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On some Hölder-type inequalities with applications

Mustapha Raïssouli[1,2] and Rabie Zine[3*]

*Correspondence:
rabie.zine@gmail.com
[3]Department of Mathematics, Faculty of Sciences and Humanities, Salman Bin Abdulaziz University, Aflaj, Kingdom of Saudi Arabia
Full list of author information is available at the end of the article

Abstract

In this paper, some mathematical inequalities of Hölder type are established. Applications for some operator inequalities as well as for functional inequalities in convex analysis are provided as well.

Keywords: Hölder-type inequalities; operator means; operator inequalities; functional inequalities; convex analysis

1 Introduction

We begin by stating some notions needed. Let E be a linear vector space over $\mathbb{K} = \mathbb{R}, \mathbb{C}$ and let C be a nonempty subset of E. Consider the two following statements:

(i) C is such that $u \in C$ and $t \geq 0 \Rightarrow tu \in C$; C is then called a cone of E.

(ii) C is such that $u \in C$ and $\lambda \in \mathbb{K} \Rightarrow \lambda u \in C$; C is sometimes called a generalized cone of E. Clearly, every generalized cone of E is a cone.

Let $f : C \to \mathbb{K}$ be a map. If C is a generalized cone, we say that f is homogeneous of degree p if $f(\lambda u) = |\lambda|^p f(u)$ for all $u \in C$ and $\lambda \in \mathbb{K}$. If C is a cone, f is called positively homogeneous of degree p if $f(tu) = t^p f(u)$ for all $u \in C$ and $t \geq 0$. Clearly, every homogeneous map of degree p (on a generalized cone) is positively homogeneous of the same degree p. The reverse is not always true.

Now, let C be a convex cone of E. A map $\Phi : C \to \mathbb{R}$ is called sub-additive if $\Phi(u + v) \leq \Phi(u) + \Phi(v)$ holds for all $u, v \in C$. If C is equipped with an order $\prec$, the map Φ is said to be monotone if for all $u, v \in C$ such that $u \prec v$ we have $\Phi(u) \leq \Phi(v)$.

Let E and F be two linear vector spaces over $\mathbb{K}$, C_1 and C_2 be two nonempty subsets of E and F, respectively, and $h : C_1 \times C_2 \to \mathbb{K}$ be a given map. If C_1 is a cone, we say that h is positively homogeneous of degree r, with respect to the first variable, if $h(tu, v) = t^r h(u, v)$ for all $u \in C_1$, $v \in C_2$ and $t \geq 0$. If C_1 and C_2 are generalized cones, we say that h is a semi-inner product if and only if

$$h(u,v) = \overline{h(v,u)}, \qquad h(\lambda u, v) = \lambda h(u,v) \quad \text{and} \quad h(u, \lambda v) = \overline{\lambda} h(u,v)$$

hold for all $\lambda \in \mathbb{C}$, $u \in C_1$ and $v \in C_2$. Clearly, every semi-inner product map is positively homogeneous of degree 1 with respect to its two variables. The reverse is, in general, false.

The remainder of this paper is organized as follows: Section 2 is devoted to the presentation of our main results together with some related consequences. Section 3 displays a lot of examples illustrating the above theoretical results. In Section 4, we investigate some operator inequalities as applications of our main results. Section 5 is focused on another application for inequalities in convex analysis.

2 The main results

We use the same notations as previously. We start this section by stating the following lemma, which will be needed in the sequel.

Lemma 2.1 *Let $a, b \geq 0$ and $p, q > 0$ be real numbers. Then we have*

$$\inf_{t>0}\left(at^p + bt^{-q}\right) = (p+q)\left(\frac{b}{p}\right)^{\frac{p}{p+q}}\left(\frac{a}{q}\right)^{\frac{q}{p+q}}.$$

Proof If $a = 0$ or $b = 0$, it is easy to see that $\inf_{t>0}(at^p + bt^{-q}) = 0$ and the desired equality holds. Assume that $a, b > 0$ and set $\phi(t) = at^p + bt^{-q}$ for $t > 0$. It is easy to see that

$$\phi'(t) = pat^{p-1} - qbt^{-q-1}$$

for all $t > 0$, with $\phi'(t) = 0$ if and only if

$$t = t_0 = (qb/pa)^{1/(p+q)}.$$

Further, simple computation leads to

$$\phi(t_0) = (p+q)\left(\frac{b}{p}\right)^{\frac{p}{p+q}}\left(\frac{a}{q}\right)^{\frac{q}{p+q}}.$$

This, with the fact that

$$\lim_{t\to 0}\phi(t) = \lim_{t\to\infty}\phi(t) = \infty,$$

yields the desired result. □

Now, our first main result may be presented.

