African Journal of Mathematics and Computer Science Research Vol. 3(6), pp. 83-92, June 2010
Available online at http://www.academicjournals.org/AJMCSR
ISSN 2006-9731 ©2010 Academic Journals

Full Length Research Paper

Subclasses of analytic functions associated with Wright generalized hypergeometric functions

G. Murugusundaramoorthy[1*] and N. Magesh[2]

School of Advanced Sciences, VIT University, Vellore - 632014, India.
Department of Mathematics, Government Arts College (Men), Krishnagiri - 635001, India.

Accepted 11 May, 2010

In this paper, we define a generalized class of starlike functions with negative coefficients and obtain coefficient estimates, distortion bounds, closure theorems and extreme points. Further we obtain modified Hadamard product, radii of close-to-convex, starlikeness and convexity for functions belonging to this class. Furthermore neighborhood results are discussed.

Key words: Univalent functions, convex functions, Starlike functions, δ-neighbourhood, inclusion relations, Hadamard product, Wright generalized hypergeometric functions.

INTRODUCTION AND PRELIMINARIES

Let A denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

which are analytic and univalent in the open disc U = {z: |z| < 1} and normalized by $f(0) = 0 = f'(0) - 1$. We denote by $S^*(\alpha)$ and $K(\alpha)$ the subclasses of A consisting of all functions which are, respectively starlike and convex of order α. Thus,

$$S^*(\alpha) = \left\{ f \in A : \mathrm{Re}\left(\frac{zf'(z)}{f(z)} \right) > \alpha,\ 0 \le \alpha < 1,\ z \in U \right\}$$

And

$$K(\alpha) = \left\{ f \in A : \mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)} \right) > \alpha,\ 0 \le \alpha < 1,\ z \in U \right\}$$

For functions $\Phi \in A$ given by $\Phi(z) = z + \sum_{n=2}^{\infty} \phi_n z^n$ and $\Psi \in A$ given $\Psi(z) = z + \sum_{n=2}^{\infty} \psi_n z^n$ we define the Hadamard product (or convolution) of Φ and Ψ by

$$(\Phi * \Psi)(z) = z + \sum_{n=2}^{\infty} \phi_n \psi_n z^n, \qquad z \in U \tag{2}$$

For positive real parameters $p_1, A_1, \cdots p_l, A_l$ and $q_1, B_1, \cdots q_m, B_m$, $(l, m \in N = 1,2,3,\cdots)$ such that

$$1 + \sum_{n=1}^{m} B_m, - \sum_{n=1}^{l} A_n \ge 0 \qquad z \in U. \tag{3}$$

The Wright generalized hypergeometric function (Wright, 1946)

$${}_l\Psi_m\left[(p_1, A_1), \cdots, (p_l, A_l); (q_1, B_1) \cdots (q_m, B_m); z\right]$$
$$= {}_l\Psi_m\left[(p_n, A_n)_{1,l}\,(q_n, B_n)_{1,m}; z\right]$$

is defined by:

*Corresponding author. E-mail: gmsmoorthy@yahoo.com.

$$ {}_l\Psi_m\left[(p_n,A_n)_{1,l}(q_n,B_n)_{1,m};z\right]= $$

$$ \sum_{n=0}^{\infty}\left\{\prod_{t=0}^{l}\Gamma(p_t+nA_t)\right\}\left\{\prod_{t=0}^{m}\Gamma(q_t+nB_t)\right\}^{-1}\frac{z^n}{n!},\quad z\in U $$

If $A_t = 1(t = 1, 2... l)$ and $B_t = 1(t = 1, 2, ...,m)$ we have the relationship:

$$ {}_l\Psi_m\left[(p_n,A_n)_{1,l}(q_n,B_n)_{1,m};z\right] = {}_lF_m(p_1,\cdots p_l;q_1\cdots;q_m;z) $$

$$ = \sum_{n=0}^{\infty}\frac{(p_1)_n,\cdots(p_l)_n}{(q_1)_n\cdots,(q_m)_n}\frac{z^n}{n!}, \tag{4} $$

$(l\le m+1; l.m\in N_0 = N\cup\{0\}; z\in U)$ is the generalized hypergeometric function (see for details (Wright, 1946)) where N denotes the set of all positive integers and $(\lambda)_n$ is the Pochhammer symbol and

$$ \Omega=\left\{\prod_{t=0}^{l}\Gamma(p_t)\right\}^{-1}\left\{\prod_{t=0}^{m}\Gamma(q_t)\right\} \tag{5} $$

By using the generalized hypergeometric function Dziok et al., (2003) introduced the linear operator. In 2004 Dziok et.al (2004) extended the linear operator by using Wright generalized hypergeometric function. First we define a function:

$$ {}_l\Phi_m\left[(p_t,A_t)_{1,l}(q_t,B_t)_{1,m};z\right]=\Omega\,{}_l\Psi_m\left[(p_t,A_t)_{1,l}(q_t,B_t)_{1,m};z\right] $$

Let $W\left[(p_t,A_t)_{1,l}(q_t,B_t)_{1,m}\right]: A\to A$ be a linear operator defined by:

$$ W\left[(p_t,A_t)_{1,l}(q_t,B_t)_{1,m}\right]f(z) := z\,{}_l\Phi_m\left[(p_t,A_t)_{1,l}(q_t,B_t)_{1,m};z\right]*f(z) $$

We observe that, for *f(z)* of the form (1.1), we have

$$ W\left[(p_t,A_t)_{1,l}(q_t,B_t)_{1,m}\right]f(z) := z+\sum_{n=2}^{\infty}\sigma_n(p_1)z^n \tag{6} $$

Where $\sigma_n(p_1)$ is defined by

$$ \sigma_n(p_1)=\frac{\Omega\Gamma(p_1+A_1(n-1))\cdots\Gamma(p_l+A_l(n-1))}{(n-1)!\Gamma(q_1+B_1(n-1))\cdots\Gamma(q_m+B_m(n-1))} \tag{7} $$

For convenience, we write:

$$ W[p_1,q_1]f(z)=W[(p_1,A_1),\cdots(p_l,A_l);(q_1,B_1)\cdots(q_m,B_m)]f(z) \tag{8} $$

introduced by Dziok et al. (2004). In view of the relationship (3), the linear operator (6) includes the Dziok-Srivastava operator (Dziok et al., 2003), so that it includes (as its special cases) various other linear operators introduced and studied by Bernardi (1969), Carlson et al. (1984), Libera (1965), Livingston (1966), Rucheweyh (1975) and Srivastava et al. (1987).

