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On generalized double statistical convergence in a random 2-normed space

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Abstract
Recently, the concept of statistical convergence has been studied in 2-normed and random 2-normed spaces by various authors. In this paper, we shall introduce the concept of λ-double statistical convergence and λ-double statistical Cauchy in a random 2-normed space. We also shall prove some new results.
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1 Introduction

The probabilistic metric space was introduced by Menger [1] which is an interesting and an important generalization of the notion of a metric space. The theory of probabilistic normed (or metric) space was initiated and developed in [2–6]; further it was extended to random/probabilistic 2-normed spaces by Goleț [7] using the concept of 2-norm which is defined by Gähler (see [8, 9]); and Gürdal and Pehlivan [10] studied statistical convergence in 2-normed spaces. Also statistical convergence in 2-Banach spaces was studied by Gürdal and Pehlivan in [11]. Moreover, recently some new sequence spaces have been studied by Savas [12–14] by using 2-normed spaces.

In order to extend the notion of convergence of sequences, statistical convergence of sequences was introduced by Fast [15] and Schoenberg [16] independently. A lot of developments have been made in this areas after the works of Šalát [17] and Fridy [18]. Over the years and under different names, statistical convergence has been discussed in the theory of Fourier analysis, ergodic theory and number theory. Recently, Mursaleen [19] studied λ-statistical convergence as a generalization of the statistical convergence, and in [20] he considered the concept of statistical convergence of sequences in random 2-normed spaces. Quite recently, Bipan and Savas [21] defined lacunary statistical convergence in a random 2-normed space, and also Savas [22] studied λ-statistical convergence in a random 2-normed space.

The notion of statistical convergence depends on the density of subsets of $\mathbf{N}$, the set of natural numbers. Let K be a subset of $\mathbf{N}$. Then the asymptotic density of K denoted by $\delta(K)$ is defined as

$$\delta(K) = \lim_{n\to\infty} \frac{1}{n}\bigl|\{k \le n : k \in K\}\bigr|,$$

where the vertical bars denote the cardinality of the enclosed set.

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A single sequence $x = (x_k)$ is said to be *statistically convergent* to ℓ if for every $\varepsilon > 0$, the set $K(\varepsilon) = \{k \leq n : |x_k - \ell| \geq \varepsilon\}$ has asymptotic density zero, *i.e.*,

$$\lim_{n\to\infty} \frac{1}{n} \left| \left\{ k \leq n : |x_k - \ell| \geq \varepsilon \right\} \right| = 0.$$

In this case we write $S - \lim x = \ell$ or $x_k \to \ell(S)$ (see [15, 18]).

2 Definitions and preliminaries

We begin by recalling some notations and definitions which will be used in this paper.

Definition 1 A function $f : \mathbf{R} \to \mathbf{R}_0^+$ is called a *distribution function* if it is a non-decreasing and left continuous with $\inf_{t\in\mathbf{R}} f(t) = 0$ and $\sup_{t\in\mathbf{R}} f(t) = 1$. By D^+, we denote the set of all distribution functions such that $f(0) = 0$. If $a \in \mathbf{R}_0^+$, then $H_a \in D^+$, where

$$H_a(t) = \begin{cases} 1, & \text{if } t > a; \\ 0, & \text{if } t \leq a. \end{cases}$$

It is obvious that $H_0 \geq f$ for all $f \in D^+$.

A *t-norm* is a continuous mapping $* : [0,1] \times [0,1] \to [0,1]$ such that $([0,1], *)$ is an Abelian monoid with unit one and $c * d \geq a * b$ if $c \geq a$ and $d \geq b$ for all $a, b, c, d \in [0,1]$. A triangle function τ is a binary operation on D^+, which is commutative, associative and $\tau(f, H_0) = f$ for every $f \in D^+$.

In [8], Gähler introduced the following concept of a 2-normed space.

