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Full Length Research Paper

Symbolic computation of multisoliton solutions to generalized sixth-order KdV equation (KdV6)

Alvaro H. Salas[1]*, Orlando García Hurtado[2] and Roberto Manuel Poveda Chaves[2]

[1]Universidad de Caldas, Department of Mathematics, Universidad Nacional de Colombia, Campus la Nubia, Manizales, Caldas, Colombia.
[2]Facultad de Ingeniería Universidad Distrital "Francisco José de Caldas" Bogotá, Colombia.

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One, two and three soliton solutions are formally derived for a generalized version of the sixth-order KdV equation (KdV6) by using the Cole-Hopf transform. A comparison with particular cases of recently discovered completely integrable KdV6 equations are considered to illustrate the results obtained.

Key words: Sixth-order KdV equation, $KdV6$, exact travelling wave solution, one soliton solution, two soliton solution, three soliton solution, multisoliton solution, solitary wave solution, *Mathematica 8*, *Maple 15*.

INTRODUCTION

The general sixth-order KdV equation (KdV6) reads

$$u_{xxxxxx}+au_xu_{xxxx}+bu_{xx}u_{xxx}+cu_x^2u_{xx}+du_{tt}+eu_{xxxt}+fu_xu_{xt}+gu_tu_{xx}=0, e\neq 0. \quad (1)$$

where a,b,c,d,e,f and g are arbitrary parameters. This equation was derived by Karasu et al. (2008). They found that there are four distinct cases of relations between the parameters for Equation 1 to pass the Painlevè test. Three of them were well-known integrable systems: A bidirectional version of the Sawada-Kotera-Caudrey-Dodd-Gibbon equation and the Kaup-Kupershmidt equation as well as the Drinfeld- Sokolov-Hirota-Satsuma equation (Kupershmidt, 2008; Caudrey et al., 1976; Ramani, 1981; Ramani et al., 2008; Kaup, 1980; Drinfeld and Sokolov, 1981). Further, Yao et al. (2008a, b) showed that the $KdV6$ equation is equivalent to the Rosochatius deformation of KdV equation with self-consistent sources. Gómez and Salas (2008) studied some exact solutions for the particular case $a=20$, $b=40$, $c=120$, $d=0$, $e=1$, $f=8$, $g=4$, that is,

$$u_{xxxxxx}+20u_xu_{xxxx}+40u_{xx}u_{xxx}+120u_x^2u_{xx}+u_{xxxt}+8u_xu_{xt}+4u_tu_{xx}=0 \quad (2)$$

by using the Cole-Hopf transformation and an improved tanh-coth method. Wazwaz (2008) obtained multiple soliton solutions and multiple singular soliton solutions of Equation 2. Moreover, in a recent work (Zhang et al., 2009), authors obtained new bilinear forms of the $KdV6$ equation and the multi-soliton solutions were derived.

In this paper, we shall study the KdV6 from a more general point of view. The main purpose is to generalize results in previous cited works. We always shall suppose that the KdV6 Equation 1 satisfies the following condition (Karasu et al., 2008):

$$c=12a+6b-360. \quad (3)$$

Observe that Equation 2 satisfies condition (3). We will derive one, two and three soliton solutions under certain additional restrictions over the coefficients a,b,d,e,f and g . To this end, we first substitute

$$u(x,t)=\exp(kx-\omega t) \quad (4)$$

into the linear terms of the Equation 1, that is, we consider the simplified equation

$$u_{xxxxxx}+du_{tt}+eu_{xxxt}=0 \quad (5)$$

to determine the dispersion relation between k and

*Corresponding author. E-mail: asalash2002@yahoo.com.

ω. This gives

$$d\omega^2 - ek^3\omega + k^6 = 0, \tag{6}$$

from which

$$\omega = \frac{e \pm \sqrt{e^2-4d}}{2d}k^3 \ \text{ if } \ d \neq 0 \tag{7}$$

and

$$\omega = \frac{1}{e}k^3 \text{ if } \ d = 0. \tag{8}$$

ONE SOLITON SOLUTIONS TO KDV6

We seek solutions to Equation 1 in the Cole-Hopf form

$$u(x,t) = R\frac{\partial_x f(x,t)}{f(x,t)}, \tag{9}$$

for the special choice

$$f(x,t) = 1 + \exp(kx - \omega t), \tag{10}$$

where R is some constant and ω is given by either Equations 7 or 8. Substituting Equation 9 to 10 into Equation 1 gives a polynomial equation in the variable $X = \exp(kx - \omega t)$. Equating the coefficients of $X^i (i = 0,1,2,...)$ to zero, we obtain an algebraic system in the variables a,b,d,e,f , g, ω, k and R. Solving it with the aid of a computer package such as Mathematica 8 or Maple 15 gives us following solutions:

$$\omega = \frac{e \pm \sqrt{e^2-4d}}{2d}k^3,$$

$$g = \frac{1}{2}\left((a+b-36)e - 2f \pm (60-a-b)\sqrt{e^2-4d}\right), \ R = 1 \ :$$

$$u(x,t) = \frac{k\exp(kx-\omega t)}{1+\exp(kx-\omega t)}. \tag{11}$$

$$\omega = \frac{e \pm \sqrt{e^2-4d}}{2d}k^3,$$

$$g = -\frac{1}{10}((a-2b-180)e + 10f) \pm \tfrac{1}{2}(a-60)\sqrt{e^2-4d},$$

$$R = \frac{60}{2a+b-60} \ :$$

$$u(x,t) = \frac{60k\exp(kx-\omega t)}{(2a+b-60)(1+\exp(kx-\omega t))}, \text{ where } 2a+b-60 \neq 0. \tag{12}$$

$$d = 0, \ \omega = \frac{k^3}{e}, \ g = (a+b-48)e - f, \ R = 1 \ :$$

$$u(x,t) = \frac{k\exp(kx-\omega t)}{1+\exp(kx-\omega t)}. \tag{13}$$

$$d = 0, \ \omega = \frac{k^3}{e}, \ g = -15((3a-b-240)e + 5f), \ R = \frac{60}{2a+b-60} \ :$$

$$u(x,t) = \frac{60k\exp(kx-\omega t)}{(2a+b-60)(1+\exp(kx-\omega t))}, \quad \text{where } 2a+b-60 \neq 0. \tag{14}$$

TWO SOLITON SOLUTIONS TO KDV6

We look for two-soliton solutions to Equation 1 in the form

$$u(x,t) = R\frac{\partial_x f(x,t)}{f(x,t)}, \tag{15}$$

for the special choice

$$f(x,t) = 1 + \exp(k_1x - \omega_1 t) + \exp(k_2 x - \omega_2 t) + S\exp((k_1+k_2)x - (\omega_1+\omega_2)t), \tag{16}$$

where R and S are some constants and the ω_i is given by

$$\omega_i = \frac{e \pm \sqrt{e^2-4d}}{2d}k_i^3 \ \text{ if } \ d \neq 0 \ \text{ for } i = 1,2. \tag{17}$$

and

$$\omega_i = \frac{1}{e}k_i^3 \text{ if } \ d = 0 \ \text{ for } i = 1,2. \tag{18}$$

Inserting Equations 15 to 16 into Equation 1 gives a polynomial equation in the variables $X = \exp(k_1x - \omega_1 t)$ and $Y = \exp(k_2 x - \omega_2 t)$. Equating the coefficients of X^iY^j $(i, j = 0,1,2,...)$ to zero yields an algebraic system in the variables a,b,d,e,f , g, k_1, k_2, R and S. Solving it with the aid of a computer gives the following solutions :

$$\rho = \frac{e \pm \sqrt{e^2-4d}}{2d}, \ \omega_1 = \rho k_1^3, \ \omega_2 = \rho k_2^3,$$

$$f = \frac{a+12e\rho-24}{\rho}, \ g = \frac{b-36}{\rho}, \ R = 1, \ S = \frac{(k_1-k_2)^2}{(k_1+k_2)^2} \ :$$

$$f(x,t)=1+\exp\left(k_1x-\rho k_1^3t\right)+\exp\left(k_2x-\rho k_2^3t\right)$$
$$+\frac{\left(k_1-k_2\right)^2}{\left(k_1+k_2\right)^2}\exp\left(\left(k_1+k_2\right)x-\rho\left(k_1^3+k_2^3\right)t\right), \quad (19)$$

$$u(x,t)=\frac{\partial_x f(x,t)}{f(x,t)}.$$

$$\rho=\frac{e\pm\sqrt{e^2-4d}}{2d},\ \omega_1=\rho k_1^3,\ \omega_2=\rho k_2^3,\ a=30,$$