Denoted by $S(\alpha,\beta,\gamma,A,B)$, the subclass of A consisting of functions f(z) of the form (1) and satisfying the condition:

$$ \left|\frac{\dfrac{z(W[p_1,q_1]f(z))'}{W[p_1,q_1]f(z)}-1}{2\gamma(B-A)\left(\dfrac{z(W[p_1,q_1]f(z))'}{W[p_1,q_1]f(z)}-\alpha\right)-B\left(\dfrac{z(W[p_1,q_1]f(z))'}{W[p_1,q_1]f(z)}-1\right)}\right|<\beta,\quad z\in U \tag{9} $$

Where $W[p_1,q_1]f(z)$ is given by (8), $0\le\alpha<1, 0<\beta\le1$

$$ \frac{B}{2(B-A)}<\gamma\le\begin{cases}\dfrac{B}{2(B-A)\alpha}, & \alpha\ne0\\ 1 & \alpha=0\end{cases} $$

For fixed $-1\le A\le B\le1$ and $0<B\le1$. We also let

$$ TS^*(\alpha,\beta,\gamma,A,B)=S(\alpha,\beta,\gamma,A,B)\cap T $$

Where

$$ T:=\left\{f\in A: f(z)=z-\sum_{n=2}^{\infty}|a_n|z^n,\ z\in U\right\}, \tag{10} $$

A subclass of A introduced and studied by Silverman (1975).

By suitably specializing the values of $A_t,B_t,l,m,p_1,\cdots p_l,q_1,q_m,A,B,\alpha,\beta$ and γ the class $TS^*(\alpha,\beta,\gamma,A,B)$ leads to known subclasses studied in (Aghalary et al., 2002; Khairanar et al., 2008) and (Owa et al., 2002) and various new subclasses. In this paper we obtain sharp result for coefficient estimates, distortion theorem, radius of starlikeness and convexity and other related results.

For convenience we consider:

$$\frac{B}{2(B-A)} < \gamma \leq \begin{cases} \frac{B}{2(B-A)\alpha}, & \alpha \neq 0 \\ 1 & \alpha = 0 \end{cases}$$

For fixed $-1 \leq A \leq B \leq 1$, $0 \leq \alpha < 1, 0 < \beta \leq 1$ and $0 < B \leq 1$, one or otherwise stated.

CHARACTERIZATION PROPERTIES

Theorem 1

Let the function f(z) be defined by (10) is in the class $TS^*(\alpha,\beta,\gamma,A,B)$ if and only

$$\sum_{n=2}^{\infty}[2\beta\gamma(B-A)(n-\alpha)+(1-B\beta)(n-1)]\sigma_n(p_1)|a_n| \leq 2\beta\gamma(1-\alpha)(B-A) \tag{11}$$

Where $\sigma_n(p_1)$ is given by (7).

Proof

Suppose,

$$\sum_{n=2}^{\infty}[2\beta\gamma(B-A)(n-\alpha)+(1-B\beta)(n-1)]\sigma_n(p_1)|a_n| \leq 2\beta\gamma(1-\alpha)(B-A)$$

We have

$$\left|z(W[p_1,q_1]f(z))' - W[p_1,q_1]f(z)\right| - \beta\left|2\gamma(B-A)\left[z(W[p_1,q_1]f(z))' - \alpha W[p_1,q_1]f(z)\right] - B\left[z(W[p_1,q_1]f(z))' - W[p_1,q_1]f(z)\right]\right| < 0$$

With the provision

$$\left|\sum_{n=2}^{\infty}(n-1)\sigma_n(p_1)a_n z^n\right| - \beta\left|2\gamma(B-A)(1-\alpha) + \sum_{n=2}^{\infty}[2\gamma(B-A)(\alpha-n)+B(n-1)]\sigma_n(p_1)a_n z^n\right| < 0,$$

For |z| = r < 1; then the above condition bounded above by

$$\sum_{n=2}^{\infty}(n-1)\sigma_n(p_1)a_n r^n - 2\gamma(B-A)(\alpha-1) - \beta\sum_{n=2}^{\infty}[2\gamma(B-A)(\alpha-n)+B(n-1)]\sigma_n(p_1)a_n r^n$$

$$=\sum_{n=2}^{\infty}\{(n-1)-\beta[2\gamma(B-A)(\alpha-n)+B(n-1)]\}\sigma_n(p_1)a_n r^n - 2\beta\gamma(B-A)(1-\alpha)$$

$$\leq\sum_{n=2}^{\infty}\{(n-1)-\beta[2\gamma(B-A)(\alpha-n)+B(n-1)]\}\sigma_n(p_1)a_n - 2\beta\gamma(B-A)(1-\alpha) \leq 0$$

Therefore $TS^*(\alpha,\beta,\gamma,A,B)$. Conversely, Let

$$\left|\frac{\frac{z(W[p_1,q_1]f(z))'}{W[p_1,q_1]f(z)} - 1}{2\gamma(B-A)\left(\frac{z(W[p_1,q_1]f(z))'}{W[p_1,q_1]f(z)} - \alpha\right) - B\left(\frac{z(W[p_1,q_1]f(z))'}{W[p_1,q_1]f(z)} - 1\right)}\right|$$

$$\left|\frac{\sum_{n=2}^{\infty}(n-1)\sigma_n(p_1)a_n z^n}{2\gamma(B-A)(1-\alpha) + \sum_{n=2}^{\infty}\{2\gamma(B-A)(\alpha-n)+B(n-1)\}\sigma_n(p_1)a_n z^n}\right| < \beta$$

