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A Korpelevich-like algorithm for variational inequalities

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Abstract
A Korpelevich-like algorithm has been introduced for solving a generalized variational inequality. It is shown that the presented algorithm converges strongly to a special solution of the generalized variational inequality.
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1 Introduction

Now it is well-known that the variational inequality of finding $x^* \in C$ such that

$$\langle Ax^*, x - x^* \rangle \geq 0, \quad \forall x \in C, \tag{1.1}$$

where C is a nonempty closed convex subset of a real Hilbert space H and $A : C \to H$ is a given mapping, is a fundamental problem in variational analysis and, in particular, in optimization theory. For related works, please see [1–20] and the references contained therein. Especially, Yao, Marino and Muglia [21] presented the following modified Korpelevich method for solving (1.1):

$$\begin{aligned} y_n &= P_C[x_n - \lambda A x_n - \alpha_n x_n], \\ x_{n+1} &= P_C\big[x_n - \lambda A y_n + \mu(y_n - x_n)\big], \quad n \geq 0. \end{aligned} \tag{1.2}$$

Recently, Aoyama, Iiduka and Takahashi [22] extended the variational inequality (1.1) to Banach spaces as follows:

$$\text{Find } x^* \in C \text{ such that } \langle Ax^*, J(x - x^*) \rangle \geq 0, \quad \forall x \in C, \tag{1.3}$$

where C is a nonempty closed convex subset of a real Banach space E. We use $S(C, A)$ to denote the solution set of (1.3). The generalized variational inequality (1.3) is connected with the fixed point problem for nonlinear mappings. For solving the above generalized variational inequality (1.3), Aoyama, Iiduka and Takahashi [22] introduced the iterative algorithm

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) Q_C[x_n - \lambda_n A x_n], \quad n \geq 0, \tag{1.4}$$

where Q_C is a sunny nonexpansive retraction from E onto C and $\{\alpha_n\} \subset (0,1)$, $\{\lambda_n\} \subset (0,\infty)$ are two real number sequences. Motivated by (1.4), Yao and Maruster [23] presented a modification of (1.4) as follows:

$$x_{n+1} = \beta_n x_n + (1-\beta_n)Q_C\big[(1-\alpha_n)(x_n - \lambda A x_n)\big], \quad n \geq 0. \tag{1.5}$$

Motivated and inspired by the above algorithms (1.2), (1.4) and (1.5), in this paper, we suggest an extragradient-type method via the sunny nonexpansive retraction for solving the variational inequalities (1.3) in Banach spaces. It is shown that the presented algorithm converges strongly to a special solution of the variational inequality (1.3).

2 Preliminaries

Let C be a nonempty closed convex subset of a real Banach space E. Recall that a mapping A of C into E is said to be *accretive* if there exists $j(x-y) \in J(x-y)$ such that

$$\big\langle Ax - Ay, j(x-y)\big\rangle \geq 0$$

for all $x, y \in C$. A mapping A of C into E is said to be α*-strongly accretive* if for $\alpha > 0$,

$$\big\langle Ax - Ay, j(x-y)\big\rangle \geq \alpha \|x-y\|^2$$

for all $x, y \in C$. A mapping A of C into E is said to be α*-inverse-strongly accretive* if for $\alpha > 0$,

$$\big\langle Ax - Ay, j(x-y)\big\rangle \geq \alpha \|Ax - Ay\|^2$$

for all $x, y \in C$.

Let $U = \{x \in E : \|x\| = 1\}$. A Banach space E is said to *uniformly convex* if for each $\epsilon \in (0,2]$, there exists $\delta > 0$ such that for any $x, y \in U$,

$$\|x-y\| \geq \epsilon \quad \text{implies} \quad \left\|\frac{x+y}{2}\right\| \leq 1-\delta.$$

