

Full Length Research Paper

The solution of Euler-Bernoulli beams using variational derivative method

Atilla Özütok^{1*} and Arife Akin¹

Selcuk University, Engineering Architectural Faculty, 42031, Konya, Turkey.

Accepted 7 April, 2010

Finite difference methods are often used for analyzing structures governed by complex differential equations. The finite difference method, well known as an efficient numerical method, was formerly applied to the case of beam and plate problems. The basic disadvantages of this method are the requirement of out-of-region points during the solution process and the difficulty of implementing the boundary conditions along irregular boundaries. In this study, the variational derivative method was proposed to solve the functional of the Euler –Bernoulli beam obtained by Gâteaux differential method. The main reason for the preference of this method over the finite difference method was the elimination of the need for the out-of-region domain points that complicate the application of the finite difference method. The moments and deflections of beams with constant and varying cross-sections and various support types were calculated in order to demonstrate the applicability of the method. The performance of this formulation is verified by comparing the obtained results with the results of the numerical examples in the literature.

Key words: Finite difference, variational derivative, beams.

INTRODUCTION

We should bear in mind that all methods of structural analysis are essentially concerned with solving the basic differential equations of equilibrium and compatibility, although, in some of the methods this fact may be obscured. Analytical solutions are limited to the cases where the load distribution, section properties and boundary conditions can be described by mathematical expressions. However, the numerical analysis methods are generally more practical for complex structures. Moreover, various methods for finding suitable approximate solutions have been under continuous development for decades. Among these methods, Finite Difference Method (FDM), Rayleigh- Ritz Method, Galerkin Method, Least-Squares Method, Finite Element Methods (FEM) have dominated the applications to problems in engineering. In FDM, a numerical solution of the differential equation for displacement or stress resultant is obtained for chosen points on the structure, referred to

as nodes or simply as point of division. During the study performed on the finite difference analysis of the systems and the comparison of the methods used with finite element techniques, we readily recognized a very close relationship between finite difference and finite element procedures. However, the FEM may appear to be a better choice than FDM owing the easier definition of the boundary conditions. The FDM has been comprehensively formulated in many studies. We can classify the FDM into two groups according to the literature namely, Conventional Finite Difference Method (CFDM) and Finite Difference Energy Method (FDEM). The solution by the CFDM is obtained by applying the finite difference expressions on the area equations directly. The disadvantages of the method include the difficulties of implementing the boundary conditions on irregular boundaries and accurately representing the irregular domains. The variational method is a powerful method that can be used for both approximate solutions and formulation of problems (Reddy, 1986).

Many researchers have developed the FDEM based on variation in order to overcome the difficulty stated previously. Houbolt used this method in 1958 to perform

*Corresponding author: E-mail: aouzutok@selcuk.edu.tr. Tel: +90.332.223.2645. Fax: +90.332.241.0635.

the static analysis of the beams and plates. In 1983, Barve and Dey and in 1990 Singh and Dey applied the FDEM to the static, vibration and instability tests of the plates. Arslan (2004) studied on the functional of the Euler-Bernoulli beam obtained by using the Gâteaux differential approach to solve the functional by the aid of the variational derivative method based on FDM, and consequently, the moments and deflections of the beams with constant and varying cross-sections and various support types were determined by considering the out-of-region points. Gelfand and Fomin (1963) carried out the buckling analysis of the elastic bars using the variational derivative method. Gürsoy (2004); Eratli and Gürsoy (2005) solved all the buckling problems of elastic bars using the methods of variational derivative and finite difference and to determined the element matrix of the bar element using variational derivative method. As a result of this study, the variational derivative method was found to be extremely similar but more advantageous when compared to FDM; due to not being in need of out-of-region points that complicate the application of FDM.

In this work, we consider the solution for the Euler-Bernoulli beam's functional obtained by using the Gâteaux differential approach. The purpose of this study is to solve the functional by the aid of the variational derivative method based on FDM. The moment and deflection values were selected as the unknowns of the functional, modified by Aköz (1985). Some significant advantages of the Gâteaux differential approach were explained by Özütok (2000). The element matrix for Euler-Bernoulli beam is derived based on the variational derivative. The performance of the matrix for bending analysis is verified with a good accuracy by comparing with the numerical examples and analytical solutions presented in the literature.