Definition 2 Let X be a real vector space of dimension $d > 1$ (d may be infinite). A real-valued function $\|\cdot,\cdot\|$ from X^2 into $\mathbf{R}$ satisfying the following conditions:

(1) $\|x_1, x_2\| = 0$ if and only if x_1, x_2 are linearly dependent,

(2) $\|x_1, x_2\|$ is invariant under permutation,

(3) $\|\alpha x_1, x_2\| = |\alpha| \|x_1, x_2\|$, for any $\alpha \in \mathbf{R}$,

(4) $\|x + \overline{x}, x_2\| \leq \|x, x_2\| + \|\overline{x}, x_2\|$

is called a 2-norm on X and the pair $(X, \|\cdot,\cdot\|)$ is called a 2-normed space.

A trivial example of a 2-normed space is $X = \mathbf{R}^2$, equipped with the Euclidean 2-norm $\|x_1, x_2\|_E$ = the area of the parallelogram spanned by the vectors x_1, x_2 which may be given explicitly by the formula

$$\|x_1, x_2\|_E = \left|\det(x_{ij})\right| = \operatorname{abs}\left(\det\left(\langle x_i, x_j\rangle\right)\right),$$

where $x_i = (x_{i1}, x_{i2}) \in \mathbf{R}^2$ for each $i = 1, 2$.

Recently, Goleț [7] used the idea of a 2-normed space to define a random 2-normed space.

Definition 3 Let X be a linear space of dimension $d > 1$ (d may be infinite), τ a triangle, and $\mathcal{F} : X \times X \to D^+$. Then $\mathcal{F}$ is called a *probabilistic 2-norm* and $(X, \mathcal{F}, \tau)$ a *probabilistic 2-normed* space if the following conditions are satisfied:

($P2N_1$) $\mathcal{F}(x,y;t) = H_0(t)$ if x and y are linearly dependent, where $\mathcal{F}(x,y;t)$ denotes the value of $\mathcal{F}(x,y)$ at $t \in \mathbf{R}$,
($P2N_2$) $\mathcal{F}(x,y;t) \neq H_0(t)$ if x and y are linearly independent,
($P2N_3$) $\mathcal{F}(x,y;t) = \mathcal{F}(y,x;t)$, for all $x,y \in X$,
($P2N_4$) $\mathcal{F}(\alpha x,y;t) = \mathcal{F}(x,y;\frac{t}{|\alpha|})$, for every $t > 0$, $\alpha \neq 0$ and $x,y \in X$,
($P2N_5$) $\mathcal{F}(x+y,z;t) \geq \tau(\mathcal{F}(x,z;t),\mathcal{F}(y,z;t))$, whenever $x,y,z \in X$.

If ($P2N_5$) is replaced by

($P2N_6$) $\mathcal{F}(x+y,z;t_1+t_2) \geq \mathcal{F}(x,z;t_1) * \mathcal{F}(y,z;t_2)$, for all $x,y,z \in X$ and $t_1,t_2 \in \mathbf{R}_0^+$;

then $(X,\mathcal{F},*)$ is called a *random 2-normed* space (for short, R2NS).

Remark 1 Every 2-normed space $(X,\|\cdot,\cdot\|)$ can be made a random 2-normed space in a natural way by setting $\mathcal{F}(x,y;t) = H_0(t - \|x,y\|)$ for every $x,y \in X$, $t > 0$ and $a*b = \min\{a,b\}$, $a,b \in [0,1]$.

Example 1 Let $(X,\|\cdot,\cdot\|)$ be a 2-normed space with $\|x,z\| = \|x_1z_2 - x_2z_1\|$, $x = (x_1,x_2)$, $z = (z_1,z_2)$ and $a*b = ab$, $a,b \in [0,1]$. For all $x \in X$, $t > 0$ and nonzero $z \in X$, consider

$$\mathcal{F}(x,z;t) = \begin{cases} \frac{t}{t+\|x,z\|}, & \text{if } t > 0; \\ 0, & \text{if } t \leq 0. \end{cases}$$

Then $(X,F,*)$ is a random 2-normed space.

