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A more accurate half-discrete Hilbert-type inequality with a non-homogeneous kernel

Bicheng Yang[1*] and Xindong Liu[2]

*Correspondence: bcyang@gdei.edu.cn
[1]Department of Mathematics, Guangdong University of Education, Guangzhou, Guangdong 510303, China
Full list of author information is available at the end of the article

Abstract

By means of weight functions and the improved Euler-Maclaurin summation formula, a more accurate half-discrete Hilbert-type inequality with a non-homogeneous kernel and a best constant factor is given. A best extension, some equivalent forms, the operator expressions as well as some particular cases are also considered.

MSC: 26D15; 47A07

Keywords: Hilbert-type inequality; weight function; equivalent form; operator

1 Introduction

Assuming that $f,g \in L^2(\mathbf{R}_+)$, $\|f\| = \{\int_0^\infty f^2(x)\,dx\}^{\frac{1}{2}} > 0$, $\|g\| > 0$, we have the following Hilbert's integral inequality (*cf.* [1]):

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}\,dx\,dy < \pi \|f\| \|g\|, \tag{1}$$

where the constant factor π is best possible. If $a = \{a_n\}_{n=1}^\infty, b = \{b_n\}_{n=1}^\infty \in l^2$, $\|a\| = \{\sum_{n=1}^\infty a_n^2\}^{\frac{1}{2}} > 0$, $\|b\| > 0$, then we have the following analogous discrete Hilbert's inequality:

$$\sum_{m=1}^\infty \sum_{n=1}^\infty \frac{a_m b_n}{m+n} < \pi \|a\| \|b\|, \tag{2}$$

with the same best constant factor π. Inequalities (1) and (2) are important in analysis and its applications (*cf.* [2–4]).

In 1998, by introducing an independent parameter $\lambda \in (0,1]$, Yang [5] gave an extension of (1). For generalizing the results from [5], Yang [6] gave some best extensions of (1) and (2) as follows. If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\lambda_1 + \lambda_2 = \lambda$, $k_\lambda(x,y)$ is a non-negative homogeneous function of degree $-\lambda$ satisfying $k(\lambda_1) = \int_0^\infty k_\lambda(t,1)t^{\lambda_1 - 1}\,dt \in \mathbf{R}_+$, $\phi(x) = x^{p(1-\lambda_1)-1}$, $\psi(x) = x^{q(1-\lambda_2)-1}$, $f(\geq 0) \in L_{p,\phi}(\mathbf{R}_+) = \{f \mid \|f\|_{p,\phi} := \{\int_0^\infty \phi(x)|f(x)|^p\,dx\}^{\frac{1}{p}} < \infty\}$, $g(\geq 0) \in L_{q,\psi}(\mathbf{R}_+)$, and $\|f\|_{p,\phi}$, $\|g\|_{q,\psi} > 0$, then

$$\int_0^\infty \int_0^\infty k_\lambda(x,y)f(x)g(y)\,dx\,dy < k(\lambda_1)\|f\|_{p,\phi}\|g\|_{q,\psi}, \tag{3}$$

where the constant factor $k(\lambda_1)$ is best possible. Moreover, if $k_\lambda(x,y)$ is finite and $k_\lambda(x,y)x^{\lambda_1-1}(k_\lambda(x,y)y^{\lambda_2-1})$ is decreasing for $x > 0$ ($y > 0$), then for $a_m, b_n \geq 0$, $a = \{a_m\}_{m=1}^\infty \in$

$l_{p,\phi} = \{a|\|a\|_{p,\phi} := \{\sum_{n=1}^{\infty}\phi(n)|a_n|^p\}^{\frac{1}{p}} < \infty\}$, and $b = \{b_n\}_{n=1}^{\infty} \in l_{q,\psi}$, $\|a\|_{p,\phi}$, $\|b\|_{q,\psi} > 0$, we have

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} k_\lambda(m,n)a_m b_n < k(\lambda_1)\|a\|_{p,\phi}\|b\|_{q,\psi}, \tag{4}$$

where the constant $k(\lambda_1)$ is still the best value. Clearly, for $p = q = 2$, $\lambda = 1$, $k_1(x,y) = \frac{1}{x+y}$, $\lambda_1 = \lambda_2 = \frac{1}{2}$, (3) reduces to (1), while (4) reduces to (2).

Some other results about integral and discrete Hilbert-type inequalities can be found in [7–16]. On half-discrete Hilbert-type inequalities with the general non-homogeneous kernels, Hardy *et al.* provided a few results in Theorem 351 of [1]. But they did not prove that the constant factors are best possible. In 2005, Yang [17] gave a result with the kernel $\frac{1}{(1+nx)^\lambda}$ by introducing a variable and proved that the constant factor is best possible. Very recently, Yang [18] and [19] gave the following half-discrete Hilbert's inequality with the best constant factor:

$$\int_0^\infty f(x)\sum_{n=1}^{\infty}\frac{a_n}{(x+n)^\lambda}\,dx < \pi\|f\|\|a\|; \tag{5}$$

Chen [20] and Yang [21] gave two more accurate half-discrete Mulholland's inequalities by using Hadamard's inequality.

In this paper, by means of weight functions and the improved Euler-Maclaurin summation formula, a more accurate half-discrete Hilbert-type inequality with a non-homogeneous kernel and a best constant factor is given as follows. For $0 < \alpha + \beta \le 2$, $\gamma \in \mathbf{R}$, $\eta \le 1 - \frac{\alpha+\beta}{8}(1 + \sqrt{3 + \frac{4}{\alpha+\beta}})$,

$$\begin{aligned}&\int_\gamma^\infty f(x)\sum_{n=1}^{\infty}\frac{(\min\{1,(x-\gamma)(n-\eta)\})^\beta}{(\max\{1,(x-\gamma)(n-\eta)\})^\alpha}a_n\,dx\\&\quad < \frac{4}{\alpha+\beta}\left\{\int_\gamma^\infty (x-\gamma)^{1-\alpha+\beta}f^2(x)\,dx\sum_{n=1}^{\infty}(n-\eta)^{1-\alpha+\beta}a_n^2\right\}^{\frac{1}{2}}.\end{aligned} \tag{6}$$

Moreover, a best extension of (6), some equivalent forms, the operator expressions as well as some particular inequalities are considered.