$$b=30,\ f=6e,\ g=6e,\ R=1,$$

$$S=\frac{\left(k_1-k_2\right)^2\left(ek_1^2\rho+ek_2k_1\rho+ek_2^2\rho-2k_1^2+k_2k_1-2k_2^2\right)}{\left(k_1+k_2\right)^2\left(ek_1^2\rho-ek_2k_1\rho+ek_2^2\rho-2k_1^2-k_2k_1-2k_2^2\right)}:$$

$$f(x,t)=1+\exp\left(k_1x-\rho k_1^3t\right)+\exp\left(k_2x-\rho k_2^3t\right)+$$
$$\frac{\left(k_1-k_2\right)^2\left(ek_1^2\rho+ek_2k_1\rho+ek_2^2\rho-2k_1^2+k_2k_1-2k_2^2\right)}{\left(k_1+k_2\right)^2\left(ek_1^2\rho-ek_2k_1\rho+ek_2^2\rho-2k_1^2-k_2k_1-2k_2^2\right)}\exp\left(\left(k_1+k_2\right)x-\rho\left(k_1^3+k_2^3\right)t\right), \quad (20)$$

$$u(x,t)=\frac{\partial_x f(x,t)}{f(x,t)},\text{ where } d\neq\frac{\left(k_1+k_2\right)^2\left(k_1^2-k_2k_1+k_2^2\right)}{\left(2k_1^2+k_2k_1+2k_2^2\right)^2}e^2.$$

$$\rho=\frac{2k_1^2+k_2k_1+2k_2^2}{e\left(k_1^2-k_2k_1+k_2^2\right)},\quad \omega_1=\rho k_1^3,\quad \omega_2=\rho k_2^3,\quad a=30,$$

$$b=30,\ f=\frac{e\left(ak_1^2-ak_2k_1+ak_2^2+36k_2k_1\right)}{2k_1^2+k_2k_1+2k_2^2},$$

$$g=\frac{(b-36)\left(k_1^2-k_2k_1+k_2^2\right)}{2k_1^2+k_2k_1+2k_2^2}e,\ R=1,\ S=\frac{\left(k_1-k_2\right)^2}{\left(k_1+k_2\right)^2}:$$

$$f(x,t)=1+\exp\left(k_1x-\rho k_1^3t\right)+\exp\left(k_2x-\rho k_2^3t\right)+$$
$$\frac{\left(k_1-k_2\right)^2}{\left(k_1+k_2\right)^2}\exp\left(\left(k_1+k_2\right)x-\rho\left(k_1^3+k_2^3\right)t\right), \quad (21)$$

$$u(x,t)=\frac{\partial_x f(x,t)}{f(x,t)},\text{ where } d=\frac{\left(k_1+k_2\right)^2\left(k_1^2-k_2k_1+k_2^2\right)}{\left(2k_1^2+k_2k_1+2k_2^2\right)^2}e^2.$$

$$\rho=e\pm\sqrt{e^2-4d}\,2d,\ \omega_1=\rho k_1^3,\ \omega_2=\rho k_2^3,$$

$$f=\tfrac{1}{5}(2a+b-60)e+a-2b+1205\rho,$$

$$g=-2(3a-b-90)5\rho,$$

$$R=\frac{60}{2a+b-60},\ S=\left(k_1-k_2\right)^2\left(k_1+k_2\right)^2:$$

$$f(x,t)=1+\exp\left(k_1x-\omega_0k_1^3t\right)+\exp\left(k_2x-\omega_0k_2^3t\right)$$
$$+\frac{\left(k_1-k_2\right)^2}{\left(k_1+k_2\right)^2}\exp\left(\left(k_1+k_2\right)x-\omega_0\left(k_1^3+k_2^3\right)t\right), \quad (22)$$

$$u(x,t)=\frac{60}{2a+b-60}\frac{\partial_x f(x,t)}{f(x,t)},\text{ where } 2a+b-60\neq 0.$$

$$d=0,\quad \omega_1=k_1^3e,\quad \omega_2=k_2^3e,\quad f=(a-12)e,$$
$$g=(b-36)e,\ S=\left(k_1-k_2\right)^2\left(k_1+k_2\right)^2,\ R=1:$$

$$f(x,t)=f(x,t)=1+\exp\left(k_1x-1ek_1^3t\right)+\exp\left(k_2x-1ek_2^3t\right)$$
$$+\frac{\left(k_1-k_2\right)^2}{\left(k_1+k_2\right)^2}\exp\left(\left(k_1+k_2\right)x-1e\left(k_1^3+k_2^3\right)t\right), \quad (23)$$