Definition 4 A sequence $x = (x_{k,l})$ in a random 2-normed space $(X,\mathcal{F},*)$ is said to be *double convergent* (or $\mathcal{F}$-convergent) to $\ell \in X$ with respect to $\mathcal{F}$ if for each $\varepsilon > 0$, $\eta \in (0,1)$, there exists a positive integer n_0 such that $\mathcal{F}(x_{k,l} - \ell,z;\varepsilon) > 1 - \eta$, whenever $k,l \geq n_0$ and for nonzero $z \in X$. In this case we write $\mathcal{F} - \lim_{k,l} x_{k,l} = \ell$, and ℓ is called the $\mathcal{F}$-limit of $x = (x_{k,l})$.

Definition 5 A sequence $x = (x_{k,l})$ in a random 2-normed space $(X,\mathcal{F},*)$ is said to be *double Cauchy* with respect to $\mathcal{F}$ if for each $\varepsilon > 0$, $\eta \in (0,1)$ there exist $N = N(\varepsilon)$ and $M = M(\varepsilon)$ such that $\mathcal{F}(x_{k,l} - x_{p,q},z;\varepsilon) > 1 - \eta$, whenever $k,p \geq N$ and $l,q \geq M$ and for nonzero $z \in X$.

Definition 6 A sequence $x = (x_{k,l})$ in a random 2-normed space $(X,\mathcal{F},*)$ is said to be *double statistically convergent* or S^{2R2N}*-convergent* to some $\ell \in X$ with respect to $\mathcal{F}$ if for each $\varepsilon > 0$, $\eta \in (0,1)$ and for nonzero $z \in X$ such that

$$\delta\left(\left\{(k,l) \in \mathbf{N} \times \mathbf{N} : \mathcal{F}(x_{k,l} - \ell,z;\varepsilon) \leq 1 - \eta\right\}\right) = 0.$$

In other words, we can write the sequence $(x_{k,l})$ *double statistically converges* to ℓ in random 2-normed space $(X,\mathcal{F},*)$ if

$$\lim_{m,n\to\infty} \frac{1}{mn} \left|\left\{k \leq m, l \leq n : \mathcal{F}(x_{k,l} - \ell,z;\varepsilon) \leq 1 - \eta\right\}\right| = 0$$

or equivalently,

$$\delta\big(\big\{k,l\in\mathbf{N}:\mathcal{F}(x_{k,l}-\ell,z;\varepsilon)>1-\eta\big\}\big)=1,$$

i.e.,

$$S^2-\lim_{k,l\to\infty}\mathcal{F}(x_{k,l}-\ell,z;\varepsilon)=1.$$

In this case we write $S^{2R2N}-\lim x=\ell$, and ℓ is called the S^{2R2N}-limit of x. Let $S^{2R2N}(X)$ denote the set of all double statistically convergent sequences in a random 2-normed space $(X,\mathcal{F},*)$.

In this article, we study λ-double statistical convergence in a random 2-normed space which is a new and interesting idea. We show that some properties of λ-double statistical convergence of real numbers also hold for sequences in random 2-normed spaces. We establish some relations related to double statistically convergent and λ-double statistically convergent sequences in random 2-normed spaces.

3 λ-double statistical convergence in a random 2-normed space

Recently, the concept of λ-double statistical convergence has been introduced and studied in [23] and [24]. In this section, we define λ-double statistically convergent sequence in a random 2-normed space $(X,\mathcal{F},*)$. Also we get some basic properties of this notion in a random 2-normed space.

Definition 7 Let $\lambda=(\lambda_n)$ and $\mu=(\mu_n)$ be two non-decreasing sequences of positive real numbers such that each is tending to ∞ and

$$\lambda_{n+1}\le\lambda_n+1,\qquad\lambda_1=1$$

and

$$\mu_{n+1}\le\mu_n+1,\qquad\mu_1=1.$$

Let $K\subseteq\mathbb{N}\times\mathbb{N}$. The number

$$\delta_{\bar\lambda}(K)=\lim_{mn}\frac{1}{\bar\lambda_{mn}}\big|\big\{k\in I_n,l\in J_m:(k,l)\in K\big\}\big|,$$

where $I_n=[n-\lambda_n+1,n]$, $J_m=[m-\mu_m+1,m]$ and $\bar\lambda_{nm}=\lambda_n\mu_m$, is said to be the λ-*double density* of K, provided the limit exists.