2 Some lemmas

Lemma 1 *If* $n_0 \in \mathbf{N}$, $s > n_0$, $g_1(y)$ $(y \in [n_0, s))$, $g_2(y)$ $(y \in [s,\infty))$ *are decreasing continuous functions satisfying* $g_1(n_0) - g_1(s-0) + g_2(s) > 0$, $g_2(\infty) = 0$, *define a function* $g(y)$ *as follows*:

$$g(y) := \begin{cases} g_1(y), & y \in [n_0, s),\\ g_2(y), & y \in [s,\infty).\end{cases}$$

Then there exists $\varepsilon \in [0,1]$ *such that*

$$\begin{aligned}&\frac{-1}{8}\bigl[g_1(n_0) + \varepsilon\bigl(g_2(s) - g_1(s-0)\bigr)\bigr]\\&\quad < \int_{n_0}^\infty \rho(y)g(y)\,dy < \frac{1-\varepsilon}{8}\bigl(g_2(s) - g_1(s-0)\bigr),\end{aligned} \tag{7}$$

where $\rho(y) = y - [y] - \frac{1}{2}$ *is a Bernoulli function of the first order. In particular, for* $g_1(y) = 0$, $y \in [n_0, s)$, *we have* $g_2(s) > 0$ *and*

$$\frac{-1}{8} g_2(s) < \int_s^\infty \rho(y) g(y)\, dy < \frac{1}{8} g_2(s); \tag{8}$$

for $g_2(y) = 0$, $y \in [s, \infty)$, *if* $g_1(s-0) \geq 0$, *then it follows* $g_1(n_0) > 0$ *and*

$$\frac{-1}{8} g_1(n_0) < \int_{n_0}^s \rho(y) g_1(y)\, dy < 0. \tag{9}$$

Proof Define a decreasing continuous function $\widetilde{g}(y)$ as

$$\widetilde{g}(y) := \begin{cases} g_1(y) + g_2(s) - g_1(s-0), & y \in [n_0, s), \\ g_2(y), & y \in [s, \infty). \end{cases}$$

Then it follows

$$\begin{aligned} \int_{n_0}^\infty \rho(y) g(y)\, dy &= \int_{n_0}^s \rho(y) g(y)\, dy + \int_s^\infty \rho(y) g(y)\, dy \\ &= \int_{n_0}^s \rho(y) \bigl(\widetilde{g}(y) - g_2(s) + g_1(s-0)\bigr)\, dy + \int_s^\infty \rho(y) \widetilde{g}(y)\, dy \\ &= \int_{n_0}^\infty \rho(y) \widetilde{g}(y)\, dy - \bigl(g_2(s) - g_1(s-0)\bigr) \int_{n_0}^s \rho(y)\, dy, \end{aligned}$$

$$\begin{aligned} \int_{n_0}^s \rho(y)\, dy &= \int_{n_0}^{[s]} \rho(y)\, dy + \int_{[s]}^s \rho(y)\, dy = \int_{[s]}^s \left(y - [s] - \frac{1}{2}\right) dy \\ &= \frac{1}{8}\left[4\left(s - [s] - \frac{1}{2}\right)^2 - 1\right] = \frac{\varepsilon - 1}{8} \quad \bigl(\varepsilon \in [0,1]\bigr). \end{aligned}$$

Since $\widetilde{g}(n_0) = g_1(n_0) + g_2(s) - g_1(s-0) > 0$, $\widetilde{g}(y)$ is a non-constant decreasing continuous function with $\widetilde{g}(\infty) = g_2(\infty) = 0$, by the improved Euler-Maclaurin summation formula (*cf.* [6], Theorem 2.2.2), it follows

$$\frac{-1}{8}\bigl(g_1(n_0) + g_2(s) - g_1(s-0)\bigr) = \frac{-1}{8} \widetilde{g}(n_0) < \int_{n_0}^\infty \rho(y) \widetilde{g}(y)\, dy < 0,$$

and then in view of the above results and by simple calculation, we have (7). □

Lemma 2 *If* $0 < \alpha + \beta \leq 2$, $\gamma \in \mathbf{R}$, $\eta \leq 1 - \frac{\alpha+\beta}{8}(1 + \sqrt{3 + \frac{4}{\alpha+\beta}})$, *and* $\omega(n)$ *and* $\varpi(x)$ *are weight functions given by*

$$\omega(n) := \int_\gamma^\infty \frac{(\min\{1, (x-\gamma)(n-\eta)\})^\beta}{(\max\{1, (x-\gamma)(n-\eta)\})^\alpha} \frac{(n-\eta)^{\frac{\alpha-\beta}{2}}}{(x-\gamma)^{1-\frac{\alpha-\beta}{2}}}\, dx, \quad n \in \mathbf{N}, \tag{10}$$

$$\varpi(x) := \sum_{n=1}^\infty \frac{(\min\{1, (x-\gamma)(n-\eta)\})^\beta}{(\max\{1, (x-\gamma)(n-\eta)\})^\alpha} \frac{(x-\gamma)^{\frac{\alpha-\beta}{2}}}{(n-\eta)^{1-\frac{\alpha-\beta}{2}}}, \quad x > \gamma, \tag{11}$$

then we have

$$\varpi(x) < \omega(n) = \frac{4}{\alpha+\beta}. \tag{12}$$

Proof Substituting $t = (x-\gamma)(n-\eta)$ in (10), and by simple calculation, we have

$$\begin{aligned}\omega(n) &= \int_0^\infty \frac{(\min\{1,t\})^\beta}{(\max\{1,t\})^\alpha} t^{\frac{\alpha-\beta}{2}-1}\,dt \\ &= \int_0^1 t^{\beta+\frac{\alpha-\beta}{2}-1}\,dt + \int_1^\infty t^{-\alpha+\frac{\alpha-\beta}{2}-1}\,dt = \frac{4}{\alpha+\beta}.\end{aligned}$$