The functional for Euler-Bernoulli beam

The equations involving the boundary conditions for Euler-Bernoulli beam which can be written in the operator form as:

$$\mathbf{Q} = \mathbf{L} \mathbf{y} - \mathbf{f} = 0 \quad (1)$$

Having obtained the field equations, one needs a method to obtain the functional. We believe that the Gâteaux differential method is suitable for this aim. Since this method was extensively used and explained in other studies (Aköz et al., 1991), for the sake of simplicity, the basic steps and definitions will be summarized briefly. The Gâteaux derivative of an operator is defined as

$$d\mathbf{Q}(\mathbf{y}; \bar{\mathbf{y}}) = \frac{\partial \mathbf{Q}(\mathbf{y} + \tau \bar{\mathbf{y}})}{\partial \tau} \Big|_{\tau=0} \quad (2)$$

Where, τ is a scalar. A necessary and sufficient condition

for \mathbf{Q} to be a potential is (Oden and Reddy, 1976)

$$\langle d\mathbf{Q}(\mathbf{y}; \bar{\mathbf{y}}), \mathbf{y}^* \rangle = \langle d\mathbf{Q}(\mathbf{y}; \mathbf{y}^*), \bar{\mathbf{y}} \rangle \quad (3)$$

Where, parentheses indicate the inner products. If the operator \mathbf{Q} is a potential, then the functional which corresponds to the field equations is given by

$$I(\mathbf{y}) = \int_0^l \langle \mathbf{Q}(s\mathbf{y}), \mathbf{y} \rangle ds \quad (4)$$

Where, s is a scalar quantity. The clear form of the functional corresponding to the field equations are obtained as in the following (Arslan, 2004),

$$I(\mathbf{y}) = [\mathbf{M}', \mathbf{v}'] - [\mathbf{q}, \mathbf{v}] - \frac{1}{2} \left[\frac{\mathbf{M}}{EI}, \mathbf{M} \right] - [\hat{\mathbf{v}}', \mathbf{M}]_{\varepsilon} - [\mathbf{v}, \hat{\mathbf{T}}]_{\sigma} - \left[(\mathbf{M} - \hat{\mathbf{M}}), \mathbf{v}' \right]_{\sigma} + [(\hat{\mathbf{v}} - \mathbf{v}), \mathbf{T}]_{\varepsilon} \quad (5)$$

Where, the subscripts ε and σ indicate the geometric and dynamic boundary conditions, respectively. As seen in Equation (5), the moment and deflection values selected as unknowns of the functional are integrated simultaneously into the functional and are obtained easily without needing the boundary conditions to be known.

Variational derivative

For a function with multiple arguments, $f(x_1, x_2, \dots, x_n)$, if the differential df can be written as

$$df = \sum_{i=1}^n g_i(x_1, x_2, \dots, x_n) dx_i, \text{ then } g_i(x_1, x_2, \dots, x_n) dx_i \text{ is}$$

called the derivative of f with respect to x_i , for $i = 1, \dots, n$ and is given by

$$\frac{\partial f}{\partial x_i} = g_i(x_1, x_2, \dots, x_n) \quad (6)$$

Notice that the derivative of function $f(x_1, \dots, x_n)$ with respect to x_i will be another function named as $g_i(x_1, \dots, x_n)$. Similarly, if the first variational derivative of a functional

$$I[y(x)] = \int_a^b F(x, y, y') dx \quad (7)$$

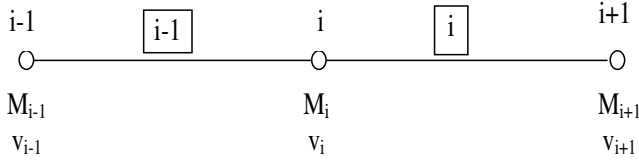


Figure 1. Nodal points and the elements.