It is known that a uniformly convex Banach space is reflexive and strictly convex. A Banach space E is said to be *smooth* if the limit

$$\lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t} \tag{2.1}$$

exists for all $x, y \in U$. It is also said to be *uniformly smooth* if the limit (2.1) is attained uniformly for $x, y \in U$. The norm of E is said to be Frechet differentiable if for each $x \in U$, the limit (2.1) is attained uniformly for $y \in U$. And we define a function $\rho : [0,\infty) \to [0,\infty)$ called the *modulus of smoothness* of E as follows:

$$\rho(\tau) = \sup\left\{\frac{1}{2}\big(\|x+y\| + \|x-y\|\big) - 1 : x, y \in X, \|x\| = 1, \|y\| = \tau\right\}.$$

It is known that E is uniformly smooth if and only if $\lim_{\tau\to 0} \rho(\tau)/\tau = 0$. Let q be a fixed real number with $1 < q \leq 2$. Then a Banach space E is said to be q-uniformly smooth if there exists a constant $c > 0$ such that $\rho(\tau) \leq c\tau^q$ for all $\tau > 0$.

We need the following lemmas for the proof of our main results.

Lemma 2.1 [24] *Let q be a given real number with $1 < q \leq 2$ and let E be a q-uniformly smooth Banach space. Then*

$$\|x+y\|^q \leq \|x\|^q + q\langle y, J_q(x)\rangle + 2\|Ky\|^q$$

for all $x, y \in E$, where K is the q-uniformly smoothness constant of E and J_q is the generalized duality mapping from E into 2^{E^*} defined by

$$J_q(x) = \{f \in E^* : \langle x, f\rangle = \|x\|^q, \|f\| = \|x\|^{q-1}\}, \quad \forall x \in E.$$

Let D be a subset of C and let Q be a mapping of C into D. Then Q is said to be *sunny* if

$$Q(Qx + t(x - Qx)) = Qx,$$

whenever $Qx + t(x - Qx) \in C$ for $x \in C$ and $t \geq 0$. A mapping Q of C into itself is called a *retraction* if $Q^2 = Q$. If a mapping Q of C into itself is a retraction, then $Qz = z$ for every $z \in R(Q)$, where $R(Q)$ is the range of Q. A subset D of C is called a *sunny nonexpansive* retract of C if there exists a sunny nonexpansive retraction from C onto D. We know the following lemma concerning sunny nonexpansive retraction.

Lemma 2.2 [25] *Let C be a closed convex subset of a smooth Banach space E, let D be a nonempty subset of C and Q be a retraction from C onto D. Then Q is sunny and nonexpansive if and only if*

$$\langle u - Qu, j(y - Qu)\rangle \leq 0$$

for all $u \in C$ and $y \in D$.

Lemma 2.3 [22] *Let C be a nonempty closed convex subset of a smooth Banach space X. Let Q_C be a sunny nonexpansive retraction from X onto C and let A be an accretive operator of C into X. Then for all $\lambda > 0$,*

$$S(C,A) = F(Q_C(I - \lambda A)),$$

where $S(C,A) = \{x^* \in C : \langle Ax^*, J(x - x^*)\rangle \geq 0, \forall x \in C\}$.

Lemma 2.4 [26] *Let C be a nonempty closed convex subset of a real 2-uniformly smooth Banach space X. Let the mapping $A : C \to X$ be α-inverse-strongly accretive. Then we have*

$$\|(I - \lambda A)x - (I - \lambda A)y\|^2 \leq \|x - y\|^2 + 2\lambda(K^2\lambda - \alpha)\|Ax - Ay\|^2.$$

In particular, if $0 \leq \lambda \leq \frac{\alpha}{K^2}$, then $I - \lambda A$ is nonexpansive.

Proof Indeed, for all $x, y \in C$, from Lemma 2.1, we have

$$\begin{aligned}\|(I - \lambda A)x - (I - \lambda A)y\|^2 &= \|(x - y) - \lambda(Ax - Ay)\|^2 \\ &\leq \|x - y\|^2 - 2\lambda\langle Ax - Ay, j(x - y)\rangle + 2K^2\lambda^2\|Ax - Ay\|^2\end{aligned}$$

$$\begin{aligned} &\le \|x-y\|^2 - 2\lambda\alpha\|Ax-Ay\|^2 + 2K^2\lambda^2\|Ax-Ay\|^2 \\ &= \|x-y\|^2 + 2\lambda\big(K^2\lambda-\alpha\big)\|Ax-Ay\|^2. \end{aligned}$$