For fixed $x > \gamma$, we find

$$\begin{aligned}h(x,y) &:= (x-\gamma)^{\frac{\alpha-\beta}{2}} \frac{(\min\{1,(x-\gamma)(y-\eta)\})^\beta}{(\max\{1,(x-\gamma)(y-\eta)\})^\alpha}(y-\eta)^{\frac{\alpha-\beta}{2}-1} \\ &= \begin{cases}(x-\gamma)^{\frac{\alpha+\beta}{2}}(y-\eta)^{\frac{\alpha+\beta}{2}-1}, & \eta < y < \eta + \frac{1}{x-\gamma}, \\ (x-\gamma)^{-\frac{\alpha+\beta}{2}}(y-\eta)^{-\frac{\alpha+\beta}{2}-1}, & y \ge \eta + \frac{1}{x-\gamma},\end{cases} \\ h'_y(x,y) &= \begin{cases}-(1-\frac{\alpha+\beta}{2})(x-\gamma)^{\frac{\alpha+\beta}{2}}(y-\eta)^{\frac{\alpha+\beta}{2}-2}, & \eta < y < \eta + \frac{1}{x-\gamma}, \\ -(\frac{\alpha+\beta}{2}+1)(x-\gamma)^{-\frac{\alpha+\beta}{2}}(y-\eta)^{-\frac{\alpha+\beta}{2}-2}, & y \ge \eta + \frac{1}{x-\gamma},\end{cases} \\ \int_\eta^\infty h(x,y)\,dy &\overset{t=(x-\gamma)(y-\eta)}{=} \int_0^\infty \frac{(\min\{1,t\})^\beta}{(\max\{1,t\})^\alpha} t^{\frac{\alpha-\beta}{2}-1}\,dt = \frac{4}{\alpha+\beta}.\end{aligned}$$

By the Euler-Maclaurin summation formula (*cf.* [6]), it follows

$$\begin{aligned}\varpi(x) &= \sum_{n=1}^\infty h(x,n) = \int_1^\infty h(x,y)\,dy + \frac{1}{2}h(x,1) + \int_1^\infty \rho(y)h'_y(x,y)\,dy \\ &= \int_\eta^\infty h(x,y)\,dy - R(x) = \frac{4}{\alpha+\beta} - R(x), \\ R(x) &:= \int_\eta^1 h(x,y)\,dy - \frac{1}{2}h(x,1) - \int_1^\infty \rho(y)h'_y(x,y)\,dy. \end{aligned} \tag{13}$$

(i) For $0 < x-\gamma < \frac{1}{1-\eta}$, we obtain $-\frac{1}{2}h(x,1) = -\frac{1}{2}(x-\gamma)^{\frac{\alpha+\beta}{2}}(1-\eta)^{\frac{\alpha+\beta}{2}-1}$, and

$$\int_\eta^1 h(x,y)\,dy = (x-\gamma)^{\frac{\alpha+\beta}{2}} \int_\eta^1 (y-\eta)^{\frac{\alpha+\beta}{2}-1}\,dy = \frac{2(1-\eta)^{\frac{\alpha+\beta}{2}}}{\alpha+\beta}(x-\gamma)^{\frac{\alpha+\beta}{2}}.$$

Setting $g(y) := -h'_y(x,y)$, wherefrom $g_1(y) = (1-\frac{\alpha+\beta}{2})(x-\gamma)^{\frac{\alpha+\beta}{2}}(y-\eta)^{\frac{\alpha+\beta}{2}-2}$, $g_2(y) = (\frac{\alpha+\beta}{2}+1)(x-\gamma)^{-\frac{\alpha+\beta}{2}}(y-\eta)^{-\frac{\alpha+\beta}{2}-2}$ and

$$\begin{aligned}&g_2\left(\eta + \frac{1}{x-\gamma}\right) - g_1\left(\left(\eta + \frac{1}{x-\gamma}\right) - 0\right) \\ &\quad = \left(\frac{\alpha+\beta}{2}+1\right)(x-\gamma)^2 - \left(1 - \frac{\alpha+\beta}{2}\right)(x-\gamma)^2 \\ &\quad = (\alpha+\beta)(x-\gamma)^2 > 0,\end{aligned}$$

then by (7), we find

$$\begin{aligned}
-\int_1^\infty \rho(y)h'_y(x,y)\,dy &= \int_1^\infty \rho(y)g(y)\,dy \\
&> \frac{-1}{8}\left[g_1(1)+g_2\left(\eta+\frac{1}{x-\gamma}\right)-g_1\left(\left(\eta+\frac{1}{x-\gamma}\right)-0\right)\right] \\
&= \frac{-1}{8}\left[\left(1-\frac{\alpha+\beta}{2}\right)(x-\gamma)^{\frac{\alpha+\beta}{2}}(1-\eta)^{\frac{\alpha+\beta}{2}-2}+(\alpha+\beta)(x-\gamma)^2\right] \\
&> \frac{-1}{8}\left[\left(1-\frac{\alpha+\beta}{2}\right)(1-\eta)^{\frac{\alpha+\beta}{2}-2}(x-\gamma)^{\frac{\alpha+\beta}{2}}\right. \\
&\quad \left.+(\alpha+\beta)(1-\eta)^{\frac{\alpha+\beta}{2}-2}(x-\gamma)^{\frac{\alpha+\beta}{2}-2}(x-\gamma)^2\right] \\
&= \frac{-1}{8}\left[\left(1+\frac{\alpha+\beta}{2}\right)(1-\eta)^{\frac{\alpha+\beta}{2}-2}(x-\gamma)^{\frac{\alpha+\beta}{2}}\right].
\end{aligned}$$

In view of (11) and the above results, since for $\eta \le 1-\frac{\alpha+\beta}{8}(1+\sqrt{3+\frac{4}{\alpha+\beta}})$, namely $1-\eta \ge \frac{\alpha+\beta}{8}(1+\sqrt{3+\frac{4}{\alpha+\beta}})$, it follows

$$\begin{aligned}
R(x) &> \frac{2}{\alpha+\beta}(1-\eta)^{\frac{\alpha+\beta}{2}}(x-\gamma)^{\frac{\alpha+\beta}{2}}-\frac{1}{2}(x-\gamma)^{\frac{\alpha+\beta}{2}}(1-\eta)^{\frac{\alpha+\beta}{2}-1} \\
&\quad -\frac{1}{8}\left(1+\frac{\alpha+\beta}{2}\right)(1-\eta)^{\frac{\alpha+\beta}{2}-2}(x-\gamma)^{\frac{\alpha+\beta}{2}} \\
&= \left[\frac{2(1-\eta)^2}{\alpha+\beta}-\frac{(1-\eta)}{2}-\frac{2+\alpha+\beta}{16}\right]\frac{(x-\gamma)^{\frac{\alpha+\beta}{2}}}{(1-\eta)^{2-\frac{\alpha+\beta}{2}}} \ge 0.
\end{aligned}$$