Definition 8 A sequence $x=(x_{k,l})$ is said to be λ-double statistically convergent or $S^2_{\bar\lambda}$-*convergent* to the number ℓ if for every $\varepsilon>0$, the set $N(\varepsilon)$ has λ-double density zero, where

$$N(\varepsilon)=\big\{k\in I_n,l\in J_m:|x_{k,l}-\ell|\ge\varepsilon\big\}.$$

In this case, we write $S^2_{\bar\lambda}-\lim x=L$.

Now we define λ-double statistical convergence in a random 2-normed space (see [25]).

Definition 9 A sequence $x = (x_{k,l})$ in a random 2-normed space $(X, \mathcal{F}, *)$ is said to be *λ-double statistically convergent* or $S^2_{\bar{\lambda}}$*-convergent* to $\ell \in X$ with respect to $\mathcal{F}$ if for every $\varepsilon > 0$, $\eta \in (0,1)$ and for nonzero $z \in X$ such that

$$\delta_{\bar{\lambda}}\left(\left\{k \in I_n, l \in J_m : \mathcal{F}(x_{k,l} - \ell, z; \varepsilon) \leq 1 - \eta\right\}\right) = 0$$

or equivalently,

$$\delta_{\bar{\lambda}}\left(\left\{k \in I_n, l \in J_m : \mathcal{F}(x_{k,l} - \ell, z; \varepsilon) > 1 - \eta\right\}\right) = 1,$$

i.e.,

$$S^2_{\bar{\lambda}} - \lim_{k,l\to\infty} \mathcal{F}(x_{k,l} - \ell, z; \varepsilon) = 1.$$

In this case we write $S^{2R2N}_{\bar{\lambda}} - \lim x = \ell$ or $x_{k,l} \to \ell(S^{2R2N}_{\bar{\lambda}})$ and

$$S^{2R2N}_{\bar{\lambda}}(X) = \left\{x = (x_{k,l}) : \exists \ell \in \mathbf{R}, S^{2R2N}_{\bar{\lambda}} - \lim x = \ell\right\}.$$

Let $S^{2R2N}_{\bar{\lambda}}(X)$ denote the set of all λ-double statistically convergent sequences in a random 2-normed space $(X, \mathcal{F}, *)$.

If $\bar{\lambda}_{mn} = mn$ for every n, m then λ-double statistically convergent sequences in a random 2-normed space $(X, \mathcal{F}, *)$ reduce to double statistically convergent sequences in a random 2-normed space $(X, \mathcal{F}, *)$.

Definition 9 immediately implies the following lemma.

Lemma 1 *Let* $(X, \mathcal{F}, *)$ *be a random* 2*-normed space. If* $x = (x_{k,l})$ *is a sequence in X, then for every* $\varepsilon > 0$, $\eta \in (0,1)$ *and for nonzero* $z \in X$*, the following statements are equivalent*:

(i) $S^{R2N}_{\bar{\lambda}} - \lim_{k,l\to\infty} x_{k,l} = \ell$;
(ii) $\delta_{\bar{\lambda}}(\{k \in I_n, l \in J_m : \mathcal{F}(x_{k,l} - \ell, z; \varepsilon) \leq 1 - \eta\}) = 0$;
(iii) $\delta_{\bar{\lambda}}(\{k \in I_n, l \in J_m : \mathcal{F}(x_{k,l} - \ell, z; \varepsilon) > 1 - \eta\}) = 1$;
(iv) $S_{\bar{\lambda}} - \lim_{k,l\to\infty} \mathcal{F}(x_{k,l} - \ell, z; \varepsilon) = 1$.

Theorem 1 *Let* $(X, \mathcal{F}, *)$ *be a random* 2*-normed space. If* $x = (x_{k,l})$ *is a sequence in X such that* $S^{2R2N}_{\bar{\lambda}} - \lim x_{k,l} = \ell$ *exists, then it is unique.*

Proof Suppose that $S^{2R2N}_{\bar{\lambda}} - \lim_{k,l\to\infty} x_{k,l} = \ell_1; S^{2R2N}_{\bar{\lambda}} - \lim_{k,l\to\infty} x_{k,l} = \ell_2$, where $(\ell_1 \neq \ell_2)$.