It is clear that if $0 \le \lambda \le \frac{\alpha}{K^2}$, then $I - \lambda A$ is nonexpansive. □

Lemma 2.5 [27] *Let C be a nonempty bounded closed convex subset of a uniformly convex Banach space E and let T be a nonexpansive mapping of C into itself. If $\{x_n\}$ is a sequence of C such that $x_n \to x$ weakly and $x_n - Tx_n \to 0$ strongly, then x is a fixed point of T.*

Lemma 2.6 [28] *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \le (1-\gamma_n)a_n + \delta_n, \quad n \ge 0,$$

where $\{\gamma_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence in R such that
(a) $\sum_{n=0}^{\infty} \gamma_n = \infty$;
(b) $\limsup_{n\to\infty} \delta_n/\gamma_n \le 0$ *or* $\sum_{n=0}^{\infty} |\delta_n| < \infty$.
Then $\lim_{n\to\infty} a_n = 0$.

3 Main results

In this section, we present our Korpelevich-like algorithm and consequently we will show its strong convergence.

3.1 Conditions assumptions

(A1) E is a uniformly convex and 2-uniformly smooth Banach space with a weakly sequentially continuous duality mapping;
(A2) C is a nonempty closed convex subset of E;
(A3) $A : C \to E$ is an α-strongly accretive and L-Lipschitz continuous mapping with $S(C,A) \neq \emptyset$;
(A4) Q_C is a sunny nonexpansive retraction from E onto C.

3.2 Parameters restrictions

(P1) λ, μ and γ are three positive constants satisfying:
  (i) $\gamma \in (0,1)$, $\lambda \in [a,b]$ for some a, b with $0 < a < b < \frac{\alpha}{K^2L^2}$;
  (ii) $\frac{\lambda}{\mu} < \frac{\alpha}{K^2L^2}$ where K is the smooth constant of E.
(P2) $\{\alpha_n\}$ is a sequence in $(0,1)$ such that $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$.

Algorithm 3.1 For given $x_0 \in C$, define a sequence $\{x_n\}$ iteratively by

$$\begin{cases} y_n = Q_C[(1-\alpha_n)x_n - \lambda A x_n], \\ x_{n+1} = (1-\gamma)x_n + \gamma Q_C[x_n - \lambda A y_n + \mu(y_n - x_n)], \quad n \ge 0. \end{cases} \tag{3.1}$$

Theorem 3.2 *The sequence $\{x_n\}$ generated by* (3.1) *converges strongly to $Q'(0)$, where Q' is a sunny nonexpansive retraction of E onto $S(C,A)$.*

Proof Let $p \in S(C,A)$. First, from Lemma 2.2, we have $p = Q_C[p - \delta Ap]$ for all $\delta > 0$. In particular, $p = Q_C[p - \lambda Ap] = Q_C[\alpha_n p + (1-\alpha_n)(p - \frac{\lambda}{1-\alpha_n}Ap)]$ for all $n \ge 0$.

Since $A : C \to E$ is α-strongly accretive and L-Lipschitzian, it must be $\frac{\alpha}{L^2}$-inverse-strongly accretive mapping. Thus, by Lemma 2.4, we have

$$\left\|(I-\lambda A)x-(I-\lambda A)y\right\|^2 \le \|x-y\|^2+2\lambda\left(K^2\lambda-\frac{\alpha}{L^2}\right)\|Ax-Ay\|^2.$$

Since $\alpha_n \to 0$ and $\lambda \in [a,b] \subset (0,\frac{\alpha}{K^2L^2})$, we get $\alpha_n < 1-\frac{K^2L^2\lambda}{\alpha}$ for enough large n. Without loss of generality, we may assume that for all $n \in \mathbb{N}$, $\alpha_n < 1-\frac{K^2L^2\lambda}{\alpha}$, *i.e.*, $\frac{\lambda}{1-\alpha_n} \in (0,\frac{\alpha}{K^2L^2})$. Hence, $I-\frac{\lambda}{1-\alpha_n}A$ is nonexpansive.