(ii) For $x-\gamma \ge \frac{1}{1-\eta}$, we obtain $-\frac{1}{2}h(x,1) = -\frac{1}{2}(x-\gamma)^{-\frac{\alpha+\beta}{2}}(1-\eta)^{-\frac{\alpha+\beta}{2}-1}$, and

$$\begin{aligned}
\int_\eta^1 h(x,y)\,dy &= \int_\eta^{\eta+\frac{1}{x-\gamma}} \frac{(x-\gamma)^{\frac{\alpha+\beta}{2}}}{(y-\eta)^{1-\frac{\alpha+\beta}{2}}}\,dy+\int_{\eta+\frac{1}{x-\gamma}}^1 \frac{(x-\gamma)^{-\frac{\alpha+\beta}{2}}}{(y-\eta)^{\frac{\alpha+\beta}{2}+1}}\,dy \\
&= \frac{4}{\alpha+\beta}-\frac{2}{\alpha+\beta}(1-\eta)^{-\frac{\alpha+\beta}{2}}(x-\gamma)^{-\frac{\alpha+\beta}{2}} \\
&\ge \frac{4(1-\eta)^{-\frac{\alpha+\beta}{2}}}{\alpha+\beta}(x-\gamma)^{-\frac{\alpha+\beta}{2}}-\frac{2(1-\eta)^{-\frac{\alpha+\beta}{2}}}{\alpha+\beta}(x-\gamma)^{-\frac{\alpha+\beta}{2}} \\
&= \frac{2}{\alpha+\beta}(1-\eta)^{-\frac{\alpha+\beta}{2}}(x-\gamma)^{-\frac{\alpha+\beta}{2}}.
\end{aligned}$$

Since for $y \ge 1$, $y-\eta \ge \frac{1}{x-\gamma}$, by the improved Euler-Maclaurin summation formula (*cf.* [6]), it follows

$$\begin{aligned}
-\int_1^\infty \rho(y)h'_y(x,y)\,dy &= \left(\frac{\alpha+\beta}{2}+1\right)(x-\gamma)^{-\frac{\alpha+\beta}{2}}\int_1^\infty \rho(y)(y-\eta)^{-\frac{\alpha+\beta}{2}-2}\,dy \\
&> -\frac{1}{12}\left(\frac{\alpha+\beta}{2}+1\right)(x-\gamma)^{-\frac{\alpha+\beta}{2}}(1-\eta)^{-\frac{\alpha+\beta}{2}-2}.
\end{aligned}$$

In view of (13) and the above results, for $1-\eta \geq \frac{\alpha+\beta}{8}(1+\sqrt{3+\frac{4}{\alpha+\beta}})$, we find

$$\begin{aligned}
R(x) &> \frac{2}{\alpha+\beta}(1-\eta)^{-\frac{\alpha+\beta}{2}}(x-\gamma)^{-\frac{\alpha+\beta}{2}} - \frac{1}{2}(1-\eta)^{-\frac{\alpha+\beta}{2}-1}(x-\gamma)^{-\frac{\alpha+\beta}{2}} \\
&\quad - \frac{1}{12}\left(\frac{\alpha+\beta}{2}+1\right)(1-\eta)^{-\frac{\alpha+\beta}{2}-2}(x-\gamma)^{-\frac{\alpha+\beta}{2}} \\
&> \left[\frac{2(1-\eta)^2}{\alpha+\beta} - \frac{1-\eta}{2} - \frac{2+\alpha+\beta}{16}\right]\frac{(x-\gamma)^{-\frac{\alpha+\beta}{2}}}{(1-\eta)^{2+\frac{\alpha+\beta}{2}}} \geq 0.
\end{aligned}$$

Hence, for $x > \gamma$, we have $R(x) > 0$, and then (12) follows. □

Lemma 3 *Let the assumptions of Lemma* 2 *be fulfilled and, additionally, let* $p > 1$, $\frac{1}{p}+\frac{1}{q} = 1$, $a_n \geq 0$, $n \in \mathbf{N}$, $f(x)$ *be a non-negative measurable function in* (γ, ∞). *Then we have the following inequalities*:

$$\begin{aligned}
J &:= \left\{\sum_{n=1}^{\infty}(n-\beta)^{\frac{p(\alpha-\beta)}{2}-1}\left[\int_{\gamma}^{\infty}\frac{(\min\{1,(x-\gamma)(n-\eta)\})^{\beta}}{(\max\{1,(x-\gamma)(n-\eta)\})^{\alpha}}f(x)\,dx\right]^p\right\}^{\frac{1}{p}} \\
&\leq \left(\frac{4}{\alpha+\beta}\right)^{\frac{1}{q}}\left\{\int_{\gamma}^{\infty}\varpi(x)(x-\gamma)^{p(1-\frac{\alpha-\beta}{2})-1}f^p(x)\,dx\right\}^{\frac{1}{p}}, \qquad (14)
\end{aligned}$$

$$\begin{aligned}
L_1 &:= \left\{\int_{\gamma}^{\infty}\frac{(x-\alpha)^{\frac{q(\alpha-\beta)}{2}-1}}{[\varpi(x)]^{q-1}}\left[\sum_{n=1}^{\infty}\frac{(\min\{1,(x-\gamma)(n-\eta)\})^{\beta}a_n}{(\max\{1,(x-\gamma)(n-\eta)\})^{\alpha}}\right]^q dx\right\}^{\frac{1}{q}} \\
&\leq \left\{\frac{4}{\alpha+\beta}\sum_{n=1}^{\infty}(n-\eta)^{q(1-\frac{\alpha-\beta}{2})-1}a_n^q\right\}^{\frac{1}{q}}. \qquad (15)
\end{aligned}$$