As $\mathrm{Re}(z) \leq |z|$ for all z, we have

$$\left|\mathrm{Re}\left\{\frac{\sum_{n=2}^{\infty}(n-1)\sigma_n(p_1)a_n z^n}{2\gamma(B-A)(1-\alpha) + \sum_{n=2}^{\infty}\{2\gamma(B-A)(\alpha-n)+B(n-1)\}\sigma_n(p_1)a_n z^n}\right\}\right| < \beta$$

Choosing values of z on real axis such that $\frac{z(W[p_1,q_1]f(z))'}{W[p_1,q_1]f(z)}$ is real and upon clearing the denominator through real values, and as $z \to 1$ we obtain

$$\sum_{n=2}^{\infty}[2\beta\gamma(B-A)(n-\alpha)+(1-B\beta)(n-1)]\sigma_n(p_1)|a_n| - 2\beta\gamma(1-\alpha)(B-A) \leq 0$$

Corollary 2

Let the function *f(z)* defined by (1.10) be in the class $TS^*(\alpha,\beta,\gamma,A,B)$. Then we have

$$a_n \leq \frac{2\beta\gamma(1-\alpha)(B-A)}{[2\beta\gamma(B-A)(n-\alpha)+(1-B\beta)(n-1)]\sigma_n(p_1)} \tag{12}$$

The equation (12) is attained for the function

$$f(z) = z - \frac{2\beta\gamma(1-\alpha)(B-A)}{[2\beta\gamma(B-A)(n-\alpha)+(1-B\beta)(n-1)]\sigma_n(p_1)}z^n, \quad (n \geq 2) \tag{13}$$

Where $\sigma_n(p_1)$ is given by (7).

Let the functions $f_j(z)(j=1,2)$ be defined by:

$$f_j(z) = z - \sum_{n=2}^{\infty} a_{n,j} z^n \quad for \quad a_{n,j} \geq 0, \ z \in U. \tag{14}$$

Theorem 2 (Closure theorem)

Let the functions $f_j(z)(j = 1,2,\cdots m)$ defined by (2.4) be in the classes $TS^*(\alpha_j, \beta, \gamma, A, B)$ *(j = 1, 2, . . .m)* respectively. Then the function h (z) defined by:

$$h(z) = z - \frac{1}{m}\sum_{n=2}^{\infty}\left(\sum_{j=1}^{m} a_{n,j}\right) z^n$$

Is in the class $TS^*(\alpha, \beta, \gamma, A, B)$, where $\alpha = \min_{1\leq j\leq m}\{\alpha_j\}$ where $0 \leq \alpha_j \leq 1$.

Proof

Since $f_j \in TS^*(\alpha_j, \beta, \gamma, A, B)$, (j = 1, 2 . . . m) by applying Theorem 1, to (4) we observe that

$$\sum_{n=2}^{\infty}[2\beta\gamma(B-A)(n-\alpha)+(1-B\beta)(n-1)]\sigma_n(p_1)\left(\frac{1}{m}\sum_{j=1}^{m} a_{n,j}\right)$$

$$= \frac{1}{m}\sum_{j=1}^{m}\left(\sum_{n=2}^{\infty}[2\beta\gamma(B-A)(n-\alpha)+(1-B\beta)(n-1)]\sigma_n(p_1)a_{n,j}\right)$$

$$\leq \frac{1}{m}\sum_{j=1}^{m} 2\beta\gamma(1-\alpha_j)(B-A) \leq 2\beta\gamma(1-\alpha)(B-A)$$

Which in view of Theorem 1, again implies that $h \in TS^*(\alpha, \beta, \gamma, A, B)$ and so the proof is complete.

Theorem 3 (Extreme points)

Let

$f_1(z) = z$ *and*

$$f_n(z) = z - \frac{2\beta\gamma(1-\alpha)(B-A)}{[2\beta\gamma(B-A)(n-\alpha)+(1-B\beta)(n-1)]\sigma_n(p_1)} z^n, \quad (n \geq 2) \tag{15}$$

where $\sigma_n(p_1)$ is given by (7). Then f (z) is in the class $TS^*(\alpha, \beta, \gamma, A, B)$ if and only if it can be expressed in the form:

$$f(z) = \sum_{n=1}^{\infty} \mu_n f_n(z) \tag{16}$$

Where $\mu_n \geq 0 (n \geq 1)$ and $\sum_{n=1}^{\infty} \mu_n = 1$.

Proof

Suppose that

$$f(z) = \mu_1 f_1(z) + \sum_{n=2}^{\infty}\mu_n f_n(z)$$

$$= \mu_1 z + \sum_{n=2}^{\infty}\mu_n\left[z - \frac{2\beta\gamma(1-\alpha)(B-A)}{[2\beta\gamma(B-A)(n-\alpha)+(1-B\beta)(n-1)]\sigma_n(p_1)} z^n\right]$$

$$= \mu_1 z + \sum_{n=2}^{\infty}\mu_n z - \sum_{n=2}^{\infty}\mu_n \frac{2\beta\gamma(1-\alpha)(B-A)}{[2\beta\gamma(B-A)(n-\alpha)+(1-B\beta)(n-1)]\sigma_n(p_1)} z^n$$

$$= z - \sum_{n=2}^{\infty} \frac{2\beta\gamma(1-\alpha)(B-A)}{[2\beta\gamma(B-A)(n-\alpha)+(1-B\beta)(n-1)]\sigma_n(p_1)} \mu_n z^n.$$

Then it follows that

$$\sum_{n=2}^{\infty}\frac{[2\beta\gamma(B-A)(n-\alpha)+(1-B\beta)(n-1)]\sigma_n(p_1)}{2\beta\gamma(1-\alpha)(B-A)}\mu_n\frac{2\beta\gamma(1-\alpha)(B-A)}{[2\beta\gamma(B-A)(n-\alpha)+(1-B\beta)(n-1)]\sigma_n(p_1)}$$

$$= \sum_{n=2}^{\infty}\mu_n = \sum_{n=2}^{\infty}\mu_n - \mu_1 = 1 - \mu_1 \leq 1$$

By Theorem 1, $f \in TS^*(\alpha, \beta, \gamma, A, B)$.