$$u(x,t)=\frac{\partial_x f(x,t)}{f(x,t)}.$$

$$d=0,\ \omega_1=k_1^3e,\ \omega_2=k_2^3e,\ a=30,\ b=30,\ f=6e,$$

$$g=6e,\ S=\left(k_1-k_2\right)^4\left(k_1+k_2\right)^4,\ R=1:$$

$$f(x,t)=f(x,t)=1+\exp\left(k_1x-1ek_1^3t\right)+\exp\left(k_2x-1ek_2^3t\right)$$
$$+\left(k_1-k_2\right)^4\left(k_1+k_2\right)^4\exp\left(\left(k_1+k_2\right)x-1e\left(k_1^3+k_2^3\right)t\right), \quad (24)$$

$$u(x,t)=\frac{\partial_x f(x,t)}{f(x,t)}.$$

THREE SOLITON SOLUTIONS

The three-soliton solutions to Equation 1 have the form

$$u(x,t)=R\frac{\partial_x f(x,t)}{f(x,t)}, \quad (25)$$

for the special choice

$$f(x,t)=1+\sum_{i=1}^{3}\exp(k_ix-\omega_it)+\sum_{1\le i<j\le 3}\left(\frac{k_i-k_j}{k_i+k_j}\right)^2\exp((k_i+k_j)x-(\omega_i+\omega_j)t)+$$
$$\left(\frac{k_1-k_2}{k_1+k_2}\cdot\frac{k_2-k_3}{k_2+k_3}\cdot\frac{k_3-k_1}{k_3+k_1}\right)^2\exp\left((k_1+k_2+k_3)x-(\omega_1+\omega_2+\omega_3)t\right). \quad (26)$$

where R is some constant and the ω_i are given by either

$$\omega_i=\frac{e\pm\sqrt{e^2-4d}}{2d}k_i^3 \text{ if } d\neq 0 \text{ for } i=1,2,3. \quad (27)$$

or

$$\omega_i=\frac{1}{e}k_i^3 \text{ if } d=0 \text{ for } i=1,2,3. \quad (28)$$

Inserting Equations 25 to 26 into Equation 1 gives a polynomial equation in the variables $X=\exp(k_1x-\omega_1t)$, $Y=\exp(k_2x-\omega_2t)$ and $Z=\exp(k_3x-\omega_3t)$. Equating the coefficients of $X^iY^jZ^l$ $(i,j,l=0,1,2,...)$ to zero yields an algebraic system. Solving it with the aid of a computer gives following solutions:

$\omega_i=\rho k_i^3$, $\rho=\frac{e\pm\sqrt{e^2-4d}}{2d}$, $f=12e-g+\frac{a+b-60}{\rho}$, $g=\frac{b-36}{\rho}$ and $R=1$.

$\omega_i=\rho k_i^3$, $\rho=\frac{e\pm\sqrt{e^2-4d}}{2d}$, $f=\frac{a+2ae\rho+(b-60)(e\rho-2)}{5\rho}$, $g=\frac{2(90-3a+b)}{5\rho}$ and $R=\frac{60}{2a+b-60}$.

$d=0$, $\omega_i=\frac{k_i^3}{e}$, $f=(a-12)e$, $g=(b-36)e$ and $R=1$.

COMPARISON OF OBTAINED RESULTS WITH KNOWN ONES

Here, we show that previous results are a generalization to some of the results in Wazwaz (2008) for one, two and three soliton solutions.

First case

$a=20$, $b=40$, $c=120$, $d=0$, $e=1$, $f=8$, $g=4$.

Observe that these values satisfy the conditions $d=0$, $g=(a+b-48)e-f$, $R=1$ that correspond to Equation 13 with $\omega=\frac{k^3}{e}=k^3$.

The Kdv6 reads

$$u_{xxxxxx}+20u_xu_{xxxx}+40u_{xx}u_{xxx}+120u_x^2u_{xx}+u_{xxxt}+8u_xu_{xt}+4u_tu_{xx}=0 \quad (29)$$

Equation 29 appears in Wazwaz (2008). From Equation 13, the following is a solution to Equation 29:

$$u(x,t)=\frac{k\exp\left(kx-k^3t\right)}{1+\exp\left(kx-k^3t\right)} \quad (30)$$

Figure 1 shows the graph of Equation 30.