Theorem 2.2 *Let E and F be two linear vector spaces over $\mathbb{K}$, C_1 is a cone of E and C_2 is a nonempty subset of F. Let $f : C_1 \to [0,\infty)$, $g : C_2 \to [0,\infty)$, and $h : C_1 \times C_2 \to \mathbb{R}$ be three maps such that*

$$\forall (u,v) \in C_1 \times C_2, \quad h(u,v) \leq f(u) + g(v). \tag{1}$$

Assume that f is positively homogeneous of degrees p and h is positively homogeneous, with respect to the first variable, of degree r, with $\min(p,0) < r < \max(p,0)$. Then the inequality

$$h(u,v) \leq \left(\frac{p}{r}f(u)\right)^{r/p}\left(\frac{p}{p-r}g(v)\right)^{(p-r)/p} \tag{2}$$

holds true for all $(u,v) \in C_1 \times C_2$.

Proof We present the proof for $p > 0$ ($0 < r < p$), and that of the case $p < 0$ ($p < r < 0$) can be stated in a similar manner. Replacing $u \in C_1$ by $tu \in C_1$, with $t > 0$, in (1) and using the positive homogeneity assumed in our statement, we obtain

$$t^r h(u,v) \leq t^p f(u) + g(v),$$

or equivalently

$$h(u,v) \leq t^{p-r} f(u) + t^{-r} g(v).$$

This means that the map $t \mapsto t^{p-r} f(u) + t^{-r} g(v)$, for $t > 0$, is bounded below and so we can write

$$h(u,v) \leq \inf_{t>0} \bigl(t^{p-r} f(u) + t^{-r} g(v)\bigr).$$

Following Lemma 2.1, with $a = f(u) \geq 0$ and $b = g(v) \geq 0$, we immediately deduce, after a simple manipulation, the desired inequality. We then finished the proof. □

Remark 2.1 (i) With the assumptions of Theorem 2.2 the inequalities (1) and (2) are in fact equivalent. The implication '(2) ⇒ (1)' follows by a simple application of the Young inequality. Similar statements can be made for the analog situation in the following results.

(ii) If the map h comes with positive values then (2) can be written in the following equivalent form:

$$\left(\frac{1}{p} h(u,v)\right)^p \leq \left(\frac{1}{r} f(u)\right)^r \left(\frac{1}{p-r} g(v)\right)^{p-r} \quad \text{if } p > 0$$

and

$$\left(\frac{1}{-p} h(u,v)\right)^p \geq \left(\frac{1}{-r} f(u)\right)^r \left(\frac{1}{r-p} g(v)\right)^{p-r} \quad \text{if } p < 0.$$

(iii) It is worth noticing that the functions f and g in the previous theorem, as well as in the following results, are not necessarily continuous.

Theorem 2.2 has many consequences whose certain of them are recited in what follows.

Corollary 2.3 *Let E, F be two linear vector spaces and C_1, C_2 be two generalized cones of E and F, respectively. Let $f : C_1 \to [0,\infty)$, $g : C_2 \to [0,\infty)$ and $h : C_1 \times C_2 \to \mathbb{C}$ be three maps such that*

$$\forall (u,v) \in C_1 \times C_2, \quad \operatorname{Re}\bigl(h(u,v)\bigr) \leq f(u) + g(v). \tag{3}$$

Assume that f is homogeneous of degree $p > 1$, g is homogeneous of degree $p^* > 1$, with $1/p + 1/p^* = 1$, and h is a semi-inner product. Then the inequality

$$\bigl|h(u,v)\bigr| \leq \bigl(pf(u)\bigr)^{1/p} \bigl(p^* g(v)\bigr)^{1/p^*} \tag{4}$$

holds for all $(u,v) \in C_1 \times C_2$.

Proof Under our assumptions, we can apply the above theorem (with $r = 1$) for obtaining, from (3),

$$\mathrm{Re}\big(h(u,v)\big) \le \big(pf(u)\big)^{1/p}\big(p^*g(v)\big)^{1/p^*}$$

for all $(u,v) \in C_1 \times C_2$, with $p^* = p/(p-1)$. If in this inequality we replace u by $(h(v,u))^{1/2}u \in C_1$ and v by $(h(u,v))^{1/2}v \in C_2$ and we use the fact that f and g are homogeneous of degree p and p^*, respectively, h being a semi-inner product, we obtain after elementary manipulation

$$\big|h(u,v)\big|^2 \le \big|h(u,v)\big|\big(pf(u)\big)^{1/p}\big(p^*g(v)\big)^{1/p^*}.$$

We can assume that $h(u,v) \neq 0$, since for $h(u,v) = 0$ the inequality (4) is obviously satisfied. We then deduce the desired result and this completes the proof. □

Remark 2.2 In Corollary 2.3, if E is a locally convex space, we can take $F = E^*$ algebraic (or topological) dual of E and h the duality map between E and E^*. As an example explaining this situation, see Theorem 5.2 in Section 5.