is written as

$$\delta I[y(x)] = \int_a^b g(x) \delta y(x) dx \quad (8)$$

Then, the *functional derivative* of I will be

$$\frac{\delta I}{\delta y(x)} = g(x) \quad (9)$$

The functional derivative of a functional $I[y(x)]$ is another function $g(x)$. Imagine that we approximate the function $y(x)$ by a piece-wise linear curve that passes through a set of points (x_i, y_i) , $i = 0, \dots, n+1$, where $x_i = i\Delta x$, $y_i = y(x_i)$, $x_0 = a$ and $x_{n+1} = b$. Let $y(x)$ be subjected to essential boundary conditions, so that $y_0 = y(a)$ and $y_{n+1} = y(b)$ are fixed. Then, the functional $I[y(x)]$ can be approximated by a sum.

$$I_n(y_1, y_2, \dots, y_n) \equiv \sum_{i=0}^n F(x_i, y_i, \frac{y_{i+1} - y_i}{\Delta x}) \Delta x \quad (10)$$

The partial derivative of I_n with respect to one of its arguments is,

$$\frac{\partial I_n}{\partial y_k} = \left(F_y \Big|_{y=y_k} + \frac{1}{\Delta x} F_{y'} \Big|_{y'=\frac{y_k - y_{k-1}}{\Delta x}} - \frac{1}{\Delta x} F_{y'} \Big|_{y'=\frac{y_{k+1} - y_k}{\Delta x}} \right) \Delta x \quad (11)$$

Comparing this with the definition of the functional derivative, the following equation is written

$$\frac{\delta I_n}{\delta y} = F_y(x, y, y') - \frac{d}{dx} (F_{y'}(x, y, y')) = \lim_{\Delta x \rightarrow 0} \frac{\partial I_n}{\partial y_k} \frac{1}{\Delta x} \quad (12)$$

Then, the above expression is called “variational

derivative” (Kang et al., 2006), and the first condition for $y(x)$ to make $I[y(x)]$ functional maximum is $\delta I = 0$. Using this condition, the $\frac{\delta I_n}{\delta y}$ expression will be equal to zero

at every point. Variational derivative method differs from FDM by not needing to use the out-of-region points (Eratli and Gürsoy, 2005). A formulation based on variational derivative method that does not require any out-of-region points is developed to calculate the bending moments and deflections of the bars making use of the functional obtained here. Then, the element matrix can be written as

$$I = \sum_{e=1}^n \left(M_e' v_e' - \frac{1}{2} \frac{M_e^2}{EI} - q v \right) \Delta \quad (13)$$

by using this equation $I[y(x)]$. Here, $\Delta = x_{i+1} - x_i$ and e is element number. If the following relationships are replaced with the derivative expressions in the functional

$$v' = \frac{v_{i+1} - v_i}{\Delta}, \quad M' = \frac{M_{i+1} - M_i}{\Delta} \quad (14)$$

and the derivatives are taken with respect to the moments and deflections defined as the unknowns at each nodal point of i th element of the bar system shown in Figure 1 and are equalized to zero, as follows

$$\frac{\delta I_n}{\delta y} = F_y(x, y, y') - \frac{d}{dx} (F_{y'}(x, y, y')) = \lim_{\Delta x \rightarrow 0} \frac{\partial I_n}{\partial y_k} \frac{1}{\Delta x} = 0 \quad (15)$$

then, the element matrix is obtained as

$$\begin{bmatrix} -\lambda^2 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \\ 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \end{bmatrix} \begin{Bmatrix} M_i \\ M_{i+1} \\ v_i \\ v_{i+1} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ q\Delta^2 \\ 0 \end{Bmatrix} \quad (16)$$

$$\text{Where, } \lambda^2 = \frac{\Delta^2}{EI}.$$

Numerical examples

Several problems of beams with various boundary conditions have been studied in the past. In order to check the performance of variational derivative method, various problems are solved and the results are compared with those of similar examples in the literature.

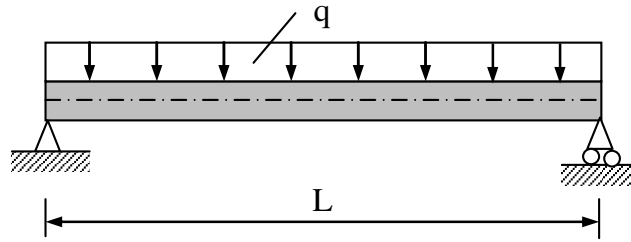


Figure 2. The simply supported beam to be analyzed under uniformly distributed transverse loading.