Proof Setting $k(x,n) := \frac{(\min\{1,(x-\gamma)(n-\eta)\})^{\beta}}{(\max\{1,(x-\gamma)(n-\eta)\})^{\alpha}}$, by Hölder's inequality (*cf.* [22]) and (12), it follows

$$\begin{aligned}
&\left[\int_{\gamma}^{\infty}\frac{(\min\{1,(x-\gamma)(n-\eta)\})^{\beta}}{(\max\{1,(x-\gamma)(n-\eta)\})^{\alpha}}f(x)\,dx\right]^p \\
&= \left\{\int_{\gamma}^{\infty}k(x,n)\left[\frac{(x-\gamma)^{(1-\frac{\alpha-\beta}{2})/q}}{(n-\eta)^{(1-\frac{\alpha-\beta}{2})/p}}f(x)\right]\left[\frac{(n-\gamma)^{(1-\frac{\alpha-\beta}{2})/p}}{(x-\gamma)^{(1-\frac{\alpha-\beta}{2})/q}}\right]dx\right\}^p \\
&\leq \int_{\gamma}^{\infty}k(x,n)\frac{(x-\gamma)^{(1-\frac{\alpha-\beta}{2})(p-1)}}{(n-\eta)^{1-\frac{\alpha-\beta}{2}}}f^p(x)\,dx \\
&\quad \times \left\{\int_{\gamma}^{\infty}k(x,n)\frac{(n-\eta)^{(1-\frac{\alpha-\beta}{2})(q-1)}}{(x-\gamma)^{1-\frac{\alpha-\beta}{2}}}\,dx\right\}^{p-1} \\
&= \left\{\omega(n)(n-\eta)^{q(1-\frac{\alpha-\beta}{2})-1}\right\}^{p-1}\int_{\gamma}^{\infty}k(x,n)\frac{(x-\gamma)^{(1-\frac{\alpha-\beta}{2})(p-1)}}{(n-\eta)^{1-\frac{\alpha-\beta}{2}}}f^p(x)\,dx \\
&= \left(\frac{4}{\alpha+\beta}\right)^{p-1}(n-\eta)^{1-\frac{p(\alpha-\beta)}{2}}\int_{\gamma}^{\infty}k(x,n)\frac{(x-\gamma)^{(1-\frac{\alpha-\beta}{2})(p-1)}}{(n-\eta)^{1-\frac{\alpha-\beta}{2}}}f^p(x)\,dx.
\end{aligned}$$

Then by the Lebesgue term-by-term integration theorem (*cf.* [23]), we have

$$\begin{aligned} J &\le \left(\frac{4}{\alpha+\beta}\right)^{\frac{1}{q}} \left\{ \sum_{n=1}^{\infty} \int_{\gamma}^{\infty} k(x,n) \frac{(x-\gamma)^{(1-\frac{\alpha-\beta}{2})(p-1)}}{(n-\eta)^{1-\frac{\alpha-\beta}{2}}} f^{p}(x)\,dx \right\}^{\frac{1}{p}} \\ &= \left(\frac{4}{\alpha+\beta}\right)^{\frac{1}{q}} \left\{ \int_{\gamma}^{\infty} \sum_{n=1}^{\infty} k(x,n) \frac{(x-\gamma)^{(1-\frac{\alpha-\beta}{2})(p-1)}}{(n-\eta)^{1-\frac{\alpha-\beta}{2}}} f^{p}(x)\,dx \right\}^{\frac{1}{p}} \\ &= \left(\frac{4}{\alpha+\beta}\right)^{\frac{1}{q}} \left\{ \int_{\gamma}^{\infty} \varpi(x)(x-\gamma)^{p(1-\frac{\alpha-\beta}{2})-1} f^{p}(x)\,dx \right\}^{\frac{1}{p}}. \end{aligned}$$

Hence, (14) follows. By Hölder's inequality again, we have

$$\begin{aligned} \left[\sum_{n=1}^{\infty} k(x,n) a_n\right]^{q} &= \left\{ \sum_{n=1}^{\infty} k(x,n) \left[\frac{(x-\gamma)^{(1-\frac{\alpha-\beta}{2})/q}}{(n-\eta)^{(1-\frac{\alpha-\beta}{2})/p}} \right] \right. \\ &\quad \left. \times \left[\frac{(n-\eta)^{(1-\frac{\alpha-\beta}{2})/p} a_n}{(x-\gamma)^{(1-\frac{\alpha-\beta}{2})/q}} \right] \right\}^{q} \le \left\{ \sum_{n=1}^{\infty} k(x,n) \frac{(x-\gamma)^{(1-\frac{\alpha-\beta}{2})(p-1)}}{(n-\eta)^{1-\frac{\alpha-\beta}{2}}} \right\}^{q-1} \\ &\quad \times \sum_{n=1}^{\infty} k(x,n) \frac{(n-\eta)^{(1-\frac{\alpha-\beta}{2})(q-1)}}{(x-\gamma)^{1-\frac{\alpha-\beta}{2}}} a_n^{q} \\ &= \frac{[\varpi(x)]^{q-1}}{(x-\gamma)^{\frac{q(\alpha-\beta)}{2}-1}} \sum_{n=1}^{\infty} k(x,n) \frac{(n-\eta)^{(1-\frac{\alpha-\beta}{2})(q-1)}}{(x-\gamma)^{1-\frac{\alpha-\beta}{2}}} a_n^{q}. \end{aligned}$$

By the Lebesgue term-by-term integration theorem, we have

$$\begin{aligned} L_1 &\le \left\{ \int_{\gamma}^{\infty} \sum_{n=1}^{\infty} k(x,n) \frac{(n-\eta)^{(1-\frac{\alpha-\beta}{2})(q-1)}}{(x-\gamma)^{1-\frac{\alpha-\beta}{2}}} a_n^{q}\,dx \right\}^{\frac{1}{q}} \\ &= \left\{ \sum_{n=1}^{\infty} \int_{\gamma}^{\infty} k(x,n) \frac{(n-\eta)^{(1-\frac{\alpha-\beta}{2})(q-1)}}{(x-\gamma)^{1-\frac{\alpha-\beta}{2}}} a_n^{q}\,dx \right\}^{\frac{1}{q}} \\ &= \left\{ \sum_{n=1}^{\infty} \omega(n)(n-\eta)^{q(1-\frac{\alpha-\beta}{2})-1} a_n^{q} \right\}^{\frac{1}{q}}, \end{aligned}$$

and in view of (12), inequality (15) follows. □

Lemma 4 *Let the assumptions of Lemma* 2 *be fulfilled and, additionally, let* $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, $0<\varepsilon<\frac{p}{2}(\alpha+\beta)$. *Setting* $\widetilde{f}(x)=(x-\gamma)^{\frac{\alpha-\beta}{2}+\frac{\varepsilon}{p}-1}$, $x\in(\gamma,\gamma+1)$; $\widetilde{f}(x)=0$, $x\in[\gamma+1,\infty)$, *and* $\widetilde{a}_n=(n-\eta)^{\frac{\alpha-\beta}{2}-\frac{\varepsilon}{q}-1}$, $n\in\mathbf{N}$, *then we have*