From (3.1), we have

$$\begin{aligned}
\|y_n-p\| &= \left\|Q_C\big[(1-\alpha_n)x_n-\lambda Ax_n\big]-Q_C\left[\alpha_n p+(1-\alpha_n)\left(p-\frac{\lambda}{1-\alpha_n}Ap\right)\right]\right\| \\
&\le \left\|\alpha_n(-p)+(1-\alpha_n)\left[\left(x_n-\frac{\lambda}{1-\alpha_n}Ax_n\right)-\left(p-\frac{\lambda}{1-\alpha_n}Ap\right)\right]\right\| \\
&\le \alpha_n\|p\|+(1-\alpha_n)\left\|\left(I-\frac{\lambda}{1-\alpha_n}A\right)x_n-\left(I-\frac{\lambda}{1-\alpha_n}A\right)p\right\| \\
&\le \alpha_n\|p\|+(1-\alpha_n)\|x_n-p\|.
\end{aligned} \tag{3.2}$$

By (3.1) and (3.2), we have

$$\begin{aligned}
\|x_{n+1}-p\| &\le (1-\gamma)\|x_n-p\|+\gamma\left\|Q_C\big[x_n-\lambda Ay_n+\mu(y_n-x_n)\big]-p\right\| \\
&= (1-\gamma)\|x_n-p\|+\gamma\left\|Q_C\left[(1-\mu)x_n+\mu\left(y_n-\frac{\lambda}{\mu}Ay_n\right)\right]\right. \\
&\quad \left.-Q_C\left[(1-\mu)p+\mu\left(p-\frac{\lambda}{\mu}Ap\right)\right]\right\| \\
&\le (1-\gamma)\|x_n-p\| \\
&\quad +\gamma\left\|(1-\mu)(x_n-p)+\mu\left[\left(y_n-\frac{\lambda}{\mu}Ay_n\right)-\left(p-\frac{\lambda}{\mu}Ap\right)\right]\right\| \\
&\le (1-\gamma)\|x_n-p\|+(1-\mu)\gamma\|x_n-p\| \\
&\quad +\mu\gamma\left\|\left(y_n-\frac{\lambda}{\mu}Ay_n\right)-\left(p-\frac{\lambda}{\mu}Ap\right)\right\| \\
&\le (1-\mu\gamma)\|x_n-p\|+\mu\gamma\|y_n-p\| \\
&\le (1-\mu\gamma)\|x_n-p\|+\mu\gamma\alpha_n\|p\|+\mu\gamma(1-\alpha_n)\|x_n-p\| \\
&= (1-\mu\gamma\alpha_n)\|x_n-p\|+\mu\gamma\alpha_n\|p\| \\
&\le \max\big\{\|x_n-p\|,\|p\|\big\} \\
&\ \ \vdots \\
&\le \max\big\{\|x_0-p\|,\|p\|\big\}.
\end{aligned} \tag{3.3}$$

Hence, $\{x_n\}$ is bounded.

Set $z_n = Q_C[x_n - \lambda Ay_n + \mu(y_n - x_n)]$. From (3.1), we have $x_{n+1} = (1-\gamma)x_n + \gamma z_n$ for all $n \geq 0$. Then we have

$$\begin{aligned}
\|y_n - y_{n-1}\| &= \big\|Q_C\big[(1-\alpha_n)x_n - \lambda Ax_n\big] - Q_C\big[(1-\alpha_{n-1})x_{n-1} - \lambda Ax_{n-1}\big]\big\| \\
&\leq \left\|(1-\alpha_n)\left(x_n - \frac{\lambda}{1-\alpha_n}Ax_n\right) - (1-\alpha_{n-1})\left(x_{n-1} - \frac{\lambda}{1-\alpha_{n-1}}Ax_{n-1}\right)\right\| \\
&\leq (1-\alpha_n)\left\|\left(x_n - \frac{\lambda}{1-\alpha_n}Ax_n\right) - \left(x_{n-1} - \frac{\lambda}{1-\alpha_n}Ax_{n-1}\right)\right\| \\
&\quad + |\alpha_n - \alpha_{n-1}|\|x_{n-1}\| \\
&\leq (1-\alpha_n)\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|x_{n-1}\|,
\end{aligned}$$

