\bigl[\alpha^2(\delta_{2s|\mathcal{I}} - t) + \beta^2(\delta_{2s|\mathcal{I}} + t) + 2\delta_{2s|\mathcal{I}}\bigr]. \end{aligned}$$

Making the choice $\alpha = \frac{(\delta_{2s|\mathcal{I}}+t)}{\sqrt{\delta_{2s|\mathcal{I}}^2 - t^2}}$, $\beta = \frac{(\delta_{2s|\mathcal{I}}-t)}{\sqrt{\delta_{2s|\mathcal{I}}^2 - t^2}}$, we derive

$$\langle Au, Aw\rangle \le \sqrt{\delta_{2s|\mathcal{I}}^2 - t^2}. \tag{31}$$

On the other hand, we also have

$$\begin{aligned}2\langle Au, Aw\rangle &\geq \frac{1}{\alpha+\beta}\big[(1-\delta_{2s|\mathcal{I}})\|\alpha u + w\|_2^2 - (1+\delta_{2s|\mathcal{I}})\|\beta u - w\|_2^2\big] \\ &\quad - \frac{1}{\alpha+\beta}\big(\alpha^2-\beta^2\big)(1+t)\|u\|_2^2 \\ &= -\frac{1}{\alpha+\beta}\big[\alpha^2(\delta_{2s|\mathcal{I}}-t) + \beta^2(\delta_{2s|\mathcal{I}}+t) + 2\delta_{2s|\mathcal{I}}\big].\end{aligned}$$

Making the choice $\alpha = \frac{(\delta_{2s|\mathcal{I}}-t)}{\sqrt{\delta_{2s|\mathcal{I}}^2-t^2}}$, $\beta = \frac{(\delta_{2s|\mathcal{I}}+t)}{\sqrt{\delta_{2s|\mathcal{I}}^2-t^2}}$, we get

$$\langle Au, Aw\rangle \geq -\sqrt{\delta_{2s|\mathcal{I}}^2 - t^2}. \tag{32}$$

Combining (31) with (32) yields the desired inequality (29). Next, noticing that $Av_S = A(v - \sum_{j\geq 1} Av_{S_j})$, we have

$$\begin{aligned}\|Av_S\|_2^2 &= \langle Av_S, Av\rangle - \sum_{j\geq 1}\langle Av_S, Av_{S_j}\rangle \\ &\leq \|Av_S\|_2\|Av\|_2 + \sum_{j\geq 1}\sqrt{\delta_{2s|\mathcal{I}}^2 - t^2}\|v_S\|_2\|v_{S_j}\|_2 \\ &= \|v_S\|_2\left(\sqrt{1+t}\|Av\|_2 + \sqrt{\delta_{2s|\mathcal{I}}^2 - t^2}\sum_{j\geq 1}\|v_{S_j}\|_2\right).\end{aligned} \tag{33}$$

According to Lemma A.1 and the setting of S_j, we have

$$\begin{aligned}\sum_{j\geq 1}\|v_{S_j}\|_{2,2} &\leq \sum_{j\geq 1}\left[\frac{1}{\sqrt{s}}\|v_{S_j}\|_{2,1} + \frac{\sqrt{s}}{4}\big(\big\|v_{S_j}[1]\big\|_2 - \big\|v_{S_j}[s]\big\|_2\big)\right] \\ &\leq \frac{1}{\sqrt{s}}\|v_{S^C}\|_{2,1} + \frac{\sqrt{s}}{4}\big\|v_{S_1}[1]\big\|_2 \\ &\leq \frac{1}{\sqrt{s}}\|v_{S^C}\|_{2,1} + \frac{1}{4}\|v_S\|_2.\end{aligned} \tag{34}$$

Substituting (34) into (33) and noticing (28), we also have

$$(1+t)\|v_S\|_2 \leq \sqrt{1+t}\|Av\|_2 + \frac{\sqrt{\delta_{2s|\mathcal{I}}^2 - t^2}}{\sqrt{s}}\|v_{S^C}\|_{2,1} + \frac{\sqrt{\delta_{2s|\mathcal{I}}^2 - t^2}}{4}\|v_S\|_2,$$

that is,

$$\|v_S\|_2 \leq \frac{\sqrt{\delta_{2s|\mathcal{I}}^2 - t^2}}{\sqrt{s}(1+t)}\|v_{S^C}\|_{2,1} + \frac{\sqrt{\delta_{2s|\mathcal{I}}^2 - t^2}}{4(1+t)}\|v_S\|_2 + \frac{1}{\sqrt{1+t}}\|Av\|_2.$$

Let

$$f(t) = \frac{\sqrt{\delta_{2s|\mathcal{I}}^2 - t^2}}{1+t}, \quad |t| \leq \delta_{2s|\mathcal{I}}, \tag{35}$$

then it is not difficult to conclude that $f(t)$ has a maximum point $t = -\delta_{2s|\mathcal{I}}^2$ in the closed interval $[-\delta_{2s|\mathcal{I}}, \delta_{2s|\mathcal{I}}]$, so for $|t| \leq \delta_{2s|\mathcal{I}}$, we have

$$f(t) = \frac{\sqrt{\delta_{2s|\mathcal{I}}^2 - t^2}}{1+t} \leq \frac{\delta_{2s|\mathcal{I}}}{\sqrt{1-\delta_{2s|\mathcal{I}}^2}}. \tag{36}$$

