Scientific Research and Essays Vol. 6(24), pp. 5261-5264, 23 October, 2011
Available online at http://www.academicjournals.org/SRE
DOI: 10.5897/SRE11.1292
ISSN 1992-2248 ©2011 Academic Journals

Full Length Research Paper

The nonabelian tensor square ($G\otimes G$) of symplectic groups and projective symplectic groups

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Accepted 17 August, 2011

The determination of $G\otimes G$ for linear groups was mentioned as an open problem by Brown et al. (1987). Hannebauer focused on the nonabelian tensor square of $SL(2, q)$, $PSL(2, q)$, $GL(2, q)$ and $PGL(2, q)$ for all $q \geq 5$ and $q = 9$ in a contribution of 1990. The aim of this paper is to determine the nonabelian tensor square $G\otimes G$ for these groups up to isomorphism by the use of the commutator subgroup and Schur multiplier.

Key words: 2000 mathematics subject classification, Primary, 20J99, secondary, 20J06, 19C09, and phrases, commutator subgroup, Schur multiplier, nonabelian tensor square.

INTRODUCTION

For a group G, the nonabelian tensor square $G\otimes G$ is the group generated the symbols $g\otimes h$ and defined by the relations

$$gg'\otimes h=({}^{g}g'\otimes {}^{g}h)(g\otimes h),\ g\otimes hh'=(g\otimes h)({}^{h}g\otimes {}^{h}h').$$

for all $g, g', h, h' \in G$, where ${}^{g}g'=gg'g^{-1}$. The nonabelian tensor square is a special case of the nonabelian tensor product which has its origin in homotopy theory and was introduced by Brown and Loday (1984, 1987). The exterior square $G\wedge G$ is obtained by imposing the additional relations $g\otimes g = 1_{\otimes}$ for all $g\in G$ on $G\otimes G$. The commutator map induces homomorphisms $\kappa : g\otimes h \in G\otimes G \rightarrow \kappa(g\otimes h)= [g, h] \in G'$ and $\kappa' : g\wedge h \in G\wedge G \rightarrow \kappa'(g\wedge h)= [g, h] \in G'$ and $J_2(G) = \ker(\kappa)$. The results of Brown and Loday (1984, 1987) give the commutative diagram given as in Figure 1 with exact rows and central extensions as columns, where G' is the commutator subgroup of G, $M(G)$ is the multiplicator of G and Γ is Whitehead's quadratic function (Whitehead, 1950).

The determination of $G\otimes G$ for $G = GL(2, q)$ and other linear groups was mentioned as an open problem by Brown et al. (1987) and was pointed out in a more general form in (Kappe, 1999). In the latter paper, there is a list of open problems on the computation of the nonabelian tensor square of finite groups. Among these, there is the problem to find an explicit value of the nonabelian tensor square of linear groups. Hannebauer (1990) determined the nonabelian tensor square of $SL(2, q)$, $PSL(2, q)$, $GL(2, q)$ and $PGL(2, q)$ for all $q \geq 5$ and $q = 9$. Later, Erfanian et al. (2008) determined the nonabelian tensor square of $SL(n, q)$, $PSL(n, q)$, $GL(n, q)$ and $PGL(n, q)$ for all $n, q \geq 2$. This work continues the investigations in the same area, focusing on symplectic groups and projective symplectic groups. We also determine this structure for special linear groups and projective special linear groups, but the method used for computing this structure is different from the method that has been used by Erfanian et al. (2008). As an application we determine the Schur multiplier of these groups. The epicentre and exterior centre of these groups are also determined in the sense of Beyl et al. (1979).

We will prove the following two main theorems:

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Theorem 1.1: Let $\mathbb{F}_q$ be a finite field with q elements, $|\mathbb{F}_q| > 4$ and $SL\,(2, 9)$ is excluded, then

(i) $SL\,(n, q) \otimes SL\,(n, q) \cong SL\,(n, q)$.

(ii) $PSL\,(n, q) \otimes PSL\,(n, q) \cong SL(n, q)$.

Theorem 1.2: Let $\mathbb{F}_q$ be a finite field with q elements and $|\mathbb{F}_q| > 4$, then

(i) $Sp\,(2n, q) \otimes Sp\,(2n, q) \cong Sp\,(2n, q)$.

(ii) $PSp\,(2n, q) \otimes PSp\,(2n, q) \cong Sp(2n, q)$.

Preliminaries

This section includes some results on the commutator subgroup, Schur multiplier and nonabelian tensor square which play an important rule for proving our main theorems.