and thus

$$\begin{aligned}
\|z_n - z_{n-1}\| &= \big\|Q_C\big[x_n - \lambda Ay_n + \mu(y_n - x_n)\big] - Q_C\big[x_{n-1} - \lambda Ay_{n-1} + \mu(y_{n-1} - x_{n-1})\big]\big\| \\
&\leq (1-\mu)\|x_n - x_{n-1}\| + \mu\left\|\left(y_n - \frac{\lambda}{\mu}Ay_n\right) - \left(y_{n-1} - \frac{\lambda}{\mu}Ay_{n-1}\right)\right\| \\
&\leq (1-\mu)\|x_n - x_{n-1}\| + \mu\|y_n - y_{n-1}\| \\
&\leq (1-\mu\alpha_n)\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|x_{n-1}\|.
\end{aligned}$$

It follows that

$$\limsup_{n\to\infty}\big(\|z_n - z_{n-1}\| - \|x_n - x_{n-1}\|\big) \leq 0.$$

This together with Lemma 2.6 implies that

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0.$$

From (3.2), we have

$$\begin{aligned}
\|y_n - p\|^2 &\leq \left\|\alpha_n(-p) + (1-\alpha_n)\left[\left(x_n - \frac{\lambda}{1-\alpha_n}Ax_n\right) - \left(p - \frac{\lambda}{1-\alpha_n}Ap\right)\right]\right\|^2 \\
&\leq \alpha_n\|p\|^2 + (1-\alpha_n)\left\|\left(x_n - \frac{\lambda}{1-\alpha_n}Ax_n\right) - \left(p - \frac{\lambda}{1-\alpha_n}Ap\right)\right\|^2 \\
&\leq \alpha_n\|p\|^2 + (1-\alpha_n)\|x_n - p\|^2 + 2\lambda\left(\frac{K^2\lambda}{1-\alpha_n} - \frac{\alpha}{L^2}\right)\|Ax_n - Ap\|^2. \qquad (3.4)
\end{aligned}$$

From (3.1), (3.3) and (3.4), we obtain

$$\begin{aligned}
&\|x_{n+1} - p\|^2 \\
&\quad\leq (1-\gamma)\|x_n - p\|^2 + \gamma\left\|(1-\mu)(x_n - p) + \mu\left[\left(y_n - \frac{\lambda}{\mu}Ay_n\right) - \left(p - \frac{\lambda}{\mu}Ap\right)\right]\right\|^2 \\
&\quad\leq (1-\gamma)\|x_n - p\|^2 + \gamma(1-\mu)\|x_n - p\|^2 + \gamma\mu\left\|\left(y_n - \frac{\lambda}{\mu}Ay_n\right) - \left(p - \frac{\lambda}{\mu}Ap\right)\right\|^2
\end{aligned}$$

$$\begin{aligned}
&\le (1-\gamma\mu)\|x_n-p\|^2+\gamma\mu\left[\|y_n-p\|^2+\frac{2\lambda}{\mu}\left(\frac{K^2\lambda}{\mu}-\frac{\alpha}{L^2}\right)\|Ay_n-Ap\|^2\right]\\
&\le \gamma\mu\left[\alpha_n\|p\|^2+(1-\alpha_n)\|x_n-p\|^2+2\lambda\left(\frac{K^2\lambda}{1-\alpha_n}-\frac{\alpha}{L^2}\right)\|Ax_n-Ap\|^2\right]\\
&\quad +(1-\gamma\mu)\|x_n-p\|^2+2\gamma\lambda\left(\frac{K^2\lambda}{\mu}-\frac{\alpha}{L^2}\right)\|Ay_n-Ap\|^2\\
&=\alpha_n\gamma\mu\|p\|^2+(1-\gamma\mu\alpha_n)\|x_n-p\|^2+2\gamma\lambda\mu\left(\frac{K^2\lambda}{1-\alpha_n}-\frac{\alpha}{L^2}\right)\|Ax_n-Ap\|^2\\
&\quad +2\gamma\lambda\mu\left(\frac{K^2\lambda}{\mu}-\frac{\alpha}{L^2}\right)\|Ay_n-Ap\|^2.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
0 &\le -2\gamma\lambda\mu\left(\frac{K^2\lambda}{1-\alpha_n}-\frac{\alpha}{L^2}\right)\|Ax_n-Ap\|^2-2\gamma\lambda\mu\left(\frac{K^2\lambda}{\mu}-\frac{\alpha}{L^2}\right)\|Ay_n-Ap\|^2\\
&\le \alpha_n\gamma\mu\|p\|^2+\|x_n-p\|^2-\|x_{n+1}-p\|^2\\
&=\alpha_n\gamma\mu\|p\|^2+\big(\|x_n-p\|+\|x_{n+1}-p\|\big)\big(\|x_n-p\|-\|x_{n+1}-p\|\big)\\
&\le \alpha_n\gamma\mu\|p\|^2+\big(\|x_n-p\|+\|x_{n+1}-p\|\big)\|x_n-x_{n+1}\|.
\end{aligned}$$