Conversely, suppose that $f \in TS^*(\alpha, \beta, \gamma, A, B)$. Then

$$a_n \leq \frac{2\beta\gamma(1-\alpha)(B-A)}{[2\beta\gamma(B-A)(n-\alpha)+(1-B\beta)(n-1)]\sigma_n(p_1)} \quad . \quad (n \geq 2).$$

We set

$$\mu_n = \frac{[2\beta\gamma(B-A)(n-\alpha)+(1-B\beta)(n-1)]\sigma_n(p_1)}{2\beta\gamma(1-\alpha)(B-A)} a_n \quad (n \geq 2)$$

and $\mu_1 = 1 - \sum_{n=2}^{\infty}\mu_n$. Then using (1.10) we obtain

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n$$

$$= z - \sum_{n=2}^{\infty}\mu_n \frac{2\beta\gamma(1-\alpha)(B-A)}{[2\beta\gamma(B-A)(n-\alpha)+(1-B\beta)(n-1)]\sigma_n(p_1)} z^n$$

$$= z - \sum_{n=2}^{\infty} \mu_n[z - f_n(z)]$$

$$= z - \sum_{n=2}^{\infty} \mu_n z + \sum_{n=2}^{\infty} \mu_n f_n(z)$$

$$= \mu_1 f_1(z) + \sum_{n=2}^{\infty} \mu_n f_n(z)$$

$$= \sum_{n=1}^{\infty} \mu_n f_n(z).$$

This completes the proof of Theorem 3.

DISTORTION BOUNDS

Theorem 4

Let the function f (z) defined by (1.10) belong to $TS^*(\alpha,\beta,\gamma,A,B)$. Then

$$|f(z)| \geq |z| \left\{1 - \frac{2\beta\gamma(1-\alpha)(B-A)}{[1+2\beta\gamma(B-A)(2-\alpha)-B\beta]\sigma_2(p_1)}|z|\right\} \tag{17}$$

and

$$|f(z)| \leq |z| \left\{1 + \frac{2\beta\gamma(1-\alpha)(B-A)}{[1+2\beta\gamma(B-A)(2-\alpha)-B\beta]\sigma_2(p_1)}|z|\right\}, \tag{18}$$

where $\sigma_2(p_1)$ is given by (7).

Proof

In the view of (11) and the fact that $\sigma_n(p_1)$ is non-decreasing for $n \geq 2,$ we have;

$$[2\beta\gamma(B-A)(2-\alpha)+(1-B\beta)]\sigma_2(p_1)\sum_{n=2}^{\infty}a_n \leq \sum_{n=2}^{\infty}[2\beta\gamma(B-A)(n-\alpha)+(1-B\beta)(n-1)]\sigma_n(p_1)a_n$$
$$\leq 2\beta\gamma(1-\alpha)(B-A)$$

which is equivalent to,

$$\sum_{n=2}^{\infty} a_n \leq \frac{2\beta\gamma(1-\alpha)(B-A)}{[1+2\beta\gamma(B-A)(2-\alpha)-B\beta]\sigma_2(p_1)} \tag{19}$$

Using (10) and (19), we obtain

$$|f(z)| \geq |z| - |z|^2 \sum_{n=2}^{\infty} a_n$$

$$= |z| - |z|^2 \frac{2\beta\gamma(1-\alpha)(B-A)}{[1+2\beta\gamma(B-A)(2-\alpha)-B\beta]\sigma_2(p_1)}$$

$$= |z| \left\{1 - \frac{2\beta\gamma(1-\alpha)(B-A)}{[1+2\beta\gamma(B-A)(2-\alpha)-B\beta]\sigma_2(p_1)}|z|\right\}$$

and

$$|f(z)| \leq |z| \left\{1 + \frac{2\beta\gamma(1-\alpha)(B-A)}{[1+2\beta\gamma(B-A)(2-\alpha)-B\beta]\sigma_2(p_1)}|z|\right\}.$$

Hence the proof is complete.

RADIUS OF STARLIKENESS AND CONVEXITY

Next we obtain the radii of close-to-convexity, star-likeness and convexity for the class $TS^*(\alpha,\beta,\gamma,A,B)$.

Theorem 5

Let the function f (z) defined by (10) belong to the class $TS^*(\alpha,\beta,\gamma,A,B)$.

Then f(z) is close-to-convex of order $\eta\,(0 \leq \eta < 1)$ in the disc $|z| < r_1$, where

$$r_1 := \inf_{n\geq 2}\left[\frac{(1-\eta)[2\beta\gamma(B-A)(n-\alpha)+(1-B\beta)(n-1)]\sigma_n(p_1)}{2n\beta\gamma(1-\alpha)(B-A)}\right]^{\frac{1}{n-1}},$$

(20)

Where $\sigma_n(p_1)$ is given by (1.7). The result is sharp, with external function f (z) given by (15).

Proof

Given $f \in T,$ and f is close-to-convex of order η, we have $|f'(z)-1| < 1-\eta.$ (21)

For the left hand side of (21) we have

$$|f'(z)-1| < \sum_{n=2}^{\infty} n a_n |z|^{n-1}.$$

The last expression is less than $1-\sigma$ if

$$\sum_{n=2}^{\infty}\frac{n}{1-\eta}a_n|z|^{n-1} < 1.$$

Using the fact, that $f \in TS^*(\alpha,\beta,\gamma,A,B)$ if and only if

$$\sum_{n=2}^{\infty}\frac{[2\beta\gamma(B-A)(n-\alpha)+(1-B\beta)(n-1)]\sigma_n(p_1)}{2\beta\gamma(1-\alpha)(B-A)}a_n \leq 1.$$

We can say (21) is true if

$$\frac{n}{1-\eta}|z|^{n-1} < \frac{[2\beta\gamma(B-A)(n-\alpha)+(1-B\beta)(n-1)]\sigma_n(p_1)}{2\beta\gamma(1-\alpha)(B-A)}.$$

or, equivalently,

$$|z|^{n-1} < \frac{(1-\eta)[2\beta\gamma(B-A)(n-\alpha)+(1-B\beta)(n-1)]\sigma_n(p_1)}{2n\beta\gamma(1-\alpha)(B-A)},$$

where $\sigma_n(p_1)$ is given by (7). This completes the proof.