Another corollary of Theorem 2.2 may be stated as well.

Corollary 2.4 *Let f, g, and h be as in Theorem* 2.2. *Then*

$$\sum_{i=1}^n h(u_i,v_i) \le \left(\frac{p}{r}\sum_{i=1}^n f(u_i)\right)^{r/p}\left(\frac{p}{p-r}\sum_{i=1}^n g(v_i)\right)^{(p-r)/p} \tag{5}$$

holds true for all $u_1, u_2, \dots, u_n \in C_1$ and $v_1, v_2, \dots, v_n \in C_2$.

Proof Condition (1) implies that

$$\tilde{h}(u,v) := \sum_{i=1}^n h(u_i,v_i) \le \sum_{i=1}^n f(u_i) + \sum_{i=1}^n g(v_i) := \tilde{f}(u) + \tilde{g}(v)$$

for all $u = (u_1, u_2, \dots, u_n) \in C_1^n$ and $v = (v_1, v_2, \dots, v_n) \in C_2^n$. It is easy to see that $\tilde{f} : C_1^n \to [0,\infty)$ is positively homogeneous of degree p and $\tilde{h} : C_1^n \times C_2^n \to \mathbb{R}$ is positively homogeneous, with respect to the first variable u, of degree r. Theorem 2.2 yields the desired inequality (5), and this completes the proof. □

We now state the following result.

Theorem 2.5 *Let E and F be as above, C_1 be a cone of E and C_2 be a nonempty subset of F. Let $f, g : C_1 \times C_2 \to [0,\infty)$ and $h : C_1 \times C_2 \to [0,\infty)$ be three maps such that*

$$\forall (u,v) \in C_1 \times C_2, \quad h(u,v) \le f(u,v) + g(u,v).$$

Assume that further f, g, and h are positively homogeneous, with respect to the first variable, of degrees p, q, and r, respectively, with $p < r < q$. Then the inequality

$$\left(\frac{1}{q-p}h(u,v)\right)^{q-p} \le \left(\frac{1}{q-r}f(u,v)\right)^{q-r}\left(\frac{1}{r-p}g(u,v)\right)^{r-p} \tag{6}$$

holds true for all $(u,v) \in C_1 \times C_2$.

Proof Analogously to the proof of Theorem 2.2, we show that

$$h(u,v) \le \inf_{t>0}\bigl(t^{p-r}f(u,v) + t^{q-r}g(u,v)\bigr).$$

The desired inequality (6) follows by application of Lemma 2.1 in a similar manner as previous. The details are simple and are omitted here. □

We end this section by stating the two following results, which extend Theorem 2.2 and Theorem 2.5, respectively.

Theorem 2.6 *Let C_1, C_2 be as in Theorem 2.2 and $(C,\prec)$ be an ordered cone of a certain linear space. Let $f: C_1 \to C$, $g: C_2 \to C$, and $h: C_1 \times C_2 \to C$ be such that*

$$\forall (u,v) \in C_1 \times C_2, \quad h(u,v) \prec f(u) + g(v). \tag{7}$$

Assume that f and h are as in Theorem 2.2. If $\Phi: C \to [0,\infty)$ is monotone sub-additive and homogeneous of degree $s > 0$, then the inequality

$$\Phi\bigl(h(u,v)\bigr) \le \left(\frac{p}{r}\Phi\bigl(f(u)\bigr)\right)^{r/p}\left(\frac{p}{p-r}\Phi\bigl(g(v)\bigr)\right)^{(p-r)/p} \tag{8}$$

holds true for all $(u,v) \in C_1 \times C_2$.

Proof With the fact that Φ is monotone and sub-additive, (7) implies that

$$\Phi\bigl(h(u,v)\bigr) \le \Phi\bigl(f(u)\bigr) + \Phi\bigl(g(v)\bigr),$$

with $\Phi \circ f: C_1 \to [0,\infty)$ homogeneous (with respect to the first variable) of degree ps and $\Phi \circ h: C_1 \times C_2 \to [0,\infty)$ homogeneous of degree rs, with $\min(ps,0) < rs < \max(ps,0)$ since $s > 0$. We can then use Theorem 2.2 and the desired inequality follows after a simple manipulation. □

The statement of Corollary 2.4 can be included in the situation of the previous theorem. We omit all details of this point, leaving them for the reader.