Therefore,

$$\|v_S\|_2 \leq \frac{\delta_{2s|\mathcal{I}}}{\sqrt{1-\delta_{2s|\mathcal{I}}^2}} \frac{1}{\sqrt{s}} \|v_{S^C}\|_{2,1} + \frac{\delta_{2s|\mathcal{I}}}{4\sqrt{1-\delta_{2s|\mathcal{I}}^2}} \|v_S\|_2 + \frac{1}{\sqrt{1-\delta_{2s|\mathcal{I}}}} \|Av\|_2,$$

that is,

$$\|v_S\|_2 \leq \frac{4\delta_{2s|\mathcal{I}}}{4\sqrt{1-\delta_{2s|\mathcal{I}}^2} - \delta_{2s|\mathcal{I}}} \frac{1}{\sqrt{s}} \|v_{S^C}\|_{2,1} + \frac{4\sqrt{1+\delta_{2s|\mathcal{I}}}}{4\sqrt{1-\delta_{2s|\mathcal{I}}^2} - \delta_{2s|\mathcal{I}}} \|Av\|_2.$$

Here, we require

$$4\sqrt{1-\delta_{2s|\mathcal{I}}^2} - \delta_{2s|\mathcal{I}} > 0, \qquad \frac{4\delta_{2s|\mathcal{I}}}{4\sqrt{1-\delta_{2s|\mathcal{I}}^2} - \delta_{2s|\mathcal{I}}} < 1, \tag{37}$$

which implies $\delta_{2s|\mathcal{I}}^2 < \frac{16}{41}$, that is, $\delta_{2s|\mathcal{I}} < \frac{4}{\sqrt{41}} \approx 0.6246$. □

Remark 1 Substituting (27) into (25) and (26), we can obtain the constants in Theorem 1.

Remark 2 Our result improves that of [1], that is, the bound of block RIC $\delta_{2s|\mathcal{I}}$ is improved from 0.4931 to 0.6246.

4 Conclusions

In this paper, we gave a new bound on the block RIC $\delta_{2s|\mathcal{I}} < 0.6246$, under this bound, every block s-sparse signal can be exactly recovered via the l_2/l_1-minimization approach in the noiseless case and is stably recovered in the noisy measurement case. The result improves the bound on the block RIC $\delta_{2s|\mathcal{I}}$ in [1].

Appendix

Lemma A.1 *Suppose that $v \in \mathbb{R}^N$ is split into m blocks, $v[1], v[2], \dots, v[m]$, which are of length $d_1, d_2, \dots, d_m$, respectively, that is,*

$$v = [\underbrace{v_1, \dots, v_{d_1}}_{v[1]}, \underbrace{v_{d_1+1}, \dots, v_{d_1+d_2}}_{v[2]}, \dots, \underbrace{v_{N-d_m+1}, \dots, v_N}_{v[m]}]^T. \tag{38}$$

Suppose that the m blocks in x are rearranged by nonincreasing order for which

$$\|v[1]\|_2 \geq \|v[2]\|_2 \geq \dots \geq \|v[m]\|_2 \geq 0.$$

Then

$$\sqrt{\|v[1]\|_2^2 + \cdots + \|v[m]\|_2^2} \le \frac{\|v[1]\|_2 + \cdots + \|v[m]\|_2}{\sqrt{m}} + \frac{\sqrt{m}}{4}\left(\|v[1]\|_2 - \|v[m]\|_2\right). \tag{39}$$

Proof See Lemma 6.14 in [30] for the details. □

Acknowledgements
This work was supported by the Science Research Project of Ningxia Higher Education Institutions of China (NGY2015152), National Natural Science Foundation of China (NSFC) (11561055), and Science Research Project of Beifang University of Nationalities (2016SXKY07).

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The two authors contributed equally to this work, and they read and approved the final manuscript.

Author details
[1]School of Mathematics and Information Science, Beifang University of Nationalities, Wenchang Road, Yinchuan, 750021, China. [2]Editorial Department of University Journal, Beifang University of Nationalities, Wenchang Road, Yinchuan, 750021, China.