Theorem 6

Let $f \in TS^*(\alpha,\beta,\gamma,A,B)$. Then f (z) is starlike of order $\eta\,(0 \le \eta < 1)$ in the disc $|z| < r_2$, that is,

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > \eta, \quad (|z| < r_2; 0 \le \eta < 1),$$

where

$$r_2 := \inf_{n\ge 2}\left[\left(\frac{1-\eta}{n-\eta}\right)\frac{[2\beta\gamma(B-A)(n-\alpha)+(1-B\beta)(n-1)]\sigma_n(p_1)}{2\beta\gamma(1-\alpha)(B-A)}\right]^{\frac{1}{n-1}}, \tag{22}$$

f(z) is convex of order $\eta\,(0 \le \eta < 1)$ in the disc $|z| < r_3$, that is, $\mathrm{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) > \eta, \ (|z| < r_3; 0 \le \eta < 1),$ where;

$$r_3 := \inf_{n\ge 2}\left[\left(\frac{1-\eta}{n(n-\eta)}\right)\frac{[2\beta\gamma(B-A)(n-\alpha)+(1-B\beta)(n-1)]\sigma_n(p_1)}{2\beta\gamma(1-\alpha)(B-A)}\right]^{\frac{1}{n-1}}, \tag{23}$$

Where $\sigma_n(p_1)$ is given by (7). Each of these results are sharp for the external function f(z) given by (15).

Proof

Given $f \in T$, and f is starlike of order η, we have

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < 1-\eta. \tag{24}$$

For the left hand side of (24) we have

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < \frac{\sum_{n=2}^{\infty}(n-1)a_n|z|^{n-1}}{1-\sum_{n=2}^{\infty}a_n|z|^{n-1}}.$$

The last expression is less than $1-\eta$ if

$$\sum_{n=2}^{\infty}\frac{n-\eta}{1-\eta}a_n|z|^{n-1} < 1.$$

Using the fact, that $f \in TS^*(\alpha,\beta,\gamma,A,B)$ if and only if

$$\sum_{n=2}^{\infty}\frac{[2\beta\gamma(B-A)(n-\alpha)+(1-B\beta)(n-1)]\sigma_n(p_1)}{2\beta\gamma(1-\alpha)(B-A)}a_n \le 1.$$

We can say (24) is true if

$$\frac{n-\eta}{1-\eta}|z|^{n-1} < \frac{[2\beta\gamma(B-A)(n-\alpha)+(1-B\beta)(n-1)]\sigma_n(p_1)}{2\beta\gamma(1-\alpha)(B-A)}.$$

or, equivalently,

$$|z|^{n-1} < \left(\frac{1-\eta}{n-\eta}\right)\frac{[2\beta\gamma(B-A)(n-\alpha)+(1-B\beta)(n-1)]\sigma_n(p_1)}{2\beta\gamma(1-\alpha)(B-A)},$$

This yields the starlikeness of the family which completes the proof.

(ii) Using the fact that f (z) is convex if and only if $zf'(z)$ is starlike, we can prove (ii), on lines similar to the proof of (i).

MODIFIED HADAMARD PRODUCTS

Let the functions $f_j(z)(j=1,2)$ be defined by (14). The modified Hadamard product of $f_1(z)$ and $f_2(z)$ is defined by:

$$(f_1 * f_2)(z) = z - \sum_{n=2}^{\infty}a_{n,j}a_{n,2}\,z^n.$$

Using the techniques of Schild et al. (1975), we prove the following results.

Theorem 7

For functions $f_j(z)(j=1,2)$ defined by (14), let $f_1 \in TS^*(\alpha,\beta,\gamma,A,B)$ and $f_2 \in TS^*(\mu,\beta,\gamma,A,B)$. Then $(f_1 * f_2) \in TS^*(\xi,\beta,\gamma,A,B)$, where

$$\xi = 1 - \frac{2\beta\gamma(B-A)(1-\alpha)(1-\mu)(1+2\beta\gamma(B-A)-B\beta)}{\Lambda_1(\alpha,\beta,\gamma,A,B,2)\Lambda_2(\mu,\beta,\gamma,A,B,2)\sigma_2(p_1)-4\beta^2\gamma^2(1-\alpha)(B-A)^2(1-\mu)}, \tag{25}$$

and

$$\Lambda_1(\alpha,\beta,\gamma,A,B,2)=2\beta\gamma(B-A)(2-\alpha)+(1-B\beta)$$
$$\Lambda_2(\mu,\beta,\gamma,A,B,2)=2\beta\gamma(B-A)(2-\mu)+(1-B\beta)$$

where $\sigma_2(p_1)$ is given by (7).

Proof

In view of Theorem 1, it suffice to prove that

$$\sum_{n=2}^{\infty}\frac{[2\beta\gamma(B-A)(n-\xi)+(1-B\beta)(n-1)]\sigma_n(p_1)}{2\beta\gamma(1-\xi)(B-A)}a_{n,1}a_{n,2}\le 1,\quad (0\le\xi<1)$$

Where ξ is defined by (25). On the other hand, under the hypothesis, it follows from (11) and the Cauchy's-Schwarz inequality that

$$\sum_{n=2}^{\infty}\frac{[\Lambda_1(\alpha,\beta,\gamma,A,B,n)]^{1/2}[\Lambda_2(\mu,\beta,\gamma,A,B,n)]^{1/2}\sigma_n(p_1)}{\sqrt{(1-\alpha)(1-\mu)}}\sqrt{a_{n,1}a_{n,2}}\le 1,$$
(26)

where

$$\Lambda_1(\alpha,\beta,\gamma,A,B,n)=2\beta\gamma(B-A)(n-\alpha)+(1-B\beta)(n-1)$$
$$\Lambda_2(\mu,\beta,\gamma,A,B,n)=2\beta\gamma(B-A)(n-\mu)+(1-B\beta)(n-1) \tag{27}$$