$$\begin{aligned} \widetilde{I} &:= \sum_{n=1}^{\infty} \widetilde{a}_n \int_{\gamma}^{\infty} \frac{(\min\{1,(x-\gamma)(n-\eta)\})^{\beta}}{(\max\{1,(x-\gamma)(n-\eta)\})^{\alpha}} \widetilde{f}(x)\,dx \\ &> \frac{1}{\varepsilon}\left[\frac{(\alpha+\beta)(1-\eta)^{-\varepsilon}}{(\frac{\alpha+\beta}{2})^{2}-(\frac{\varepsilon}{p})^{2}} - \varepsilon O(1) \right], \end{aligned} \tag{16}$$

$$\begin{aligned}\widetilde{H} &:= \left\{\int_\gamma^\infty (x-\gamma)^{p(1-\frac{\alpha-\beta}{2})-1}\widetilde{f}^p(x)\,dx\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^\infty (n-\eta)^{q(1-\frac{\alpha-\beta}{2})-1}\widetilde{a}_n^q\right\}^{\frac{1}{q}} \\ &< \frac{1}{\varepsilon}\left[\varepsilon(1-\eta)^{-\varepsilon-1}+(1-\eta)^{-\varepsilon}\right]^{\frac{1}{q}}. \end{aligned} \tag{17}$$

Proof We find

$$\begin{aligned}\widetilde{I} &= \sum_{n=1}^\infty (n-\eta)^{\frac{\alpha-\beta}{2}-\frac{\varepsilon}{q}-1} \\ &\quad\times \int_\gamma^{\gamma+1} \frac{(\min\{1,(x-\gamma)(n-\eta)\})^\beta}{(\max\{1,(x-\gamma)(n-\eta)\})^\alpha}(x-\gamma)^{\frac{\alpha-\beta}{2}+\frac{\varepsilon}{p}-1}\,dx \\ &= \sum_{n=1}^\infty (n-\eta)^{\frac{\alpha-\beta}{2}-\frac{\varepsilon}{q}-1}\left[(n-\eta)^\beta\int_\gamma^{\gamma+\frac{1}{n-\eta}}(x-\gamma)^{\frac{\alpha+\beta}{2}+\frac{\varepsilon}{p}-1}\,dx\right. \\ &\quad\left.+\frac{1}{(n-\eta)^\alpha}\int_{\gamma+\frac{1}{n-\eta}}^{\gamma+1}(x-\gamma)^{-\frac{\alpha+\beta}{2}+\frac{\varepsilon}{p}-1}\,dx\right] \\ &= \frac{\alpha+\beta}{(\frac{\alpha+\beta}{2})^2-(\frac{\varepsilon}{p})^2}\sum_{n=1}^\infty (n-\eta)^{-\varepsilon-1} - \frac{1}{\frac{\alpha+\beta}{2}-\frac{\varepsilon}{p}}\sum_{n=1}^\infty (n-\eta)^{-\frac{\alpha+\beta}{2}-\frac{\varepsilon}{q}-1} \\ &> \frac{\alpha+\beta}{(\frac{\alpha+\beta}{2})^2-(\frac{\varepsilon}{p})^2}\int_1^\infty \frac{dy}{(y-\eta)^{\varepsilon+1}} - \frac{1}{\frac{\alpha+\beta}{2}-\frac{\varepsilon}{p}}\sum_{n=1}^\infty (n-\eta)^{-\frac{\alpha+\beta}{2}-\frac{\varepsilon}{q}-1} \\ &= \frac{1}{\varepsilon}\left[\frac{(\alpha+\beta)(1-\eta)^{-\varepsilon}}{(\frac{\alpha+\beta}{2})^2-(\frac{\varepsilon}{p})^2} - \frac{\varepsilon}{\frac{\alpha+\beta}{2}-\frac{\varepsilon}{p}}\sum_{n=1}^\infty (n-\eta)^{-\frac{\alpha+\beta}{2}-\frac{\varepsilon}{q}-1}\right], \end{aligned}$$

and then (16) is valid. We obtain

$$\begin{aligned}\widetilde{H} &= \left\{\int_\gamma^{\gamma+1}(x-\gamma)^{\varepsilon-1}\,dx\right\}^{\frac{1}{p}}\left\{(1-\eta)^{-\varepsilon-1}+\sum_{n=2}^\infty (n-\eta)^{-\varepsilon-1}\right\}^{\frac{1}{q}} \\ &< \left(\frac{1}{\varepsilon}\right)^{\frac{1}{p}}\left\{(1-\eta)^{-\varepsilon-1}+\int_1^\infty (y-\eta)^{-\varepsilon-1}\,dy\right\}^{\frac{1}{q}} \\ &= \frac{1}{\varepsilon}\left\{\varepsilon(1-\eta)^{-\varepsilon-1}+(1-\eta)^{-\varepsilon}\right\}^{\frac{1}{q}}, \end{aligned}$$

and so (17) is valid. □

3 Main results

We introduce the functions

$$\Phi(x) := (x-\gamma)^{p(1-\frac{\alpha-\beta}{2})-1}\quad \bigl(x\in(\gamma,\infty)\bigr),\qquad \Psi(n) := (n-\eta)^{q(1-\frac{\alpha-\beta}{2})-1}\quad (n\in\mathbf{N}),$$

wherefrom $[\Phi(x)]^{1-q} = (x-\gamma)^{q\frac{\alpha-\beta}{2}-1}$ and $[\Psi(n)]^{1-p} = (n-\eta)^{p\frac{\alpha-\beta}{2}-1}$.