Table 1. Comparison of the VD and exact solutions.

α Values ($v_{exact} = \alpha \times \frac{5}{384} \frac{q \times L^4}{EI}$)					
Number of members	CFD [†]	FDEM [†]	VD [†] of (Arslan, 2004)	VD [†] of this study	Exact
2	1.2	1.2	1.058	1.2	1
4	1.05	1.05	1.02	1.05	1
8	1.01	1.01	1.011	1.012	1
16	1.003	1.003	1.009	1.003	1

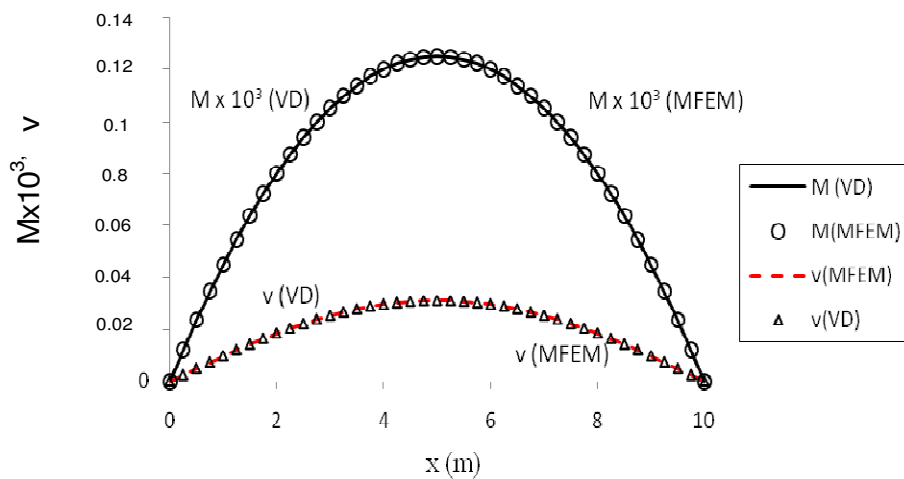


Figure 3. The distribution of moment and deflection along the beam obtained with VD and MFEM.

Simply-supported beam with uniform loading

First of all, we decided to consider the accuracy and convergence of the variational derivative element matrix on a simply supported beam subjected to a uniform vertical loading of $q = 10 \text{ kN/m}$ as shown in Figure 2. The material properties and dimensions are $E = 2 \times 10^7 \text{ kN/m}^2$, $L = 10 \text{ m}$, $h = 0.5 \text{ m}$ and $b = 0.2 \text{ m}$. The maximum deflections obtained from various types of analyses including CFD, FDEM and VD of Arslan (2004) given in

the literature and the current study are presented in Table 1 to allow comparison with the exact solution. As it is seen, the deflections found from the current study converge to the exact results. [†] CFD: conventional finite difference method; FDEM: finite difference energy method, VD: Variational derivative.

In addition to the maximum deflection values, the distribution of moment and deflection along the beam axis from VD of this study and MFEM is plotted and shown in Figure 3. The effect of the number of elements