Theorem 2.7 *Let C_1, C_2, and C be as in Theorem 2.6. Let $f: C_1 \times C_2 \to C$, $g: C_1 \times C_2 \to C$, and $h: C_1 \times C_2 \to C$ be such that*

$$\forall (u,v) \in C_1 \times C_2, \quad h(u,v) \prec f(u,v) + g(u,v). \tag{9}$$

Assume that f and h are as in Theorem 2.5. If $\Phi : C \to [0,\infty)$ is monotone sub-additive and homogeneous of degree $s > 0$, then the inequality

$$\left(\frac{1}{q-p}\Phi\big(h(u,v)\big)\right)^{q-p} \le \left(\frac{1}{q-r}\Phi\big(f(u,v)\big)\right)^{q-r}\left(\frac{1}{r-p}\Phi\big(g(u,v)\big)\right)^{r-p} \tag{10}$$

holds true for all $(u,v) \in C_1 \times C_2$.

Proof It is similar to that of Theorem 2.6. We omit all details leaving them to the reader. □

For an application of the previous theorem, see Section 4 below.

3 Some examples

This section is devoted to the presentation of some examples illustrating the above theoretical results. We need more notations. In what follows, H denotes a complex Hilbert space with its inner product $\langle\cdot,\cdot\rangle$ and its associate norm $\|\cdot\|$. The notation $\mathcal{B}(H)$ refers to the algebra of linear bounded operators defined from H into itself. A self-adjoint operator $T \in \mathcal{B}(H)$ is positive (in short, $T \ge 0$) if $\langle Tu,u\rangle \ge 0$ for all $u \in H$. We denote by $\mathcal{B}^+(H)$ (resp. $\mathcal{B}^{+*}(H)$) the convex cone of all self-adjoint positive (resp. invertible) operators $T \in \mathcal{B}(H)$. As usual, for $T,S \in \mathcal{B}(H)$ we write $T \le S$ if and only if T, S are self-adjoint and $S - T \in \mathcal{B}^+(H)$. The space $\mathcal{B}(H)$ is endowed with the classical operator norm, namely

$$\|T\| = \sup_{\|u\|=1} \|Tu\|.$$

It is well known that if T is positive then

$$\|T\| = \sup_{\|u\|=1} \langle Tu,u\rangle.$$

A norm $|||\cdot|||$ on $\mathcal{B}(H)$ is said to be unitarily invariant if it satisfies the invariance property $|||UTV||| = |||T|||$ for all $T \in \mathcal{B}(H)$ and for all unitary operators U and V.

Now we are in a position to state the following list of examples.

Example 3.1 Let $T,S \in \mathcal{B}(H)$. Then we have

$$0 \le \|Tu - Sv\|^2 = \big\langle (T^*T)u,u\big\rangle - 2\,\mathrm{Re}\langle Tu,Sv\rangle + \big\langle (S^*S)v,v\big\rangle,$$

which, by Corollary 2.3 with $p = p^* = 2$, yields

$$\big|\langle Tu,Sv\rangle\big|^2 \le \big\langle |T|^2u,u\big\rangle\big\langle |S|^2v,v\big\rangle,$$

where as usual $|T| = (T^*T)^{1/2}$. If $S = T^*$ then

$$\big|\big\langle T^2u,v\big\rangle\big|^2 \le \big\langle |T|^2u,u\big\rangle\big\langle \big|T^*\big|^2v,v\big\rangle,$$

see [1]. If T is positive (self-adjoint) then

$$\big|\langle Tu,v\rangle\big|^2 \le \langle Tu,u\rangle\langle Tv,v\rangle,$$

which is a well-known extension of the Cauchy-Schwarz inequality.

Example 3.2 Let $T, S, X \in \mathcal{B}(H)$ be three operators. Then we have

$$M := \begin{pmatrix} T & X^* \\ X & S \end{pmatrix} \text{ is positive in } \mathcal{B}(H \oplus H)$$
$$\Leftrightarrow \quad \forall u, v \in H, \quad \left|\langle Xu, v\rangle\right|^2 \leq \langle Tu, u\rangle\langle Sv, v\rangle.$$

See [2], p.284, for a direct method. Here, we simply proceed as follows. By definition, M is positive in $\mathcal{B}(H \oplus H)$ if and only if

$$\langle Tu, u\rangle + 2\operatorname{Re}\langle Xu, v\rangle + \langle Sv, v\rangle \geq 0$$

for all $u, v \in H$. According to Corollary 2.3, with Remark 2.1, we immediately deduce the desired aim.

Now, let us observe the two following examples illustrating, particularly, the situation of Corollary 2.4.