A group G is perfect if $G = G'$. Special linear groups $SL\,(n, q)$ and projective special linear groups $PSL\,(n, q)$ are perfect groups, except $(n, q) = (2, 2), (2, 3)$. Moreover, $Sp(2n, q)$ and $PSp(2n, q)$ are perfect groups, except $(n, q) = (2, 2), (4, 2), (2, 3)$, where $PSp(2, 2) \cong PSL(2, 2)$, $PSp(4, 2)$ and $PSp(2, 3) \cong PSL(2, 3)$ (Huppert, 1967).

A group G is capable, if there exists a group H such that $G \cong H/Z(H)$ (Hall, 1964). Ellis (1995) proved that a group G is capable if and only if its exterior center $Z^{\square}(G)$ is trivial, where $Z^{\square}(G) = \{g \in G \mid g \wedge x = 1\square$ for all $x \in G\}$. Here, $1\square$ denotes the identity in $G \wedge G$.

A central extension of a group G is a short exact sequence of groups

$$1 \to A \xrightarrow{\varphi} E \xrightarrow{\psi} G \to 1$$

Such that $\varphi(A) \subseteq Z\,(E)$ is in the center of E. Given a central extension

$$1 \to A \to H \to G \to 1$$

And a central extension

$$1 \to B \to K \to G \to 1$$

We say that the first extension covers, (respectively. uniquely covers) the second extension, if there exists a homomorphism such that the following diagram is commutative:

$$\begin{tikzcd}
1 \arrow[r] & A \arrow[r] \arrow[d] & H \arrow[r] \arrow[d] & G \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & B \arrow[r] & K \arrow[r] & G \arrow[r] & 1
\end{tikzcd}$$

A central extension is universal, if it uniquely covers any central extension of G. Beyl et al. (1979) established that a group is capable, if and only if its epicenter $Z^*(G) = \cap\{\varphi Z\,(E) \mid (E, \varphi)$ is a central extension of $G\}$ is trivial. They showed that a perfect group is capable if and only if $Z\,(G) = 1$. According to Karpilovsky (1987), a group G^* is said to be a covering group of G if $Z^*(G)$ has a subgroup A such that

(i) $A \subseteq Z\,(G^*) \cap [G^*, G^*]$,

(ii) $A \cong M\,(G)$,

(iii) $G \cong G^*/A$.

In the following theorem, the Schur multiplier and covering group of a finite perfect group is stated.

Theorem 2.1: Karpilovsky (1987) Let G be a finite perfect group and $1 \to A \to G^* \to G \to 1$ be a universal central extension. Then $A \cong M\,(G)$ and G^* is a covering group of G. Steinberg (1968) obtained a universal central extension for $PSL\,(n, q)$ and $PSp\,(2n, q)$ in the next theorem.

Theorem 2.2: Steinberg (1968) If q is finite, $|\mathbb{F}_q| > 4$ and $SL\,(2, 9)$ is excluded, then the natural extension

(i) $1 \to Z(SL(n, q)) \to SL(n, q) \to PSL(n, q) \to 1$ is universal.

(ii) $1 \to \{-1, +1\} \to Sp(2n, q) \to PSp(2n, q) \to 1$ is universal.

Corollary 2.3: If q is finite, $|\mathbb{F}_q| > 4$ and $SL(2, 9)$ is excluded, then

(i) $SL(n, q)$ is a covering group of $PSL(n, q)$.

(ii) $Sp(2n, q)$ is a covering group of $PSp(2n, q)$.

Proof: (i) By Theorem 2.2, $1 \to Z(SL(n, q)) \to SL(n, q) \to PSL(n, q) \to 1$ is a universal central extension. By Theorem 2.1, $SL(n, q)$ is a covering group of $PSL(n, q)$.

(ii) By the similar way, $1 \to \{-1, +1\} \to Sp(2n, q) \to PSp(2n, q) \to 1$ is a universal central extension. Then $Sp(2n, q)$ is a covering group of $PSp(2n, q)$.