Theorem 5 *If* $0<\alpha+\beta\le 2$, $\gamma\in\mathbf{R}$, $\eta\le 1-\frac{\alpha+\beta}{8}(1+\sqrt{3+\frac{4}{\alpha+\beta}})$, $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, $f(x), a_n\ge 0$, $f\in L_{p,\Phi}(\gamma,\infty)$, $a=\{a_n\}_{n=1}^\infty\in l_{q,\Psi}$, $\|f\|_{p,\Phi}>0$ *and* $\|a\|_{q,\Psi}>0$, *then we have the following* *equivalent inequalities*:

$$I := \sum_{n=1}^{\infty} a_n \int_{\gamma}^{\infty} \frac{(\min\{1,(x-\gamma)(n-\eta)\})^{\beta}}{(\max\{1,(x-\gamma)(n-\eta)\})^{\alpha}} f(x)\,dx$$
$$= \int_{\gamma}^{\infty} f(x) \sum_{n=1}^{\infty} \frac{(\min\{1,(x-\gamma)(n-\eta)\})^{\beta} a_n}{(\max\{1,(x-\gamma)(n-\eta)\})^{\alpha}}\,dx < \frac{4}{\alpha+\beta}\|f\|_{p,\Phi}\|a\|_{q,\Psi}, \tag{18}$$

$$J = \left\{\sum_{n=1}^{\infty}\bigl[\Psi(n)\bigr]^{1-p}\left[\int_{\gamma}^{\infty} \frac{(\min\{1,(x-\gamma)(n-\eta)\})^{\beta} f(x)}{(\max\{1,(x-\gamma)(n-\eta)\})^{\alpha}}\,dx\right]^{p}\right\}^{\frac{1}{p}}$$
$$< \frac{4}{\alpha+\beta}\|f\|_{p,\Phi}, \tag{19}$$

$$L := \left\{\int_{\gamma}^{\infty}\bigl[\Phi(x)\bigr]^{1-q}\left[\sum_{n=1}^{\infty} \frac{(\min\{1,(x-\gamma)(n-\eta)\})^{\beta} a_n}{(\max\{1,(x-\gamma)(n-\eta)\})^{\alpha}}\right]^{q} dx\right\}^{\frac{1}{q}}$$
$$< \frac{4}{\alpha+\beta}\|a\|_{q,\Psi}, \tag{20}$$

where the constant $\frac{4}{\alpha+\beta}$ *is the best possible in the above inequalities.*

Proof The two expressions for I in (18) follow from the Lebesgue term-by-term integration theorem. By (14) and (12), we have (19). By Hölder's inequality, we have

$$I = \sum_{n=1}^{\infty}\left[\Psi^{\frac{-1}{q}}(n)\int_{\gamma}^{\infty} \frac{(\min\{1,(x-\gamma)(n-\eta)\})^{\beta} f(x)}{(\max\{1,(x-\gamma)(n-\eta)\})^{\alpha}}\,dx\right]\bigl[\Psi^{\frac{1}{q}}(n)a_n\bigr]$$
$$\le J\|a\|_{q,\Psi}.$$

Then by (19), we have (18). On the other hand, assume that (18) is valid. Setting

$$a_n := \bigl[\Psi(n)\bigr]^{1-p}\left[\int_{\gamma}^{\infty} \frac{(\min\{1,(x-\gamma)(n-\eta)\})^{\beta} f(x)}{(\max\{1,(x-\gamma)(n-\eta)\})^{\alpha}}\,dx\right]^{p-1}, \quad n \in \mathbf{N},$$

where $J^{p-1} = \|a\|_{q,\Psi}$. By (14), we find $J < \infty$. If $J = 0$, then (19) is trivially valid; if $J > 0$, then by (18) we have

$$\|a\|_{q,\Psi}^{q} = J^{q(p-1)} = J^{p} = I < \frac{4}{\alpha+\beta}\|f\|_{p,\Phi}\|a\|_{q,\Psi},$$

therefore $\|a\|_{q,\Psi}^{q-1} = J < \frac{4}{\alpha+\beta}\|f\|_{p,\Phi}$; that is, (19) is equivalent to (18). On the other hand, by (12) we have $[\varpi(x)]^{1-q} > (\frac{4}{\alpha+\beta})^{1-q}$. Then in view of (15), we have (20). By Hölder's inequality, we find

$$I = \int_{\gamma}^{\infty}\bigl[\Phi^{\frac{1}{p}}(x)f(x)\bigr]\left[\Phi^{\frac{-1}{p}}(x)\sum_{n=1}^{\infty} \frac{(\min\{1,(x-\gamma)(n-\eta)\})^{\beta} a_n}{(\max\{1,(x-\gamma)(n-\eta)\})^{\alpha}}\right]dx$$
$$\le \|f\|_{p,\Phi}L.$$

Then by (20), we have (18). On the other hand, assume that (18) is valid. Setting

$$f(x) := \left[\Phi(x)\right]^{1-q}\left[\sum_{n=1}^{\infty}\frac{(\min\{1,(x-\gamma)(n-\eta)\})^{\beta}a_n}{(\max\{1,(x-\gamma)(n-\eta)\})^{\alpha}}\right]^{q-1}, \quad x\in(\gamma,\infty),$$

then $L^{q-1} = \|f\|_{p,\Phi}$. By (15), we find $L < \infty$. If $L = 0$, then (20) is trivially valid; if $L > 0$, then by (18), we have

$$\|f\|_{p,\Phi}^{p} = L^{p(q-1)} = I < \frac{4}{\alpha+\beta}\|f\|_{p,\Phi}\|a\|_{q,\Psi},$$

therefore $\|f\|_{p,\Phi}^{p-1} = L < \frac{4}{\alpha+\beta}\|a\|_{q,\Psi}$; that is, (20) is equivalent to (18). Hence, (18), (19) and (20) are equivalent.

If there exists a positive number k ($\leq \frac{4}{\alpha+\beta}$) such that (18) is valid as we replace $\frac{4}{\alpha+\beta}$ with k, then, in particular, it follows that $\widetilde{I} < k\widetilde{H}$. In view of (16) and (17), we have

$$\frac{(\alpha+\beta)(1-\eta)^{-\varepsilon}}{(\frac{\alpha+\beta}{2})^2-(\frac{\varepsilon}{p})^2} - \varepsilon O(1) < k\left[\varepsilon(1-\eta)^{-\varepsilon-1}+(1-\eta)^{-\varepsilon}\right]^{\frac{1}{q}},$$

and $\frac{4}{\alpha+\beta} \leq k$ ($\varepsilon \to 0^+$). Hence, $k = \frac{4}{\alpha+\beta}$ is the best value of (18).