Example 3.3 Let a, b be complex numbers and $p, p^* > 1$ with $1/p + 1/p^* = 1$. The inequality

$$|a||b| \leq \frac{1}{p}|a|^p + \frac{1}{p^*}|b|^{p^*}$$

is known as the Young inequality. We are in the situation of Corollary 2.4 with $r = 1$. We then immediately deduce the following Hölder inequality (in $\mathbb{C}^n$):

$$\sum_{i=1}^n |a_i||b_i| \leq \left(\sum_{i=1}^n |a_i|^p\right)^{1/p}\left(\sum_{i=1}^n |b_i|^{p^*}\right)^{1/p^*},$$

valid for all complex numbers a_i and b_i, $1 \leq i \leq n$.

Example 3.4 Let Ω be a nonempty open subset of $\mathbb{R}^n$ and $f, g : \Omega \to \mathbb{C}$. The Young inequality asserts that

$$\forall s \in \Omega, \quad \left|f(s)\right|\left|g(s)\right| \leq \frac{1}{p}\left|f(s)\right|^p + \frac{1}{p^*}\left|g(s)\right|^{p^*}.$$

The map Φ defined by $\Phi(\psi) = \int_\Omega \psi(s)\,ds$, for ψ Lebesgue-integrable on Ω, is linear and monotone. It follows that if $f \in L^p(\Omega)$ and $g \in L^{p^*}(\Omega)$ then we have

$$\int_\Omega \left|(fg)(s)\right|ds \leq \int_\Omega \left|f(s)\right|^p ds + \int_\Omega \left|g(s)\right|^{p^*} ds.$$

By Theorem 2.6, we then deduce the Hölder inequality in integration:

$$\int_\Omega \left|(fg)(s)\right|ds \leq \left(\int_\Omega \left|f(s)\right|^p ds\right)^{1/p}\left(\int_\Omega \left|g(s)\right|^{p^*} ds\right)^{1/p^*}.$$

See also Example 5.1 (Section 5 below) for another point of view for proving this inequality. We leave to the reader the routine task of obtaining the Hölder inequality in l_p, the space of p-convergent series.

Example 3.5 For all (Hermitian) positive definite matrices A and B and every $p \in (1,\infty)$, with $p^* = p/(p-1)$, we have [3]

$$\operatorname{tr}(AB) \le \frac{1}{p} trA^p + \frac{1}{p^*} \operatorname{tr} B^{p^*}.$$

According to Corollary 2.4, we deduce that for all $A_i, B_i, i = 1,2,\dots,m$ (Hermitian) positive definite matrices, we have

$$\operatorname{tr}\sum_{i=1}^m A_iB_i \le \left(\sum_{i=1}^m \operatorname{tr} A_i^p\right)^{1/p}\left(\sum_{i=1}^m \operatorname{tr} B_i^{p^*}\right)^{1/p^*}. \tag{11}$$

See [4], Theorem 4.1, pp.3-4, for a direct (but long) proof of (11) by using the spectral mapping theorem and some existing lemmas.

Example 3.6 Let $|||\cdot|||$ be a unitarily invariant norm. The inequality [5–7]

$$|||TXS||| \le \frac{1}{p}\left|\left|\left|T^pX\right|\right|\right| + \frac{1}{p^*}\left|\left|\left|XS^{p^*}\right|\right|\right| \tag{12}$$

holds for all $T, S \in \mathcal{B}^+(H)$, $X \in \mathcal{B}(H)$, with $1/p + 1/p^* = 1$. By Corollary 2.3, (12) is equivalent to

$$|||TXS||| \le \left|\left|\left|T^pX\right|\right|\right|^{1/p}\left|\left|\left|XS^{p^*}\right|\right|\right|^{1/p^*},$$

which is stronger than (12). According to Corollary 2.4, (12) implies that

$$\sum_{i=1}^n |||T_iXS_i||| \le \left(\sum_{i=1}^n \left|\left|\left|T_i^pX\right|\right|\right|\right)^{1/p}\left(\sum_{i=1}^n \left|\left|\left|XS_i^{p^*}\right|\right|\right|\right)^{1/p^*}$$

for all $T_i, S_i \in \mathcal{B}^+(H)$, $i = 1,2,\dots,n$, and $X \in \mathcal{B}(H)$.

4 Application to operator inequalities

We preserve the same notation as in the previous section. The following result, which is an operator version of Theorem 2.5, may be stated.

Theorem 4.1 *Let C be a cone of $\mathcal{B}^+(H)$ and $f, g, h : C \times C \to C$ be three operator maps such that*

$$\forall T, S \in C, \quad h(T,S) \le f(T,S) + g(T,S). \tag{13}$$

Assume that further f, g, and h are positively homogeneous, with respect to the first variable, of degrees p, q, and r, respectively, with $p < r < q$. Then the inequality

$$\left(\frac{1}{q-p}\langle h(T,S)u,u\rangle\right)^{q-p} \le \left(\frac{1}{q-r}\langle f(T,S)u,u\rangle\right)^{q-r}\left(\frac{1}{r-p}\langle g(T,S)u,u\rangle\right)^{r-p} \tag{14}$$

holds true for all $T, S \in C$.