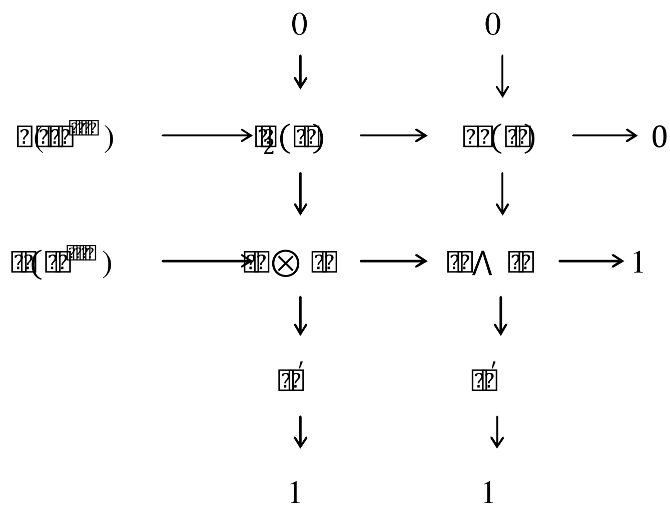

Figure 1. The commutative diagram.

In 1987, Brown et al. (1987) proved that the nonabelian tensor square is the (unique) covering group for a perfect group as follows:

Theorem 2.4: Brown et al. (1987) If G is a perfect group, then $G \otimes G$ is the (unique) covering group G^{*} of G. Since $SL(n, q)$, $PSL(n, q)$, $Sp(2n, q)$ and $PSp(2n, q)$ are perfect, Theorem 2.4 motivates us to concentrate on the covering group for these groups.

The proofs of main theorems

We prove our main theorems mentioned in section 1. First, we compute the Schur multiplier of the special linear groups, projective special linear groups, symplectic groups and projective symplectic groups.

Lemma 3.1: Karpilovsky (1987) If q is finite, $|\mathbb{F}_q| > 4$ and $SL(2, 9)$ is excluded, then
(i) $M(SL(n, q)) = 1$.
(ii) $M(PSL(n, q)) \cong \mathbb{Z}_m$ where $m = \gcd(q - 1, n)$.

Lemma 3.2: If q is finite, $|\mathbb{F}_q| > 4$, then
(i) $M(Sp(2n, q)) = 1$.
(ii) $M(PSp(2n, q)) = \mathbb{Z}_2$.

Proof : (i) Refer to Steinberg (1968).
(ii) Since $1 \rightarrow \{-1, +1\} \rightarrow Sp(2n, q) \rightarrow PSp(2n, q) \rightarrow 1$ is a universal central extension. By Theorem 2.1, $M(PSp(2n, q)) = \mathbb{Z}_2$. □

Proof of Theorem 1.1

Let $\mathbb{F}_q$ be a finite field with q elements, $|\mathbb{F}_q| > 4$ and $SL(2, 9)$ is excluded;
(i) Since $SL(n, q)$ is a perfect group and $M(SL(n, q)) = 1$, then $\Gamma(G^{ab}) = 1$. Thus the diagram in Figure 1 implies that $J_2(SL(n, q)) = M(SL(n, q)) = 1$. The same diagram shows that

$$SL(n, q) \otimes SL(n, q) \cong (SL(n, q))' \cong SL(n, q).$$

(ii) According to Corollary 2.3, $SL(n, q)$ is a covering group of $PSL(n, q)$. Since $PSL(n, q)$ is perfect, this covering group is the nonabelian tensor square of $PSL(n, q)$, that is,

$$PSL(n, q) \otimes PSL(n, q) \cong (PSL(n, q))^{*} \cong SL(n, q).$$

Proof of Theorem 1.2

Let $\mathbb{F}_q$ be a finite field with q elements, $|\mathbb{F}_q| > 4$;
(i) Since $Sp(2n, q)$ is a perfect group and $M(Sp(2n, q)) = 1$, then $\Gamma(G^{ab}) = 1$. Thus the diagram in Figure 1.1 implies that $J_2(Sp(2n, q)) = M(Sp(2n, q)) = 1$. Therefore, the same diagram shows that

$$Sp(2n, q) \otimes Sp(2n, q) \cong (Sp(2n, q))' \cong Sp(2n, q).$$

(ii) As we know $PSp(2n, q)$ is a perfect group and by Corollary 2.3, $Sp(2n, q)$ is a covering group of $PSp(2n, q)$. Therefore,

$$PSp(2n, q) \otimes PSp(2n, q) \cong (PSp(2n, q))^{*} \cong Sp(2n, q).$$

It is clear that special linear groups and symplectic groups are not capable, but projective special linear groups and projective symplectic groups are. The following corollary can be obtained easily.

Corollary 3.3: Let $\mathbb{F}_q$ be a finite field with q elements, $|\mathbb{F}_q| > 4$ with $SL(2, 9)$ excluded and G a projective special linear groups $PSL(n, q)$ or projective symplectic groups $PSp(2n, q)$. Then $Z^{\square}(G) = Z^{*}(G) = 1$.

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