On the other hand, the values $a=20$, $b=40$, $c=120$, $d=0$, $e=1$, $f=8$, $g=4$ meet conditions $f=(a-12)e$ and $g=(b-36)e$ which correspond to Equation 23 with $\omega_1=k_1^3$ and $\omega_2=k_2^3$. Thus, a two-soliton solution to Equation 29 is

$$u(x,t)=\frac{k_1\exp\left(k_1x-k_1^3t\right)+k_2\exp\left(k_2x-k_2^3t\right)+\frac{(k_1-k_2)^2}{(k_1+k_2)^2}\exp\left((k_1+k_2)x-(k_1^3+k_2^3)t\right)}{1+\exp\left(k_1x-k_1^3t\right)+\exp\left(k_2x-k_2^3t\right)+\frac{(k_1-k_2)^2}{(k_1+k_2)^2}\exp\left((k_1+k_2)x-(k_1^3+k_2^3)t\right)} \quad (31)$$

Figure 2 shows the graph of Equation 31.

A three-soliton solution to Equation 29 is given by Equations 25 and 26 with $R=1$ and $\omega_i=k_i^3$ for $i=1,2,3$.

The graph of Equations 25 to 26 is shown in Figure 3.

Second case

$a=30$, $b=30$, $c=180$, $d=-\frac{1}{5}$, $e=1$, $f=6$, $g=6$.

Observe that these values meet the condition:

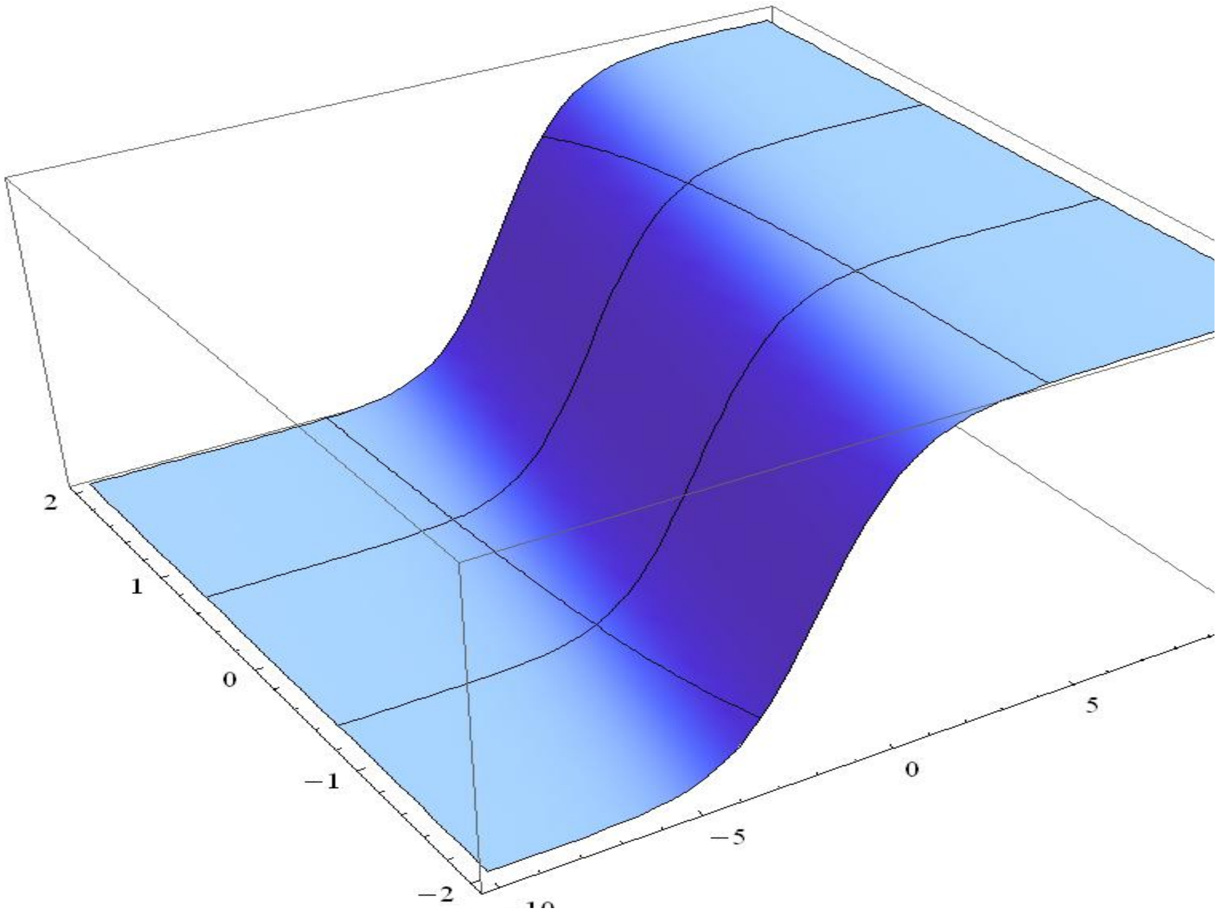

Figure 1. Graph of solution of Equations 30 to 29 for $k=1$, $-10 \le x \le 10$ and $-2 \le t \le 2$.