Proof By definition, the operator inequality (13) is equivalent to

$$\forall u \in H, \quad \langle h(T,S)u,u\rangle \le \langle f(T,S)u,u\rangle + \langle g(T,S)u,u\rangle. \tag{15}$$

The inequality (14) is true for $u = 0$. Now, fixing $0 \neq u \in E$, this inequality can be written as

$$h_u(T,S) \le f_u(T,S) + g_u(T,S),$$

where $h_u, f_u, g_u : \mathcal{B}^+(H) \times \mathcal{B}^+(H) \to (0,\infty)$ are the three quadratic forms of (15), respectively. Obviously, we can then apply Theorem 2.5 here for obtaining the desired result after a simple manipulation, and this completes the proof. □

From the above theorem, we immediately deduce the following corollary.

Corollary 4.2 *With the same hypotheses as in Theorem* 4.1 *we have*

$$\left(\frac{1}{q-p}\|h(T,S)\|\right)^{q-p} \le \left(\frac{1}{q-r}\|f(T,S)\|\right)^{q-r}\left(\frac{1}{r-p}\|g(T,S)\|\right)^{r-p}. \tag{16}$$

Now, we will illustrate the above theorem with some applications.

• Let λ be a real number such that $0 \le \lambda \le 1$ and $T,S \in \mathcal{B}^{+*}(H)$. The power geometric mean $T \sharp_\lambda S$ of T and S is defined by

$$T \sharp_\lambda S = S^{1/2}\left(S^{-1/2}TS^{-1/2}\right)^{1-\lambda}S^{1/2} = T^{1/2}\left(T^{-1/2}ST^{-1/2}\right)^{\lambda}T^{1/2} = S \sharp_{1-\lambda} T,$$

while their weighted arithmetic mean is

$$T \oplus_\lambda S = (1-\lambda)T + \lambda S = S \oplus_{1-\lambda} T.$$

For $\lambda = 1/2$, we simply write $T \sharp S$ and $T \oplus S$, respectively.

The Heinz operator mean of T and S is defined by

$$H_\lambda(T,S) = \frac{T \sharp_\lambda S + T \sharp_{1-\lambda} S}{2}.$$

This operator mean interpolates $T \sharp S$ and $T \oplus S$ [8], in the sense that

$$T \sharp S \le H_\lambda(T,S) \le T \oplus S \tag{17}$$

holds true for all $\lambda \in [0,1]$ and $T,S \in \mathcal{B}^{+*}(H)$.

The first result of application here may be stated as well.

Theorem 4.3 *With the above, the inequalities*

$$\left(\langle (T \sharp S)u,u\rangle\right)^2 \le \langle (T \sharp_\lambda S)u,u\rangle\langle (T \sharp_{1-\lambda} S)u,u\rangle \le \left(\langle (T \oplus S)u,u\rangle\right)^2 \tag{18}$$

hold true for all $u \in H$.

Proof Let us set

$$h(T,S) = 2T \sharp S, \qquad f(T,S) = T \sharp_\lambda S, \qquad g(T,S) = T \sharp_{1-\lambda} S.$$

It is easy to see that h, f, and g are positively homogeneous, with respect to the first variable T, of degrees $r = 1/2$, $p = 1-\lambda$, and $q = \lambda$, respectively. If $\lambda = 1/2$, the left side of (18) is an equality. Now, considering the two cases $0 \le \lambda < 1/2$ and $1/2 < \lambda \le 1$, Theorem 4.1 yields

$$\begin{aligned}&\left(\frac{1}{|2\lambda-1|}\langle (T\sharp S)u,u\rangle\right)^{|2\lambda-1|}\\&\quad\le \left(\frac{1}{|2\lambda-1|}\langle (T\sharp_\lambda S)u,u\rangle\right)^{|\lambda-1/2|}\left(\frac{1}{|2\lambda-1|}\langle (T\sharp_{1-\lambda} S)u,u\rangle\right)^{|\lambda-1/2|},\end{aligned}$$

which after simple reduction yields the left side of (5). The right side of (18) follows by a simple application of the arithmetic-geometric mean inequality with (17), and this completes the proof. □

From the above theorem, we immediately deduce the following inequality:

$$\|T\sharp S\|^2 \le \|T\sharp_\lambda S\|\,\|T\sharp_{1-\lambda} S\| \le \|T\oplus S\|^2.$$

Remark 4.1 The above inequalities can be written, respectively, in the following forms:

$$\begin{aligned}&\langle (T\sharp S)u,u\rangle \le \langle (T\sharp_\lambda S)u,u\rangle \sharp \langle (T\sharp_{1-\lambda} S)u,u\rangle \le \langle (T\oplus S)u,u\rangle,\\&\|T\sharp S\| \le \|T\sharp_\lambda S\| \sharp \|T\sharp_{1-\lambda} S\| \le \|T\oplus S\|,\end{aligned}$$

where for two real numbers $a, b > 0$, $a\sharp b = \sqrt{ab}$ is the geometric mean of a and b.