Let $\varepsilon > 0$ be given. Choose $a > 0$ such that $(1 - a) * (1 - a) > 1 - \varepsilon$.

Then, for any $t > 0$ and for nonzero $z \in X$, we define

$$K_1(a,t) = \left\{k \in I_n, l \in J_m : \mathcal{F}\left(x_{k,l} - \ell_1, z; \frac{t}{2}\right) \leq 1 - a\right\};$$

$$K_2(a,t) = \left\{k \in I_n, l \in J_m : \mathcal{F}\left(x_{k,l} - \ell_2, z; \frac{t}{2}\right) \leq 1 - a\right\}.$$

Since $S_{\bar{\lambda}}^{2R2N} - \lim_{k,l\to\infty} x_{k,l} = \ell_1$ and $S_{\bar{\lambda}}^{2R2N} - \lim_{k,l\to\infty} x_{k,l} = \ell_2$, we have Lemma 1 $\delta_{\bar{\lambda}}(K_1(a,t)) = 0$ and $\delta_{\bar{\lambda}}(K_2(a,t)) = 0$ for all $t > 0$.

Now, let $K(a,t) = K_1(a,t) \cup K_2(a,t)$, then it is easy to observe that $\delta_{\bar{\lambda}}(K(a,t)) = 0$. But we have $\delta_{\bar{\lambda}}(K^c(r,t)) = 1$.

Now, if $(k,l) \in K^c(a,t)$, then we have

$$\mathcal{F}(\ell_1 - \ell_2, z; t) \geq \mathcal{F}\left(x_{k,l} - \ell_1, z; \frac{t}{2}\right) * \mathcal{F}\left(x_{k,l} - \ell_2, z; \frac{t}{2}\right) > (1-a) * (1-a).$$

It follows that

$$\mathcal{F}(\ell_1 - \ell_2, z; t) > (1 - \varepsilon).$$

Since $\varepsilon > 0$ was arbitrary, we get $\mathcal{F}(\ell_1 - \ell_2, z; t) = 0$ for all $t > 0$ and nonzero $z \in X$. Hence $\ell_1 = \ell_2$.

This completes the proof. □

Next theorem gives the algebraic characterization of λ-statistical convergence on random 2-normed spaces. We give it without proof.

Theorem 2 *Let $(X, \mathcal{F}, *)$ be a random 2-normed space, and $x = (x_{k,l})$ and $y = (y_{k,l})$ be two sequences in X.*

(a) *If $S_{\bar{\lambda}}^{2R2N} - \lim x_{k,l} = \ell$ and $c(\neq 0) \in \mathbf{R}$, then $S_{\bar{\lambda}}^{2R2N} - \lim cx_{k,l} = c\ell$.*

(b) *If $S_{\bar{\lambda}}^{2R2N} - \lim x_{k,l} = \ell_1$ and $S_{\bar{\lambda}}^{R2N} - \lim y_{k,l} = \ell_2$, then $S_{\bar{\lambda}}^{2R2N} - \lim(x_{k,l} + y_{k,l}) = \ell_1 + \ell_2$.*

Theorem 3 *Let $(X, \mathcal{F}, *)$ be a random 2-normed space. If $x = (x_{k,l})$ is a sequence in X such that $\mathcal{F} - \lim x_{k,l} = \ell$, then $S_{\bar{\lambda}}^{2R2N} - \lim x_{k,l} = \ell$.*

Proof Let $\mathcal{F} - \lim x_{k,l} = \ell$. Then for every $\varepsilon > 0$, $t > 0$ and nonzero $z \in X$, there is a positive integer n_0 and m_0 such that

$$\mathcal{F}(x_k - \ell, z; t) > 1 - \varepsilon$$

for all $k \geq n_0$. Since the set

$$K(\varepsilon, t) = \left\{k \in I_n, l \in J_m : \mathcal{F}(x_{k,l} - \ell, z; t) \leq 1 - \varepsilon\right\}$$

has at most finitely many terms. Since every finite subset of $\mathbf{N} \times \mathbf{N}$ has $\delta_{\bar{\lambda}}$-density zero, finally we have $\delta_{\bar{\lambda}}(K(\varepsilon, t)) = 0$. This shows that $S_{\bar{\lambda}}^{2R2N} - \lim x_{k,l} = \ell$. □

Remark 2 The converse of the above theorem is not true in general. It follows from the following example.