$$g=\frac{1}{2}\left((a+b-36)e-2f-(60-a-b)\sqrt{e^2-4d}\right) \tag{32}$$

that corresponds to Equation 11 with $\omega=-\frac{1}{2}\left(5\pm 3\sqrt{5}\right)k^3$.

The KdV6 reads

$$u_{xxxxxx}+30u_x u_{xxxx}+30u_{xx}u_{xxx}+180u_x^2 u_{xx}-\frac{1}{5}u_{tt}+u_{xxxt}+6u_x u_{xt}+6u_t u_{xx}=0. \tag{33}$$

From Equation 11, we see that one soliton solution tom Equation 33 is

$$u(x,t)=\frac{k\exp\left(kx+\frac{1}{2}\left(5\pm 3\sqrt{5}\right)k^3 t\right)}{1+\exp\left(kx+\frac{1}{2}\left(5\pm 3\sqrt{5}\right)k^3 t\right)}. \tag{34}$$

Observe also that Equations 15 to 20 is a two soliton solution to Equation 33 with

$$\rho=-\frac{1}{2}\left(5\pm 3\sqrt{5}\right). \tag{36}$$

Third case

$a=18$, $b=36$, $c=72$, $d=-2$, $e=1$, $f=0$, $g=0$.

Observe that these values meet the condition

$$g=\frac{1}{2}\left((a+b-36)e-2f-(60-a-b)\sqrt{e^2-4d}\right)$$

that corresponds to Equation 11 with

$$\omega = \frac{e-\sqrt{e^2-4d}}{2d}k^3 = \frac{1}{2}k^3.$$

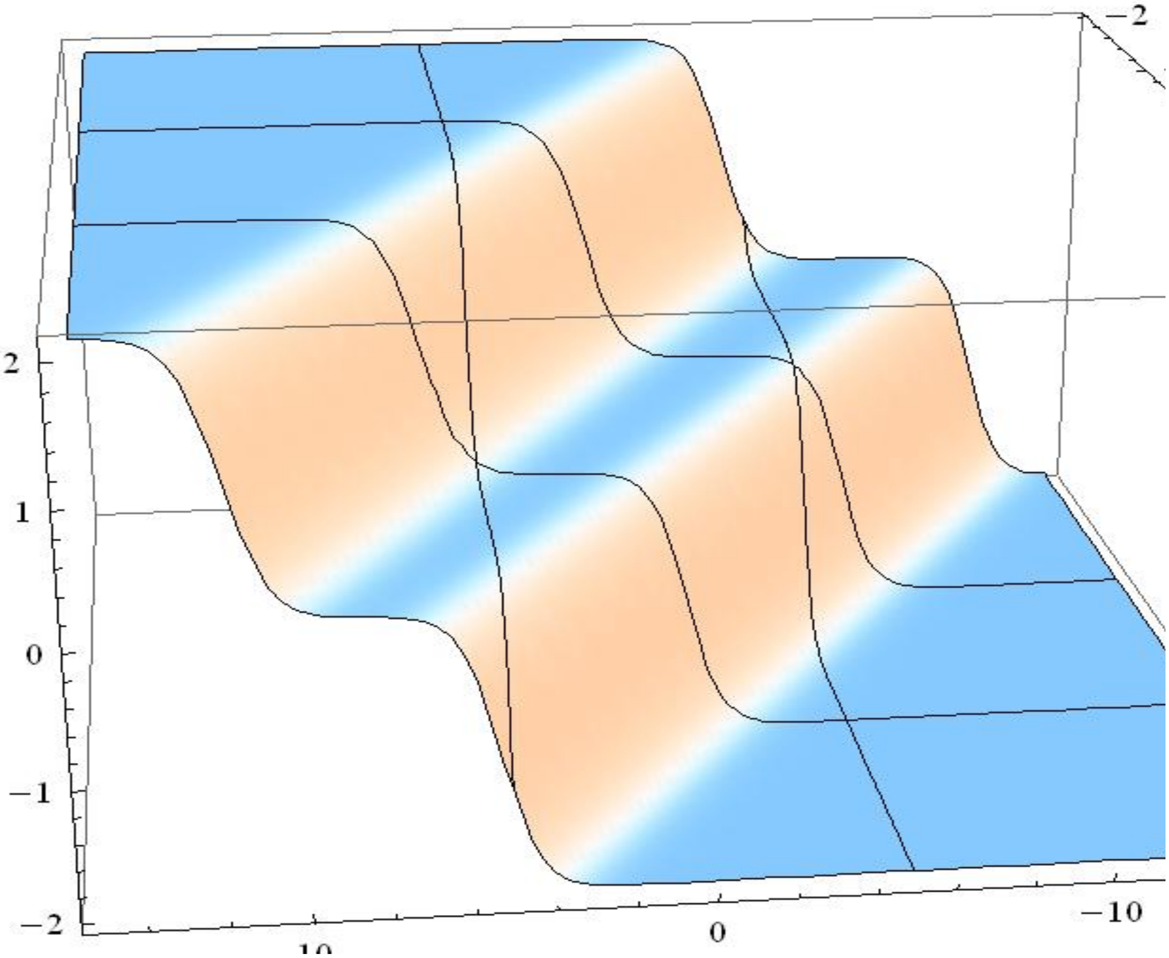

Figure 2. Graph of solution of Equations 31 to 29 for $k_1 = -2$, $k_2 = 2.1$, $-2 \le x \le 2$ and $-2 \le t \le 2$.

The Kdv6 reads

$$u_{xxxxxx} + 18u_x u_{xxxx} + 36u_{xx}u_{xxx} + 72u_x^2 u_{xx} - 2u_{tt} + u_{xxxt} = 0. \quad (37)$$