Since $\alpha_n\to 0$ and $\|x_n-x_{n+1}\|\to 0$, we obtain

$$\lim_{n\to\infty}\|Ax_n-Ap\|=\lim_{n\to\infty}\|Ay_n-Ap\|=0.$$

It follows that

$$\lim_{n\to\infty}\|Ay_n-Ax_n\|=0.$$

Since A is α-strongly accretive, we deduce

$$\|Ay_n-Ax_n\|\ge\alpha\|y_n-x_n\|,$$

which implies that

$$\lim_{n\to\infty}\|y_n-x_n\|=0,$$

that is,

$$\lim_{n\to\infty}\big\|Q_C\big[(1-\alpha_n)x_n-\lambda Ax_n\big]-x_n\big\|=0.$$

It follows that

$$\lim_{n\to\infty}\big\|Q_C[x_n-\lambda Ax_n]-x_n\big\|=0. \tag{3.5}$$

Next, we show that

$$\limsup_{n\to\infty}\big\langle Q'(0),j\big(x_n-Q'(0)\big)\big\rangle\ge 0. \tag{3.6}$$

To show (3.6), since $\{x_n\}$ is bounded, we can choose a sequence $\{x_{n_i}\}$ of $\{x_n\}$ converging weakly to z such that

$$\limsup_{n\to\infty}\langle Q'(0), j(x_n - Q'(0))\rangle = \limsup_{i\to\infty}\langle Q'(0), j(x_{n_i} - Q'(0))\rangle. \tag{3.7}$$

We first prove $z \in S(C,A)$. It follows that

$$\lim_{i\to\infty}\|Q_C(I-\lambda A)x_{n_i} - x_{n_i}\| = 0. \tag{3.8}$$

By Lemma 2.5 and (3.8), we have $z \in F(Q_C(I-\lambda A))$, it follows from Lemma 2.3 that $z \in S(C,A)$.

Now, from (3.7) and Lemma 2.2, we have

$$\begin{aligned}\limsup_{n\to\infty}\langle Q'(0), j(x_n - Q'(0))\rangle &= \limsup_{i\to\infty}\langle Q'(0), j(x_{n_i} - Q'(0))\rangle \\ &= \langle Q'(0), j(z - Q'(0))\rangle \\ &\geq 0.\end{aligned}$$

Noticing that $\|x_n - y_n\| \to 0$, we deduce that

$$\limsup_{n\to\infty}\langle Q'(0), j(y_n - Q'(0))\rangle \geq 0.$$

Since $y_n = Q_C[(1-\alpha_n)(x_n - \frac{\lambda}{1-\alpha_n}Ax_n)]$ and $Q'(0) = Q_C[\alpha_n Q'(0) + (1-\alpha_n)(Q'(0) - \frac{\lambda}{1-\alpha_n}AQ'(0))]$ for all $n \geq 0$, we can deduce from Lemma 2.2 that

$$\left\langle Q_C\left[(1-\alpha_n)\left(x_n - \frac{\lambda}{1-\alpha_n}Ax_n\right)\right] - \left[(1-\alpha_n)\left(x_n - \frac{\lambda}{1-\alpha_n}Ax_n\right)\right], j(y_n - Q'(0))\right\rangle \leq 0$$