• A second application here is stated as follows. Let T, S, and $T\sharp_\lambda S$ be as above, $0 \le \lambda \le 1$. The inequality

$$(1+\lambda)T \le \lambda T S^{-1} T + T\sharp_\lambda S \tag{19}$$

is known as the operator entropy inequality; see [6] for instance. The following result may be stated.

Theorem 4.4 *With the above, for all $u \in H$ we have*

$$\begin{aligned}&\big(\langle Tu,u\rangle\big)^{1+\lambda} \le \big(\langle TS^{-1}Tu,u\rangle\big)^{\lambda}\langle (T\sharp_\lambda S)u,u\rangle,\\&\|T\|^{1+\lambda} \le \big\|TS^{-1}T\big\|^{\lambda}\|T\sharp_\lambda S\|.\end{aligned}$$

Proof Setting $h(T,S) = (1+\lambda)T$, $f(T,S) = \lambda TS^{-1}T$, and $g(T,S) = T\sharp_\lambda S$, it is easy to see that h, f, and g are homogeneous, with respect to the first variable T, of degrees 1, 2, and $1-\lambda$. The remainder of the proof is similar to that of Theorem 4.3. The details are simple and are omitted here. □

5 An application in convex analysis

We need here more notions. Let E be a locally convex space and E^* its topological dual with the bracket duality $\langle\cdot,\cdot\rangle$. Let $f : E \to \widetilde{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$ be a functional not identically equal to ∞. The effective domain of f is $\operatorname{dom} f = \{u \in E, f(u) < \infty\}$, and the conjugate of f is the functional $f^* : E^* \to \widetilde{\mathbb{R}}$ defined through [9]

$$\forall u^* \in E, \quad f^*\bigl(u^*\bigr) = \sup_{u\in E}\bigl(\operatorname{Re}\bigl\langle u, u^*\bigr\rangle - f(u)\bigr). \tag{20}$$

Later, we shall need the following lemma.

Lemma 5.1 *Let $f : E \to \widetilde{\mathbb{R}}$ be a functional not identically equal to ∞.*

(i) *If $f(0) \leq 0$ then f^* is a positive functional.*

(ii) *If f is homogeneous of degree $p > 1$ then f^* is homogeneous of degree $p^* > 1$, with $p^* = p/(p-1)$.*

Proof (i) From (20) we immediately deduce that $f^*(u^*) \geq \operatorname{Re}\langle u^*, u\rangle - f(u)$ for all $u \in E$ and $u^* \in E^*$. Taking $u = 0$ in this inequality we obtain $f^*(u^*) \geq 0$ for all $u^* \in E^*$.

(ii) Let $0 \neq \lambda \in \mathbb{R}, \mathbb{C}$ be fixed. By definition we have

$$\begin{aligned} f^*\bigl(\lambda u^*\bigr) &= \sup_{u\in E}\bigl(\operatorname{Re}\bigl\langle u, \lambda u^*\bigr\rangle - f(u)\bigr) \\ &= \sup_{u\in E}\bigl(\operatorname{Re}\bigl\langle \lambda|\lambda|^{p^*-2}u, \lambda u^*\bigr\rangle - f\bigl(\lambda|\lambda|^{p^*-2}u\bigr)\bigr) \\ &= \sup_{u\in E}\bigl(|\lambda|^{p^*}\operatorname{Re}\bigl\langle u, u^*\bigr\rangle - |\lambda|^{(p^*-1)p}f(u)\bigr). \end{aligned}$$

It is easy to verify that $(p^*-1)p = p^*$ and so

$$f^*\bigl(\lambda u^*\bigr) = |\lambda|^{p^*}\sup_{u\in E}\bigl(\operatorname{Re}\bigl\langle u, u^*\bigr\rangle - f(u)\bigr) = |\lambda|^{p^*}f^*\bigl(u^*\bigr).$$

Summarizing, we have shown that

$$\forall u^* \in \operatorname{dom} f^*, \forall \lambda \neq 0, \quad f^*\bigl(\lambda u^*\bigr) = |\lambda|^{p^*}f^*\bigl(u^*\bigr). \tag{21}$$

From this equality, with the fact that f^* is always lower semi-continuous, we deduce

$$\forall u^* \in \operatorname{dom} f^*, \quad f^*(0) \leq \liminf_{\lambda\to 0} f^*\bigl(\lambda u^*\bigr) = \liminf_{\lambda\to 0}\bigl(|\lambda|^{p^*}f^*\bigl(u^*\bigr)\bigr) = 0. \tag{22}$$

We then have $f^*(0) \leq 0$. Since f is homogeneous, $f(0) = 0$ and, by (i) we have $f^*(0) \geq 0$. This, with (22), implies that $f^*(0) = 0$ and (21) is also satisfied for $\lambda = 0$, and this completes the proof. □

Our main result of application in this section is the following.