From Equation 11, it is clear that the following is a one-soliton solution to Equation 37:

$$u(x,t) = \frac{k\exp\left(kx - \frac{1}{2}k^3 t\right)}{1+\exp\left(kx - \frac{1}{2}k^3 t\right)}. \quad (38)$$

Now, it is easy to see that the values meet conditions $f = \frac{a+12e\rho-24}{\rho}$, $g = \frac{b-36}{\rho}$ which correspond to Equation 19. We conclude that Equation 19 is a two-soliton solution to Equation 37 with $\rho = 1$ or $\rho = -12.$

Finally, we observed that the values $a = 18$, $b = 36$, $c = 72$, $d = -2$, $e = 1$, $f = 0$, $g = 0$ satisfy the conditions, $f = 12e - g + \frac{a+b-60}{\rho}$, $g = \frac{b-36}{\rho}$ which correspond to Equations 25 to 26 with $\omega_i = \rho k_i^3$ $(i = 1,2,3)$, $\rho = \frac{e-\sqrt{e^2-4d}}{2d} = \frac{1}{2}$ and $R = 1.$

DISCUSSION

We obtained solutions to a generalized version of KdV6 Equation 1 for the special choice Equation 3. This condition is necessary for the Equation 1 to pass the Painlevè test for integrability (Karasu et al., 2008). Our results were obtained by using the Cole-Hopf transformation Equation 15. There exists another useful method to obtain solutions to nonlinear equations. This is the Hirota's bilinear method. This method was applied in a recent work (Zhang et al., 2009) to obtain N-soliton