Thus we need to find the largest ξ such that

$$\sum_{n=2}^{\infty}\frac{2\beta\gamma(B-A)(n-\xi)+(1-B\beta)(n-1)\sigma_n(p_1)}{2\beta\gamma(B-A)(1-\xi)}a_{n,1}a_{n,2}$$
$$\le\sum_{n=2}^{\infty}\frac{[\Lambda_1(\alpha,\beta,\gamma,A,B,n)]^{\frac{1}{2}}[\Lambda_2(\mu,\beta,\gamma,A,B,n)]^{\frac{1}{2}}\sigma_n(p_1)}{\sqrt{(1-\alpha)(1-\mu)}}\sqrt{a_{n,1}a_{n,2}},$$

or, equivalently that:

$$\sqrt{a_{n,1}a_{n,2}}\le\frac{1-\xi}{\sqrt{(1-\alpha)(1-\mu)}}\frac{[\Lambda_1(\alpha,\beta,\gamma,A,B,n)]^{\frac{1}{2}}[\Lambda_2(\mu,\beta,\gamma,A,B,n)]^{\frac{1}{2}}}{2\beta\gamma(B-A)(n-\xi)+(1-B\beta)(n-1)},$$
$(n\ge 2)$.

In view of (26) it is sufficient to find largest ξ such that

$$\frac{2\beta\gamma(B-A)\sqrt{(1-\alpha)(1-\mu)}(\sigma_n(p_1))^{-1}}{[\Lambda_1(\alpha,\beta,\gamma,A,B,n)]^{1/2}[\Lambda_2(\mu,\beta,\gamma,A,B,n)]^{\frac{1}{2}}}$$
$$\le\frac{1-\xi}{\sqrt{(1-\alpha)(1-\mu)}}\frac{[\Lambda_1(\alpha,\beta,\gamma,A,B,n)]^{\frac{1}{2}}[\Lambda_2(\mu,\beta,\gamma,A,B,n)]^{\frac{1}{2}}}{2\beta\gamma(B-A)(n-\xi)+(1-B\beta)(n-1)}$$

which yields

$$\xi=\Psi(n)=1-$$
$$\frac{2\beta\gamma(B-A)(1-\alpha)(1-\mu)(n-1)(1+2\beta\gamma(B-A)-B\beta)}{[\Lambda_1(\alpha,\beta,\gamma,A,B,n)]\,[\Lambda_2(\mu,\beta,\gamma,A,B,n)]\,\sigma_n(p_1)-4\beta^2\gamma^2(B-A)^2(1-\alpha)(1-\mu)} \tag{28}$$

for $n\ge 2$ is an increasing function of n ($n\ge 2$) and letting n = 2 in (29), we have

$$\xi=\Psi(2)=1-$$
$$\frac{2\beta\gamma(B-A)(1-\alpha)(1-\mu)(1+2\beta\gamma(B-A)-B\beta)}{[\Lambda_1(\alpha,\beta,\gamma,A,B,2)]\,[\Lambda_2(\mu,\beta,\gamma,A,B,2)]\,\sigma_2(p_1)-4\beta^2\gamma^2(B-A)^2(1-\alpha)(1-\mu)}$$

Where $[\Lambda_1(\alpha,\beta,\gamma,A,B,2)]$ and $[\Lambda_2(\mu,\beta,\gamma,A,B,2)]$ as defined in (27), where $\sigma_2(p_1)$ is given by (7).

Theorem 8

Let the functions $f_j(z)(j=1,2)$ defined by (14), be in the class $TS^*(\alpha,\beta,\gamma,A,B)$. Then $(f_1 * f_2)\in TS^*(\rho,\beta,\gamma,A,B)$, where $\sigma_2(p_1)$ is given by (7).

Proof

By taking μ = α, in the above theorem, the result follows.

Theorem 9

Let the function f(z) defined by (10) be in the class $TS^*(\alpha,\beta,\gamma,A,B)$. Also let $g(z)=z-\sum_{n=2}^{\infty}b_n z^n$ for $|b_n|\le 1$. Then $(f*g)\in TS^*(\alpha,\beta,\gamma,A,B)$.

Proof

Since

$$\sum_{n=2}^{\infty}[2\beta\gamma(B-A)(n-\alpha)+(1-B\beta)(n-1)]\sigma_n(p_1)\,|\,a_n b_n\,|$$
$$\le\sum_{n=2}^{\infty}[2\beta\gamma(B-A)(n-\alpha)+(1-B\beta)(n-1)]\sigma_n(p_1)a_n\,|\,b_n\,|$$
$$\le\sum_{n=2}^{\infty}[2\beta\gamma(B-A)(n-\alpha)+(1-B\beta)(n-1)]\sigma_n(p_1)a_n$$
$$\le 2\beta\gamma(1-\alpha)(B-A)$$

it follows that $(f*g) \in TS^*(\alpha,\beta,\gamma,A,B)$, by the view of Theorem 1.

Theorem 10

Let the functions $f_j(z)(j=1,2)$ defined by (14), be in the class $TS^*(\alpha,\beta,\gamma,A,B)$. Then the function h(z) defined by $h(z)=z-\sum_{n=2}^{\infty}(a^2_{n,1}+a^2_{n,2})z^n$ is in the class $TS^*(\xi,\beta,\gamma,A,B)$ where

$$\xi=1-\frac{4\beta\gamma(B-A)(1-\alpha)^2(1+2\beta\gamma(B-A)-B\beta)}{\sigma_2(p_1)[1+2\beta\gamma(B-A)(2-\alpha)-B\beta]^2-8\beta^2\gamma^2(1-\alpha)^2(B-A)^2},$$

where $\sigma_2(p_1)$ is given by (7).