and

$$\begin{aligned}&\left\langle\left[\alpha_n Q'(0) + (1-\alpha_n)\left(Q'(0) - \frac{\lambda}{1-\alpha_n}AQ'(0)\right)\right]\right. \\ &\quad \left. - Q_C\left[\alpha_n Q'(0) + (1-\alpha_n)\left(Q'(0) - \frac{\lambda}{1-\alpha_n}AQ'(0)\right)\right], j(y_n - Q'(0))\right\rangle \leq 0.\end{aligned}$$

Therefore, we have

$$\begin{aligned}&\|y_n - Q'(0)\|^2 \\ &\quad = \left\| Q_C\left[(1-\alpha_n)\left(x_n - \frac{\lambda}{1-\alpha_n}Ax_n\right)\right]\right. \\ &\qquad \left. - Q_C\left[\alpha_n Q'(0) + (1-\alpha_n)\left(Q'(0) - \frac{\lambda}{1-\alpha_n}AQ'(0)\right)\right]\right\|^2 \\ &\quad \leq \left\langle \alpha_n(-Q'(0)) + (1-\alpha_n)\left[\left(x_n - \frac{\lambda}{1-\alpha_n}Ax_n\right)\right.\right. \\ &\qquad \left.\left. - \left(Q'(0) - \frac{\lambda}{1-\alpha_n}AQ'(0)\right)\right], j(y_n - Q'(0))\right\rangle\end{aligned}$$

$$\begin{aligned}
&\leq -\alpha_n\langle Q'(0), j(y_n - Q'(0))\rangle + (1-\alpha_n)\left\| \left(x_n - \frac{\lambda}{1-\alpha_n}Ax_n\right)\right. \\
&\quad \left. - \left(Q'(0) - \frac{\lambda}{1-\alpha_n}AQ'(0)\right)\right\| \|y_n - Q'(0)\| \\
&\leq -\alpha_n\langle Q'(0), j(y_n - Q'(0))\rangle + (1-\alpha_n)\|x_n - Q'(0)\|\|y_n - Q'(0)\| \\
&\leq -\alpha_n\langle Q'(0), j(y_n - Q'(0))\rangle + \frac{1-\alpha_n}{2}\left(\|x_n - Q'(0)\|^2 + \|y_n - Q'(0)\|^2\right),
\end{aligned}$$

which implies that

$$\|y_n - Q'(0)\|^2 \leq (1-\alpha_n)\|x_n - Q'(0)\|^2 + 2\alpha_n\langle -Q'(0), j(y_n - Q'(0))\rangle. \tag{3.9}$$

Finally, we will prove that the sequence $x_n \to Q'(0)$. As a matter of fact, from (3.1) and (3.9), we have

$$\begin{aligned}
&\|x_{n+1} - Q'(0)\|^2 \\
&\quad \leq (1-\gamma)\|x_n - Q'(0)\|^2 \\
&\qquad + \gamma\left\|(1-\mu)(x_n - Q'(0)) + \mu\left[\left(y_n - \frac{\lambda}{\mu}Ay_n\right) - \left(Q'(0) - \frac{\lambda}{\mu}AQ'(0)\right)\right]\right\|^2 \\
&\quad \leq (1-\gamma\mu)\|x_n - Q'(0)\|^2 + \gamma\mu\left\|\left(y_n - \frac{\lambda}{\mu}Ay_n\right) - \left(Q'(0) - \frac{\lambda}{\mu}AQ'(0)\right)\right\|^2 \\
&\quad \leq (1-\gamma\mu)\|x_n - Q'(0)\|^2 + \gamma\mu\|y_n - Q'(0)\|^2 \\
&\quad \leq (1-\gamma\mu\alpha_n)\|x_n - Q'(0)\|^2 + 2\gamma\mu\alpha_n\langle -Q'(0), j(y_n - Q'(0))\rangle.
\end{aligned}$$

Applying Lemma 2.6 to the last inequality, we conclude that x_n converges strongly to $Q'(0)$. This completes the proof. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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