Theorem 5.2 *Let $f : E \to \widetilde{\mathbb{R}}$ be a positive functional such that f^* is positive, too. Assume that f is homogeneous of degree $p > 1$. Then, for all $u \in E$ and $u^* \in E^*$, we have*

$$\bigl|\bigl\langle u, u^*\bigr\rangle\bigr| \leq \bigl(pf(u)\bigr)^{1/p}\bigl(p^*f^*\bigl(u^*\bigr)\bigr)^{1/p^*}. \tag{23}$$

Proof From (20) we immediately deduce that

$$\operatorname{Re}\langle u,u^*\rangle - f(u) \le f^*(u^*) \tag{24}$$

for all $u \in E$ and $u^* \in E^*$. If $f(u) < \infty$ then (24) is equivalent to

$$\operatorname{Re}\langle u,u^*\rangle \le f(u) + f^*(u^*). \tag{25}$$

If $f(u) = \infty$ then $f(u) + f^*(u^*) = \infty$ and so (25) is also satisfied. In all cases we have

$$\forall u \in E, \forall u^* \in E^*, \quad \operatorname{Re}\langle u,u^*\rangle \le f(u) + f^*(u^*).$$

We can apply Corollary 2.3 with $h(u,u^*) = \langle u,u^*\rangle$ and $g = f^*$. The desired result follows. □

Now, we will illustrate the above result with the following example.

Example 5.1 Let $p > 1$ and $E = L^p(\Omega)$ be equipped with the classical norm

$$\forall u \in L^p(\Omega), \quad \|u\|_p = \left(\int_\Omega |u(t)|^p\,dt\right)^{1/p}.$$

The topological dual of E is $E^* = L^{p^*}(\Omega)$, with $1/p + 1/p^* = 1$. Take $f(u) = \frac{1}{p}\|u\|_p^p$ for which we have $f^*(u^*) = \frac{1}{p^*}\|u^*\|_{p^*}^{p^*}$ [10]. According to (23), we immediately obtain the classical Hölder inequality in $L^p(\Omega)$, namely: $|\langle u,u^*\rangle| \le \|u\|_p\|u^*\|_{p^*}$ for all $u \in L^p(\Omega)$ and $u^* \in L^{p^*}(\Omega)$. Similarly we can obtain the Hölder inequality in $\mathbb{C}^n$ and in l_p, the space of p-convergent series. For $p = 2$, the above is reduced to the Cauchy-Schwarz inequality.

The following example is also of interest.

Example 5.2 Let E be a Hilbert space and T be a (self-adjoint) positive operator from E into itself. Take $f = f_T$ defined by

$$\forall u \in E, \quad f(u) = f_T(u) := (1/2)\langle Tu,u\rangle = (1/2)\|T^{1/2}u\|^2.$$

We know that [10]

$$(f_T)^*(u^*) = (1/2)\left\|\left(T^{1/2}\right)^+u^*\right\|^2 \quad \text{if } u^* \in \operatorname{ran} T^{1/2}, \qquad (f_T)^*(u^*) = \infty \quad \text{otherwise}, \tag{26}$$

where T^+ denotes the pseudo-inverse of T. This, with (23), implies that

$$\left|\langle u,u^*\rangle\right| \le \|T^{1/2}u\|\left\|\left(T^{1/2}\right)^+u^*\right\|$$

holds for all $u \in E$ and $u^* \in \operatorname{ran} T^{1/2}$. In particular, if, moreover, T is invertible then

$$\left|\langle u,u^*\rangle\right|^2 \le \langle Tu,u\rangle\langle T^{-1}u^*,u^*\rangle$$

holds for all $u, u^* \in E$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Both authors worked in coordination. Both authors carried out the proof, read, and approved the final version of the manuscript.

Author details
[1]Department of Mathematics, Science Faculty, Taibah University, P.O. Box 30097, Al Madinah Al Munawwarah, 41477, Kingdom of Saudi Arabia. [2]Department of Mathematics, Science Faculty, Moulay Ismail University, Meknes, Morocco. [3]Department of Mathematics, Faculty of Sciences and Humanities, Salman Bin Abdulaziz University, Aflaj, Kingdom of Saudi Arabia.

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