Example 2 Let $X = \mathbf{R}^2$, with the 2-norm $\|x, z\| = |x_1 z_2 - x_2 z_1|$, $x = (x_1, x_2)$, $z = (z_1, z_2)$ and $a * b = ab$ for all $a, b \in [0,1]$. Let $\mathcal{F}(x, y; t) = \frac{t}{t + \|x,y\|}$, for all $x, z \in X$, $z_2 \neq 0$, and $t > 0$. We define a sequence $x = (x_k)$ by

$$x_{k,l} = \begin{cases} (kl, 0), & \text{if } n - [\sqrt{\lambda_n}] + 1 \leq k \leq n \text{ and } m - [\sqrt{\mu_m}] + 1 \leq k \leq m; \\ (0,0), & \text{otherwise.} \end{cases}$$

Now for every $0 < \varepsilon < 1$ and $t > 0$, we write

$$K_n(\varepsilon,t) = \{k \in I_n, l \in J_m : \mathcal{F}(x_{k,l} - \ell, z; t) \leq 1 - \varepsilon\}.$$

Therefore, we get

$$\delta_{\bar{\lambda}}\big(K(\varepsilon,t)\big) = \lim_{nm\to\infty} \frac{[\sqrt{\bar{\lambda}_{nm}}]}{\bar{\lambda}_{nm}} = 0.$$

This shows that $S_{\bar{\lambda}}^{2R2N} - \lim x_{k,l} = 0$, while it is obvious that $\mathcal{F} - \lim x_{k,l} \neq 0$.

Theorem 4 *Let $(X, \mathcal{F}, *)$ be a random 2-normed space. If $x = (x_{k,l})$ is a sequence in X, then $S_{\bar{\lambda}}^{2R2N} - \lim x_{k,l} = \ell$ if and only if there exists a subset $K = \{(k_n, l_n) : k_1 < k_2, \ldots; l_1 < l_2, \ldots\} \subseteq \mathbf{N} \times \mathbf{N}$ such that $\delta_{\bar{\lambda}}(K) = 1$ and $\mathcal{F} - \lim_{n\to\infty} x_{k_n,l_n} = \ell$.*

Proof Suppose first that $S_{\bar{\lambda}}^{2R2N} - \lim x_{k,l} = \ell$. Then for any $t > 0$, $a = 1, 2, 3, \ldots$ and nonzero $z \in X$, let

$$A(a,t) = \left\{k \in I_n; l \in J_m : \mathcal{F}(x_{k,l} - \ell, z; t) > 1 - \frac{1}{a}\right\}$$

and

$$K(a,t) = \left\{k \in I_n; l \in J_m : \mathcal{F}(x_{k,l} - \ell, z; t) \leq 1 - \frac{1}{a}\right\}.$$

Since $S_{\bar{\lambda}}^{2R2N} - \lim x_{k,l} = \ell$, it follows that

$$\delta_{\bar{\lambda}}\big(K(a,t)\big) = 0.$$

Now, for $t > 0$ and $a = 1, 2, 3, \ldots$, we observe that

$$A(a,t) \supset A(a+1,t)$$

and

$$\delta_{\bar{\lambda}}\big(A(a,t)\big) = 1. \tag{3.1}$$

Now we have to show that for $(k,l) \in A(a,t), \mathcal{F} - \lim x_{k,l} = \ell$. Suppose that for some $(k,l) \in A(a,t)$, $(x_{k,l})$ is not convergent to ℓ with respect to $\mathcal{F}$. Then there exist some $s > 0$ and a positive integer k_0, l_0 such that

$$\{k \in I_n; l \in J_m : \mathcal{F}(x_{k,l} - \ell, z; t) \leq 1 - s\}$$

for all $k \geq k_0$ and $l \geq l_0$. Let

$$A(s,t) = \{k \in I_n; l \in J_m : \mathcal{F}(x_{k,l} - \ell, z; t) > 1 - s\}$$