Proof

In view of Theorem 1, it suffice to prove that

$$\sum_{n=2}^{\infty}\frac{[2\beta\gamma(B-A)(n-\xi)+(1-B\beta)(n-1)]\sigma_n(p_1)}{2\beta\gamma(1-\xi)(B-A)}(a^2_{n,1}+a^2_{n,2})\le 1, \tag{28}$$

where $f_j \in TS^*(\alpha,\beta,\gamma,A,B)$ we find from (2.4) and Theorem 1, that:

$$\sum_{n=2}^{\infty}\left[\frac{[2\beta\gamma(B-A)(n-\alpha)+(1-B\beta)(n-1)]\sigma_n(p_1)}{2\beta\gamma(1-\alpha)(B-A)}\right]^2 a^2_{n,j}$$
$$\le, \sum_{n=2}^{\infty}\left[\frac{[2\beta\gamma(B-A)(n-\alpha)+(1-B\beta)(n-1)]\sigma_n(p_1)}{2\beta\gamma(1-\alpha)(B-A)}a_{n,j}\right]^2$$
(29)

this yields

$$\sum_{n=2}^{\infty}\frac{1}{2}\left[\frac{[2\beta\gamma(B-A)(n-\alpha)+(1-B\beta)(n-1)]\sigma_n(p_1)}{2\beta\gamma(1-\alpha)(B-A)}\right]^2 (a^2_{n,1}+a^2_{n,2})\le 1,$$
(30)

On comparing (30) and (31), it is easily seen that the inequality (29) will be satisfied if

$$\frac{[2\beta\gamma(B-A)(n-\xi)+(1-B\beta)(n-1)]\sigma_n(p_1)}{2\beta\gamma(1-\xi)(B-A)}$$
$$\le, \frac{1}{2}\left[\frac{[2\beta\gamma(B-A)(n-\alpha)+(1-B\beta)(n-1)]\sigma_n(p_1)}{2\beta\gamma(1-\alpha)(B-A)}\right]^2, \qquad n\ge 2.$$

That is

$$\xi=1-\frac{4\beta\gamma(B-A)(1-\alpha)^2(n-1)(1+2\beta\gamma(B-A)-B\beta)}{\sigma_n(p_1)[2\beta\gamma(B-A)(n-\alpha)+(1-B\beta)(n-1)]^2-8\beta^2\gamma^2(1-\alpha)^2(B-A)^2}. \tag{31}$$

since

$$\Psi(n)=1-\frac{4\beta\gamma(B-A)(1-\alpha)^2(n-1)(1+2\beta\gamma(B-A)-B\beta)}{\sigma_n(p_1)[2\beta\gamma(B-A)(n-\alpha)+(1-B\beta)(n-1)]^2-8\beta^2\gamma^2(1-\alpha)^2(B-A)^2}$$

is an increasing function of n ($n\ge 2$). Taking n = 2 in (32), we have:

$$\xi=\Psi(2)=1-\frac{4\beta\gamma(B-A)(1-\alpha)^2(1+2\beta\gamma(B-A)-B\beta)}{\sigma_2(p_1)[1+2\beta\gamma(B-A)(2-\alpha)-B\beta]^2-8\beta^2\gamma^2(1-\alpha)^2(B-A)^2},$$

this completes the proof.

INCLUSION RELATIONS INVOLVING N δ (E)

Following (Goodman, 1957; Rucheweyh, 1981), we define the δ - neighbourhood of function $f \in T$, by

$$N_\delta(f):=\left\{h\in T: h(z)=z-\sum_{n=2}^{\infty}b_n z^n \quad and \quad \sum_{n=2}^{\infty}n\,|\,a_n-b_n\,|\le\delta\right\}. \tag{32}$$

Particularly for the identity function e (z) = z, we have

$$N_\delta(e):=\left\{h\in T: h(z)=z-\sum_{n=2}^{\infty}b_n z^n \quad and \quad \sum_{n=2}^{\infty}n\,|\,b_n\,|\le\delta\right\}. \tag{33}$$

Now we obtain inclusion relations of the class $TS^*(\alpha,\beta,\gamma,A,B)$.

Theorem 11

If

$$\delta:=\frac{4\beta\gamma(B-A)(1-\alpha)}{\sigma_2(p_1)[1+2\beta\gamma(B-A)(2-\alpha)-B\beta]}, \tag{34}$$

where $\sigma_2(\alpha_1)$ is given by (7). Then $TS^*(\alpha,\beta,\gamma,A,B)\subset N_\delta(e)$.

Proof

For $f\in TS^*(\alpha,\beta,\gamma,A,B)$, Theorem 2.1 immediately yields

$$\left[2\beta\gamma(B-A)(2-\alpha)+(1-B\beta)\right]\sigma_2(p_1)\sum_{n=2}^{\infty}a_n \le 2\beta\gamma(1-\alpha)(B-A),$$

so that $\sum_{n=2}^{\infty}a_n \le \frac{2\beta\gamma(1-\alpha)(B-A)}{\left[2\beta\gamma(B-A)(2-\alpha)+(1-B\beta)\right]\sigma_2(p_1)}$. (35)

On the other hand, from (11) and (36) that

$$\left[2\beta\gamma(B-A)+(1-B\beta)\right]\sigma_2(p_1)\sum_{n=2}^{\infty}na_n$$

$$\le 2\beta\gamma(1-\alpha)(B-A)+\left[2\beta\gamma\alpha(B-A)+(1-B\beta)\right]\sigma_2(p_1)\sum_{n=2}^{\infty}a_n$$

$$\le 2\beta\gamma(1-\alpha)(B-A)+\left[2\beta\gamma\alpha(B-A)+(1-B\beta)\right]\sigma_2(p_1)\frac{2\beta\gamma(1-\alpha)(B-A)}{\left[2\beta\gamma(B-A)(2-\alpha)+(1-B\beta)\right]\sigma_2(p_1)}$$

$$\left[2\beta\gamma(B-A)+(1-B\beta)\right]\sigma_2(p_1)\sum_{n=2}^{\infty}na_n \le 2\beta\gamma(1-\alpha)(B-A)\left[1+\frac{2\beta\gamma\alpha(B-A)+(1-B\beta)}{\left[2\beta\gamma(B-A)(2-\alpha)+(1-B\beta)\right]}\right]$$

$$\le \frac{2\left[2\beta\gamma(1-\alpha)(B-A)\right]\left[2\beta\gamma\alpha(B-A)+(1-B\beta)\right]}{\left[2\beta\gamma(B-A)(2-\alpha)+(1-B\beta)\right]}$$

That is

$$\sum_{n=2}^{\infty}na_n \le \frac{4\beta\gamma(1-\alpha)(B-A)}{\left[2\beta\gamma(B-A)(2-\alpha)+(1-B\beta)\right]\sigma_2(p_1)} := \delta \tag{36}$$

which, in view of (33) completes the proof of Theorem 11.