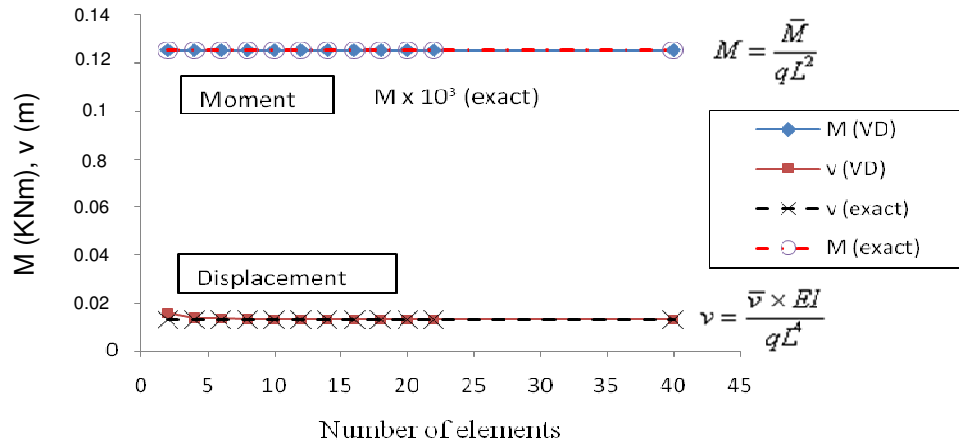


Figure 4. Deflection and moment convergence vs. number of elements.

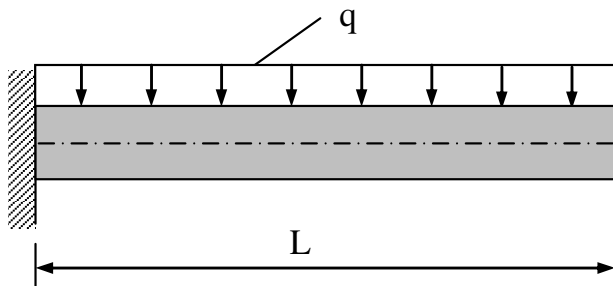


Figure 5. A cantilever beam with uniform distributed loading.

used in beam discretization on the deflection and moment are shown in Figure 4, where the nondimensionalized maximum deflection $v = \bar{v} EI / qL^4$ and moment $M = \bar{M} / qL^2$ of a simply supported beam subjected to a uniformly distributed load is plotted.

Cantilever beam with uniform distributed loading: Constant cross-section case

Considering the problem of finding the transverse deflection of a cantilever beam under a uniform distributed loading by using variational derivative method in Figure 5; the deflection and moment of the free tip versus the number of elements are plotted as shown in Figure 6. In the figure, the deflection and moment converge to the exact solution results with an error proportional to the number of elements used.

Cantilever beam with uniform distributed loading: Variable cross-section case

In order to test the effect of height variation on the

Table 2. Comparison of VD results with the literature and exact results.

$m = h_b/h_s$	(v x 10) Deflection		
	Gul, 2003 MFEM	VD of this study	Exact
1	1.252	1.266	1.250
1.25	-	1.433	-
1.667	1.667	1.664	-
2.5	-	2.00	-
5	2.677	2.618	-

deflection and moment, a cantilever beam subjected to a uniform distributed loading as shown in Figure 7 was analyzed for different heights. In Figure 7, h_b and h_s are the height of beam cross-section at the left and right hand sides of the beam, respectively. The variation of the moment of inertia along the beam can be defined for element i by

$$I_i = bh^3 \left[1 + n \frac{z}{L} \right]^3 \quad (17)$$

Where, $n = m-1$, $m = h_b/h_s$ and z is the length of element i . The maximum deflections for four cases with different height ratios are shown in Table 2. In the table, results of the study by (Gul, 2003) and exact results are given for comparison.

DISCUSSION AND CONCLUSION

In this study, the solutions of Euler–Bernoulli beams of constant and varying cross-sections under uniformly distributed loading are carried out using variational derivative method without the need of out-of-region points

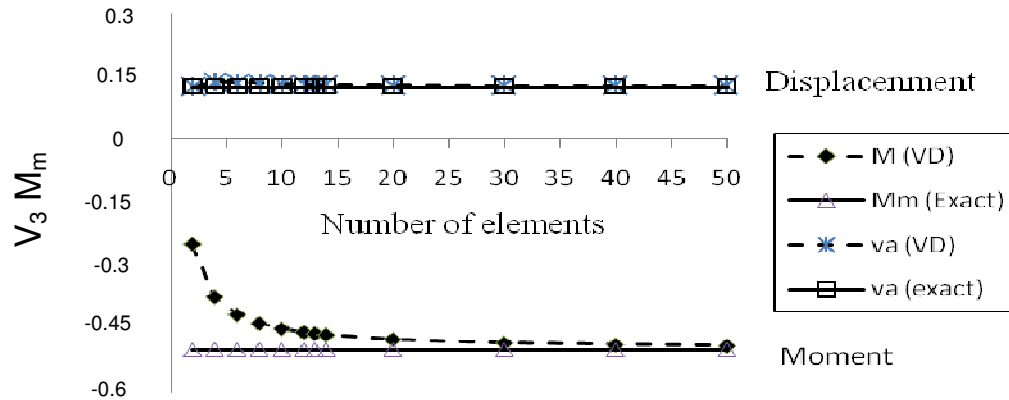


Figure 6. Deflection and moment convergences with respect to number of elements.