for $k < k_0$ and $l < l_0$ and

$$s > \frac{1}{a}, \quad a = 1, 2, 3, \ldots.$$

Then we have

$$\delta_{\bar{\lambda}}\big(A(s,t)\big) = 0.$$

Furthermore, $A(a,t) \subset A(s,t)$ implies that $\delta_{\bar{\lambda}}(A(a,t)) = 0$, which contradicts (3.1) as $\delta_{\bar{\lambda}}(A(a,t)) = 1$. Hence $\mathcal{F} - \lim x_{k,l} = \ell$.

Conversely, suppose that there exists a subset $K = \{(k_n, l_n) : k_1 < k_2, \ldots; l_1 < l_2, \ldots\} \subseteq \mathbf{N} \times \mathbf{N}$ such that $\delta_{\bar{\lambda}}(K) = 1$ and $\mathcal{F} - \lim_{n,m\to\infty} x_{k_n,l_n} = \ell$. Then for every $\varepsilon > 0$, $t > 0$ and nonzero $z \in X$, we can find a positive integer n_0 such that

$$\mathcal{F}(x_{k,l}, z; t) > 1 - \varepsilon$$

for all $k, l \geq n_0$. If we take

$$K(\varepsilon, t) = \big\{k \in I_n; l \in J_m : \mathcal{F}(x_{k,l} - \ell, z; t) \leq 1 - \varepsilon\big\},$$

then it is easy to see that

$$K(\varepsilon, t) \subset \mathbf{N} \times \mathbf{N} - \big\{(k_{n_0+1}, l_{n_0+1}), (k_{n_0+2}, l_{n_0+2}), \ldots\big\},$$

and finally,

$$\delta_{\bar{\lambda}}\big(K(\varepsilon, t)\big) \leq 1 - 1 = 0.$$

Thus $S_{\bar{\lambda}}^{R2N} - \lim x_{k,l} = \ell$. This completes the proof. □

We now have

Definition 10 A sequence $x = (x_{k,l})$ in a random 2-normed space $(X, \mathcal{F}, *)$ is said to be *λ-double statistically Cauchy* with respect to $\mathcal{F}$ if for each $\varepsilon > 0$, $\eta \in (0,1)$ and for nonzero $z \in X$, there exist $N = N(\varepsilon)$ and $M = M(\varepsilon)$ such that for all $k, m > N$ and $l, n > M$,

$$\delta_{\bar{\lambda}}\big(\big\{k \in I_n; l \in J_m : \mathcal{F}(x_{k,l} - x_{MN}, z; \varepsilon) \leq 1 - \eta\big\}\big) = 0,$$

or equivalently,

$$\delta_{\bar{\lambda}}\big(\big\{k \in I_n; l \in J_m : \mathcal{F}(x_{k,l} - x_{MN}, z; \varepsilon) > 1 - \eta\big\}\big) = 1.$$

Theorem 5 *Let $(X, \mathcal{F}, *)$ be a random 2-normed space. Then a sequence $(x_{k,l})$ in X is λ-double statistically convergent if and only if it is λ-double statistically Cauchy in random 2-normed space X.*

Proof Let $(x_{k,l})$ be a λ-double statistically convergent to ℓ with respect to random 2-normed space, *i.e.*, $S_{\bar{\lambda}}^{2R2N} - \lim x_k = \ell$. Let $\varepsilon > 0$ be given. Choose $a > 0$ such that

$$(1-a) * (1-a) > 1 - \varepsilon. \tag{3.2}$$

For $t > 0$ and for nonzero $z \in X$, define

$$A(a,t) = \left\{ k \in I_n; l \in J_m : \mathcal{F}\left(x_{k,l} - \ell, z; \frac{t}{2} \right) \leq 1 - a \right\}.$$

Then

$$A^c(a,t) = \left\{ k \in I_n; l \in J_m : \mathcal{F}\left(x_{k,l} - \ell, z; \frac{t}{2} \right) > 1 - a \right\}.$$