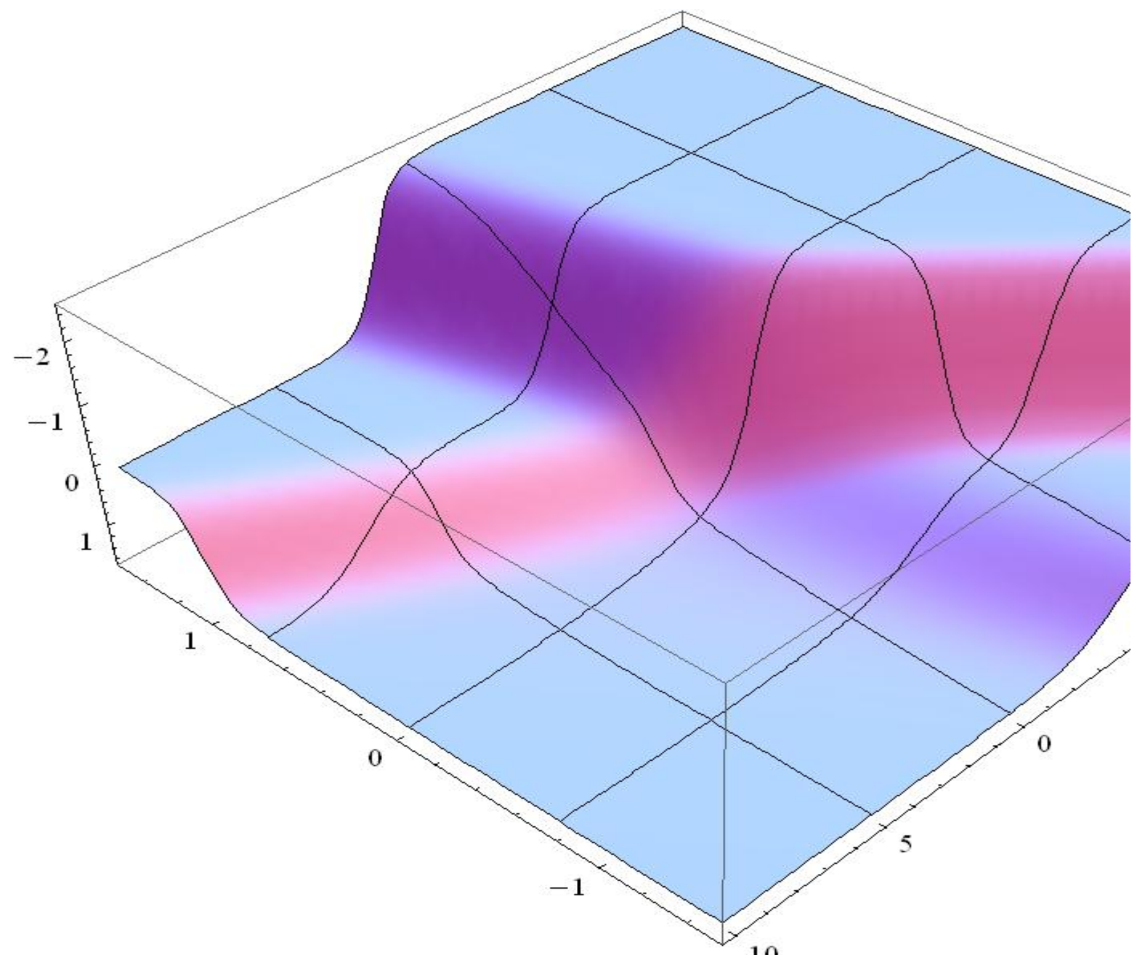

Figure 3. A three-soliton solution to Equation 29 for $k_1 = -1$, $k_2 = -1.5$, $k_3 = 2.6$, $-10 \le x \le 10$ and $-1.5 \le t \le 1.5$.

solutions for the particular case of the KdV6 given by Equation 2. The same results were obtained (Wazwaz, 2008) by using the simplified Hirota's method. The advantage of the Cole-Hopf transformation Equation 15 over Hirota's method is that, we do not need to put the equation in its bilinear form which requires the use of Hirota's bilinear operators and a great amount of complicated algebraic calculations. Bilinear form to Equation 2 is not easily obtained. On the other hand, the Cole-Hopf transformation gives exact solutions to many nonlinear partial differential equations. The difference between the method we use and Hirota's bilinear method is that, this last method is applied to bilinear form. However, both methods use the same choice for the function $f(x,t)$ given by Equations 10, 16 and 26. Previously, we showed that the known solutions in Wazwaz (2008) and Zhang (2009) are covered by the solutions we obtained in the case of one- and two-solitons solutions. The same is valid for three-soliton solutions.

On the other hand, we obtained one, two and three-soliton solutions to a generalized version of the completely integrable KdV6 Equation 2. This more general equation is given by Equation 1 with:

$d = 0$, $\omega_i = \frac{k_i^3}{e}$, $f = (a-12)e$, $g = (b-36)e$ and $c = 12a + 6b - 360$. (38)

Conclusion

We have derived one, two and three-soliton solutions for a generalized KdV6 equation with the aid of symbolic computation. Some of the results in this work are a generalization of recent results. We also may obtain similar one, two and three singular soliton solutions. We believe that the generalized version Equation 1 of the KdV6 subject to Equation 38 is completely integrable. Finally, we think that some of the results in this work are

new in the open literature. Other results concerning KdV6 equation may be found in Salas and Gomez (2010a, b), Salas (2010a) and Salas (2010b).

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