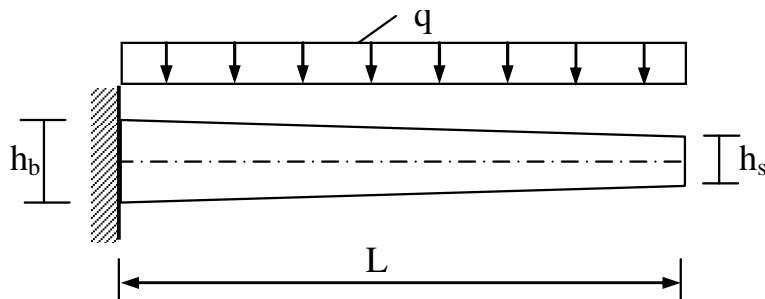


Figure 7. A cantilever beam with variable cross-section under uniformly distributed loading.

of FDM. The element matrix of the beam is obtained by applying the functional (Arslan, 2004) to the variational derivative method which was derived for straight-axis bars using Gâteaux differential method. Besides, observing the similarity of the proposed element matrix to that of FDM, the obtained matrix is superior in beam problems since it does not require any out-of-region points. A computer code is developed in Fortran to compute the proposed element matrix by considering the boundary conditions. Using this computer code, the unknown moments and the displacements at the nodal points are calculated and found to be in agreement with the results of the previous studies in the literature and the exact results. In the entire example problems considered, the proposed approach converges to the exact solution as long as the number of members is increased. Therefore, it is expected that the method will provide a better approach to the case of higher degree beam problems.

REFERENCES

- Aköz AY (1985). A new functional for bars and its applications , (in Turkish), IV. National Mechanics Meeting, Turkey pp. 100-112.
- Aköz AY, Omurtag MH, Doğruoğlu AN (1991). The mixed finite element formulation for three dimensional bars. *Int. J. Solids Struct.* Engine 28: 225-234.
- Aköz AY, Özutok A (2000). A functional for shells of arbitrary geometry and a mixed finite element method for parabolic and circular cylindrical shells, *Int. J. Numer. Meth. Engine* 47: 1933-1981.
- Arslan A (2004). The solution of Euler-Bernoulli beams by variational derivative method, (in Turkish), M.Sc. Dissertation, Selcuk University, Konya.
- Barve VD, Dey SS (1983). Isoparametric finite difference energy method for plate bending problems, *Computer Struct.* 17: 459-465.
- Eratli N, Gürsoy A (2005). Variational derivative for buckling loads of beams and frames, *Int. Appl. Mech.* 41(7): 825-830.
- Gelfand IM, Fomin SV (1963). *Calculus of variations*, Prentice-Hall, New Jersey.
- Gul M (2003). Dynamics analysis of stiffened plate, (in Turkish), M.Sc. Dissertation, İ.T.Ü., İstanbul.
- Gürsoy A (2004). The analysis of the buckling loads for beams and frames with variational derivative (in Turkish), M.Sc. Dissertation, İ.T.Ü., İstanbul.
- Houbolt JC (1958). A study of several aerothermoelastic problems of aircraft structures in High-Speed Flight, Leeman, Zurich
- Kang K, Weinberger C, Cai W (2006). A short essay on variational calculus, Stanford University.
- Oden JT, Reddy JN (1976). *Variational Method in Theoretical Mechanics*. Springer, Berlin.
- Reddy JN (1986). *An Introduction to the Finite Element Method*, McGraw-Hill, New York.
- Singh JP, Dey SS (1990). Variational finite difference method for free vibration of sector plates, *J. Sound Vib.* 136: 91-104.