Since $S_{\bar{\lambda}}^{2R2N} - \lim x_{k,l} = \ell$, it follows that $\delta_{\bar{\lambda}}(A(a,t)) = 0$, and finally, $\delta_{\bar{\lambda}}(A^c(a,t)) = 1$. Let $p, q \in A^c(a,t)$. Then

$$\mathcal{F}\left(x_{p,q} - \ell, z; \frac{t}{2} \right) > 1 - a. \tag{3.3}$$

If we take

$$B(\varepsilon,t) = \left\{ k \in I_n; l \in J_m : \mathcal{F}(x_{k,l} - x_{p,q}, z; t) \leq 1 - \varepsilon \right\},$$

then to prove the result it is sufficient to prove that $B(\varepsilon,t) \subseteq A(a,t)$.

Let $(k,l) \in B(\varepsilon,t) \cap A^c(a,t)$, then for nonzero $z \in X$, we have

$$\mathcal{F}(x_{k,l} - x_{p,q}, z; t) \leq 1 - \varepsilon \quad \text{and} \quad \mathcal{F}\left(x_{k,l} - \ell, z; \frac{t}{2} \right) > 1 - a. \tag{3.4}$$

Now, from (3.1), (3.3) and (3.4), we get

$$\begin{aligned} 1 - \varepsilon \geq \mathcal{F}(x_{k,l} - x_{p,q}, z; t) &\geq \mathcal{F}\left(x_{k,l} - \ell, z; \frac{t}{2} \right) * \mathcal{F}\left(x_p - \ell, z; \frac{t}{2} \right) \\ &> (1-a) * (1-a) > (1-\varepsilon), \end{aligned}$$

which is not possible. Thus $B(\varepsilon,t) \subset A(a,t)$. Since $\delta_{\bar{\lambda}}(A(a,t)) = 0$, it follows that $\delta_{\bar{\lambda}}(B(\varepsilon,t)) = 0$. This shows that $(x_{k,l})$ is λ-double statistically Cauchy.

Conversely, suppose $(x_{k,l})$ is λ-double statistically Cauchy but not λ-double statistically convergent with respect to $\mathcal{F}$. Then for each $\varepsilon > 0$, $t > 0$ and for nonzero $z \in X$, there exist a positive integer $N = N(\varepsilon)$ and $M = M(\varepsilon)$ such that

$$A(\varepsilon,t) = \left\{ k \in I_n; l \in J_m : \mathcal{F}(x_{k,l} - x_{NM}, z; t) \leq 1 - \varepsilon \right\}.$$

Then

$$\delta_{\bar{\lambda}}\big(A(\varepsilon,t)\big) = 0$$

and

$$\delta_{\bar{\lambda}}\big(A^c(\varepsilon,t)\big) = 1. \tag{3.5}$$

For $t > 0$, choose $a > 0$ such that

$$(1-a)*(1-a) > 1-\varepsilon \tag{3.6}$$

is satisfied, and we take

$$B(a,t) = \left\{k \in I_n; l \in J_m : \mathcal{F}\left(x_{k,l} - \ell, z; \frac{t}{2}\right) > 1-a\right\}.$$

If $N, M \in B(a,t)$, then $\mathcal{F}(x_{N,M} - \ell, z; \frac{t}{2}) > 1-a$.

Since

$$\mathcal{F}(x_{k,l} - x_{NM}, z; t) \geq \mathcal{F}\left(x_{k,l} - \ell, z; \frac{t}{2}\right) * \mathcal{F}\left(x_{N,M} - \ell, z; \frac{t}{2}\right) > (1-a)*(1-a) > 1-\varepsilon,$$

then we have

$$\delta_{\bar{\lambda}}\big(\{x_{k,l} : \mathcal{F}(x_{k,l} - x_{NM}, z; t) > 1-\varepsilon\}\big) = 0,$$

i.e., $\delta_{\bar{\lambda}}(A^c(\varepsilon,t)) = 0$, which contradicts (3.5) as $\delta_{\bar{\lambda}}(A^c(\varepsilon,t)) = 1$. Hence $(x_{k,l})$ is λ-double statistically convergent.

This completes the proof. □

Competing interests

The author declares that they have no competing interests.

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