Next we determine the neighborhood for the class $TS^{*(\rho)}(\alpha,\beta,\gamma,A,B)$ which we define as follows;

A function $f \in T$ is said to be in the class $TS^{*(\rho)}(\alpha,\beta,\gamma,A,B)$ if there exists a function $h \in TS^{*(\rho)}(\alpha,\beta,\gamma,A,B)$ such that

$$\left|\frac{f(z)}{h(z)}-1\right| < 1-\rho, \qquad (z \in U, 0 \le \rho < 1). \tag{37}$$

Theorem 12

If $h \in TS^{*(\rho)}(\alpha,\beta,\gamma,A,B)$ and

$$\rho = 1-\frac{\left[1+2\beta\gamma(B-A)(2-\alpha)-B\beta\right]\delta\sigma_2(p_1)}{\left[2+4\beta\gamma(B-A)(2-\alpha)-B\beta\right]\sigma_2(p_1)-4\beta\gamma(1-\alpha)(B-A)}. \tag{38}$$

Then, $N_\delta(e) \subset TS^{*(\rho)}(\alpha,\beta,\gamma,A,B)$. (39)

Proof

Suppose that $f \in N_\delta(h)$, we then find from (33) that $\sum_{n=2}^{\infty}n|a_n-b_n| \le \delta$ which implies that the coefficient inequality is $\sum_{n=2}^{\infty}|a_n-b_n| \le \frac{\delta}{2}$

Next, since $h \in TS^{*(\rho)}(\alpha,\beta,\gamma,A,B)$, we have

$$\sum_{n=2}^{\infty}b_n \le \frac{2\beta\gamma(1-\alpha)(B-A)}{\left[1+2\beta\gamma(B-A)(2-\alpha)-B\beta\right]\sigma_2(p_1)}.$$

so that

$$\left|\frac{f(z)}{h(z)}-1\right| < \frac{\sum_{n=2}^{\infty}|a_n-b_n|}{1-\sum_{n=2}^{\infty}b_n} \le \frac{\frac{\delta}{2}}{1-\frac{2\beta\gamma(1-\alpha)(B-A)}{\left[1+2\beta\gamma(B-A)(2-\alpha)-B\beta\right]\sigma_2(p_1)}}$$

$$= \frac{\left[1+2\beta\gamma(B-A)(2-\alpha)-B\beta\right]\delta\sigma_2(p_1)}{\left[2+4\beta\gamma(B-A)(2-\alpha)-B\beta\right]\sigma_2(p_1)-4\beta\gamma(1-\alpha)(B-A)} = 1-\rho$$

provided that ρ is given precisely by (40). Thus by definition, $f \in TS^{*(\rho)}(\alpha,\beta,\gamma,A,B)$ for ρ given by (40), which completes the proof.

ACKNOWLEDGEMENTS

The authors thank the referee's for their valuable comments.

REFERENCES

Aghalary R, Kullkarni S (2002). Some theorems on univalent functions. J. Indian Acad. Math., 24(1): 81–93.
Bernardi SD (1969). Convex and starlike univalent functions. Trans. Amer. Math. Soc., 135: 29–446.
Carlson BC, Shaffer DB (1984). Starlike and prestarlike hypergeometric functions. SIAM. J. Math. Anal., 15: 737–745.
Dziok J, Srivastava HM (2003). Certain subclasses of analytic functions associated with the generalized hypergeometric function. Intergral Transform Spec. Funct., 14: 7-18.
Dziok J, Raina RK (2004). Families of analytic functions associated with the Wright generalized hypergeometric function. Demonstratio Math., 37(3): 533–542.
Goodman AW (1957). Univalent functions and nonanalytic curves, Proc. Amer. Math. Soc., 8: 598-601.
Khairanar SM, Meena M (2008). Certain family of analytic and univalent functions with normalized conditions. Acta Math. Acade. Paeda. Nyire., 24: 333–344.
Libera RJ (1965). Some classes of regular univalent functions. Proc. Amer. Math. Soc., 16: 755–758.
Livingston AE (1966). On the radius of univalence of certain analytic functions. Proc. Amer. Math. Soc., 17: 352–357.
Owa S, Nishiwaki J (2002). Coefficient estimates for certain classes of analytic functions. J. Inequal. Pure. Appl. Math., 3(5): 1-8.
Rucheweyh St. (1981). Neighborhoods of univalent functions. Proc. Amer. Math. Soc., 81: 521-527.
Rucheweyh St. (1975). New criteria for univalent functions. Proc. Amer. Math. Soc., 49: 109–115.
Schild A, Silverman H (1975). Convolution of univalent functions with negative coefficients. Annales Univ. Mariae Curie., 29: 99 -107.
Silverman H (1975). Univalent functions with negative coefficients. Proc. Ame. Math. Soc., 51: 109–116.

Srivastava HM, Owa S (1987). Some characterization and distortion theorems involving fractional calculus, generalized hypergeometric functions, Hadamard products, linear operators and certain subclasses of analytic functions. Nagoya Math. J., 106: 1–28.

Wright EM (1946). The asymptotic expansion of the generalized hypergeometric function. Proc. London. Math. Soc., 46: 389–408.