By the equivalence of the inequalities, the constant factor $\frac{4}{\alpha+\beta}$ in (19) and (20) is the best possible. □

Remark 1 (i) Define the first type half-discrete Hilbert-type operator $T_1 : L_{p,\Phi}(\gamma,\infty) \to l_{p,\Psi^{1-p}}$ as follows. For $f \in L_{p,\Phi}(\gamma,\infty)$, we define $T_1 f \in l_{p,\Psi^{1-p}}$ by

$$T_1 f(n) = \int_{\gamma}^{\infty}\frac{(\min\{1,(x-\gamma)(n-\eta)\})^{\beta}}{(\max\{1,(x-\gamma)(n-\eta)\})^{\alpha}}f(x)\,dx, \quad n\in\mathbf{N}.$$

Then by (19), $\|T_1 f\|_{p,\Psi^{1-p}} \leq \frac{4}{\alpha+\beta}\|f\|_{p,\Phi}$ and so T_1 is a bounded operator with $\|T_1\| \leq \frac{4}{\alpha+\beta}$. Since by Theorem 5 the constant factor in (19) is best possible, we have $\|T_1\| = \frac{4}{\alpha+\beta}$.

(ii) Define the second type half-discrete Hilbert-type operator $T_2 : l_{q,\Psi} \to L_{q,\Phi^{1-q}}(\gamma,\infty)$ as follows. For $a \in l_{q,\Psi}$, we define $T_2 a \in L_{q,\Phi^{1-q}}(\gamma,\infty)$ by

$$T_2 a(x) = \sum_{n=1}^{\infty}\frac{(\min\{1,(x-\gamma)(n-\eta)\})^{\beta}}{(\max\{1,(x-\gamma)(n-\eta)\})^{\alpha}}a_n, \quad x\in(\gamma,\infty).$$

Then by (20), $\|T_2 a\|_{q,\Phi^{1-q}} \leq \frac{4}{\alpha+\beta}\|a\|_{q,\Psi}$ and so T_2 is a bounded operator with $\|T_2\| \leq \frac{4}{\alpha+\beta}$. Since by Theorem 5 the constant factor in (20) is best possible, we have $\|T_2\| = \frac{4}{\alpha+\beta}$.

Remark 2 (i) For $p = q = 2$, (18) reduces to (6). Since we find

$$\begin{aligned}
&\min_{0<\alpha+\beta\leq 2}\left\{1-\frac{\alpha+\beta}{8}\left(1+\sqrt{3+\frac{4}{\alpha+\beta}}\right)\right\} \\
&\quad = \min_{0<\alpha+\beta\leq 2}\left\{1-\frac{\alpha+\beta}{8}-\frac{1}{8}\sqrt{3(\alpha+\beta)^2+4(\alpha+\beta)}\right\} \\
&\quad = \frac{3-\sqrt{5}}{4} = 0.19^{+} > 0,
\end{aligned}$$

then for $\eta = \gamma = 0$ in (18), we have the following inequality:

$$\sum_{n=1}^{\infty} a_n \int_0^{\infty} \frac{(\min\{1, xn\})^{\beta}}{(\max\{1, xn\})^{\alpha}} f(x)\,dx$$
$$< \frac{4}{\alpha+\beta}\left\{\int_0^{\infty} x^{p(1-\frac{\alpha-\beta}{2})-1} f^p(x)\,dx\right\}^{\frac{1}{p}} \left\{\sum_{n=1}^{\infty} n^{q(1-\frac{\alpha-\beta}{2})-1} a_n^q\right\}^{\frac{1}{q}}. \qquad (21)$$

Hence, (18) is a more accurate inequality of (21).

(ii) For $\beta = 0$ in (18), we have $0 < \alpha \leq 2$, $\gamma \in \mathbf{R}$, $\eta \leq 1 - \frac{\alpha}{8}(1 + \sqrt{3 + \frac{4}{\alpha}})$, and

$$\sum_{n=1}^{\infty} a_n \int_{\gamma}^{\infty} \frac{f(x)\,dx}{(\max\{1, (x-\gamma)(n-\eta)\})^{\alpha}}$$
$$< \frac{4}{\alpha}\left\{\int_{\gamma}^{\infty} (x-\gamma)^{p(1-\frac{\alpha}{2})-1} f^p(x)\,dx\right\}^{\frac{1}{p}} \left\{\sum_{n=1}^{\infty} (n-\eta)^{q(1-\frac{\alpha}{2})-1} a_n^q\right\}^{\frac{1}{q}}; \qquad (22)$$

for $\alpha = 0$ in (18), we have $0 < \beta \leq 2$, $\gamma \in \mathbf{R}$, $\eta \leq 1 - \frac{\beta}{8}(1 + \sqrt{3 + \frac{4}{\beta}})$, and

$$\sum_{n=1}^{\infty} a_n \int_{\gamma}^{\infty} \left(\min\left\{1, (x-\gamma)(n-\eta)\right\}\right)^{\beta} f(x)\,dx$$
$$< \frac{4}{\beta}\left\{\int_{\gamma}^{\infty} (x-\gamma)^{p(1+\frac{\beta}{2})-1} f^p(x)\,dx\right\}^{\frac{1}{p}} \left\{\sum_{n=1}^{\infty} (n-\eta)^{q(1+\frac{\beta}{2})-1} a_n^q\right\}^{\frac{1}{q}}; \qquad (23)$$

for $\beta = \alpha = \lambda$ in (18), we have $0 < \lambda \leq 1$, $\gamma \in \mathbf{R}$, $\eta \leq 1 - \frac{\lambda}{4}(1 + \sqrt{3 + \frac{2}{\lambda}})$, and

$$\sum_{n=1}^{\infty} a_n \int_{\gamma}^{\infty} \left[\frac{\min\{1, (x-\gamma)(n-\eta)\}}{\max\{1, (x-\gamma)(n-\eta)\}}\right]^{\lambda} f(x)\,dx$$
$$< \frac{2}{\lambda}\left\{\int_{\gamma}^{\infty} (x-\gamma)^{p-1} f^p(x)\,dx\right\}^{\frac{1}{p}} \left\{\sum_{n=1}^{\infty} (n-\eta)^{q-1} a_n^q\right\}^{\frac{1}{q}}. \qquad (24)$$

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

BY carried out the molecular genetic studies participated in the sequence alignment and drafted the manuscript. XL conceived of the study and participated in its design and coordination. All authors read and approved the final manuscript.

Author details

[1]Department of Mathematics, Guangdong University of Education, Guangzhou, Guangdong 510303, China.
[2]Department of Mathematics, Zhaoqing University, Zhaoqing, Guangdong 526061, China.

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