Publisher's Note

Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 28 April 2017 Accepted: 10 July 2017 Published online: 01 August 2017

References

1. Lin, J, Li, S: Block sparse recovery via mixed l_2/l_1 minimization. Acta Math. Sin. Engl. Ser. **29**(7), 1401-1412 (2013)
2. Donoho, D: Compressed sensing. IEEE Trans. Inf. Theory **52**(4), 1289-1306 (2006)
3. Candès, E, Romberg, J, Tao, T: Robust uncertainty principles: exact signal reconstruction from highly incomplete frequency information. IEEE Trans. Inf. Theory **52**(2), 489-509 (2006)
4. Candès, E, Tao, T: Near-optimal signal recovery from random projections: universal encoding strategies. IEEE Trans. Inf. Theory **52**(12), 5406-5425 (2006)
5. Natarajan, BK: Sparse approximate solutions to linear systems. SIAM J. Comput. **24**, 227-234 (1995)
6. Donoho, D, Huo, X: Uncertainty principles and ideal atomic decompositions. IEEE Trans. Inf. Theory **47**(4), 2845-2862 (2001)
7. Elad, M, Bruckstein, A: A generalized uncertainty principle and sparse representation in pairs of bases. IEEE Trans. Inf. Theory **48**(9), 2558-2567 (2002)
8. Gribonval, R, Nielsen, M: Sparse representations in unions of bases. IEEE Trans. Inf. Theory **49**(5), 3320-3325 (2003)
9. Candès, E, Tao, T: Decoding by linear programming. IEEE Trans. Inf. Theory **51**(12), 4203-4215 (2005)
10. Candès, E: The restricted isometry property and its implications for compressed sensing. C. R. Math. Acad. Sci. Paris, Sér. I **346**, 589-592 (2008)
11. Baraniuk, R, Davenport, M, DeVore, R, Wakin, M: A simple proof of the restricted isometry property for random matrices. Constr. Approx. **28**, 253-263 (2008)
12. Parvaresh, F, Vikalo, H, Misra, S, Hassibi, B: Recovering sparse signals using sparse measurement matrices in compressed DNA microarrays. IEEE J. Sel. Top. Signal Process. **2**(3), 275-285 (2008)
13. Elhamifar, E, Vidal, R: Block-sparse recovery via convex optimization. IEEE Trans. Signal Process. **60**(8), 4094-4107 (2012)
14. Majumdar, A, Ward, R: Compressed sensing of color images. Signal Process. **90**, 3122-3127 (2010)
15. Huang, J, Huang, X, Metaxas, D: Learning with dynamic group sparsity. In: IEEE 12th International Conference on Computer Vision, pp. 64-71 (2009)
16. Simila, T, Tikka, J: Input selection and shrinkage in multiresponse linear regression. Comput. Stat. Data Anal. **52**, 406-422 (2007)
17. Eldar, Y, Kuppinger, P, Bolcskei, H: Block-sparse signals: uncertainty relations and efficient recovery. IEEE Trans. Signal Process. **58**(6), 3042-3054 (2010)
18. Eldar, Y, Mishali, M: Robust recovery of signals from a structured union of subspaces. IEEE Trans. Inf. Theory **55**(11), 5302-5316 (2009)
19. Fu, Y, Li, H, Zhang, Q, Zou, J: Block-sparse recovery via redundant block OMP. Signal Process. **97**, 162-171 (2014)
20. Afdideh, F, Phlypo, R, Jutten, C: Recovery guarantees for mixed norm $l_{p1,p2}$ block sparse representations. In: 24th European Signal Processing Conference (EUSIPCO), pp. 378-382 (2016)
21. Wen, J, Zhou, Z, Liu, Z, Lai, M-J, Tang, X: Sharp sufficient conditions for stable recovery of block sparse signals by block orthogonal matching pursuit (2016). arXiv:1605.02894v1

22. Karanam, S, Li, Y, Radke, RJ: Person re-identification with block sparse recovery. Image Vis. Comput. (2017). doi:10.1016/j.imavis.2016.11.015
23. Gao, Y, Peng, J, Yue, S: Stability and robustness of the l_2/l_q-minimization for block sparse recovery. Signal Process. **137**, 287-297 (2017)
24. Stojnic, M, Parvaresh, F, Hassibi, B: On the reconstruction of block-sparse signals with an optimal number of measurements. IEEE Trans. Signal Process. **57**(8), 3075-3085 (2009)
25. Baraniuk, R, Cevher, V, Duarte, M, Hegde, C: Model-based compressive sensing. IEEE Trans. Inf. Theory **56**(4), 1982-2001 (2010)
26. Mo, Q, Li, S: New bounds on the restricted isometry constant δ_{2k}. Appl. Comput. Harmon. Anal. **31**, 460-468 (2011)
27. Lin, J, Li, S, Shen, Y: New bounds for restricted isometry constants with coherent tight frames. IEEE Trans. Signal Process. **61**(3), 611-621 (2013)
28. Cohen, A, Dahmen, W, DeVore, A: Compressed sensing and best k-term approximation. J. Am. Math. Soc. **22**(1), 211-231 (2009)
29. Pinkus, A: On L_1-Approximation. Cambridge University Press, Cambridge (1989)
30. Foucart, S, Rauhut, H: A Mathematical Introduction to Compressive Sensing. Springer, New York (2013)
31. Gao, Y, Peng, J, Yue, S, Zhao, Y: On the null space property of l_q-minimization for $0 < q \leq 1$ in compressed sensing. J. Funct. Spaces **2015**, Article ID 579853 (2015). doi:10.1155/2015/579853