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Precise large deviations of aggregate claims in a discrete-time risk model with Poisson ARCH claim-number process

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Abstract

In this paper, we consider a discrete-time risk model with the claim number following a Poisson ARCH process. In this model, the mean of the current claim number depends on the previous observations. We study the large deviations for the aggregate amount for claims. For a heavy-tailed case, we obtain a precise large deviation formula, which agrees with existing ones in the literature. In computing the moderate deviation principle required by the structure of the claim-number process, our treatment substantially relies on an algorithm specifically designed for the autoregressive structure of our models.

MSC: 60F10; 62J05; 90B30

Keywords: Poisson ARCH process; precise large deviations; moderate deviation principle

1 Introduction

The goal of this paper is studying the precise large deviations for the aggregate claims

$$S_n = \sum_{t=1}^{n} \sum_{j=1}^{N_t} X_{t,j}, \tag{1.1}$$

where N_t is the number of claims in period t and $\{X_{t,j}, j = 1, 2, \dots, t = 1, 2, \dots, n\}$ form an array of independent identically distributed (i.i.d.) claim-size random variables independent of N_t with distribution $F_X = 1 - \overline{F}_X$. Here the claim-number process $\{N_t, t = 1, 2, \dots\}$ is described by the Poisson first-order Autoregressive Conditional Heteroscedasticity (ARCH(1)) process, which is defined as

$$\begin{cases} N_t | \mathscr{F}_{t-1} \sim \text{Poisson}(\lambda_t), \\ \lambda_t = a_0 + a_1 N_{t-1}, \end{cases} \tag{1.2}$$

where $a_0 > 0$, $0 \le a_1 < 1$, $N_0 \ge 0$ is a deterministic integer, and $\mathscr{F}_{t-1} = \sigma(N_0, N_1, \dots, N_{t-1})$ is the σ-field generated by $\{N_0, N_1, \dots, N_{t-1}\}$.

In the last few years, research on the time series models for count data has become a popular topic in the literature. Cossette *et al.* [1] used two integer-valued time series, namely the Poisson moving average (MA) and Poisson autoregressive (AR) processes, to model the claim frequency in the risk model. Li [2] proposed a discrete-time risk model with the claim number being an integer-valued ARCH (INARCH) process with Poisson deviates, namely the model (1.1) and derived some statistical properties and adjustment coefficient for the risk model.

The Poisson INARCH process was first considered by Rydberg and Shephard [3] and applied to finance to model the number of transactions taking place during a short time interval. In model (1.2), it is assumed that the conditional mean of the current claim number has a linear relationship with the previous values of observations. Streett [4] and Ferland *et al.* [5] point out that the process N_t in (1.2) is stationary if $0 \leq a_1 < 1$. In particular, the expectation and variance of N_t are

$$\mathbb{E}N_t = \frac{a_0}{1-a_1} \quad \text{and} \quad \operatorname{Var}(N_t) = \frac{\mathbb{E}N_t}{1-a_1^2}.$$

We study the precise large deviations for $\{S_n, n \geq 0\}$ in (1.1). We are only interested in the case of heavy-tailed claims. Heavy-tailed distributions belong to the core issues in actuarial science, because they are more in accordance with claims' reality than light-tailed ones. A useful heavy-tailed class is the class $\mathscr{C}$ of distribution functions with consistent variation (also called intermediate regular variation), characterized by the relations $\overline{F}(x) > 0$ for all $x \geq 0$ and

$$\lim_{y\uparrow 1} \limsup_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} = \lim_{y\downarrow 1} \liminf_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} = 1.$$

Our main result is given below.

Theorem 1.1 *Consider the aggregate amount of claims* (1.1), *assume that* $F_X(x) \in \mathscr{C}$, $\mathbb{E}X = \mu \in (0,\infty)$, *and* $N_0 \geq 0$ *a deterministic integer. Then, for every fixed* $\gamma > 0$, *uniformly for all* $x \geq \gamma n$,

$$\mathbb{P}\left(S_n - \frac{a_0}{1-a_1} n\mu > x\right) \sim \frac{na_0}{1-a_1} \overline{F}_X(x), \quad n \to \infty. \tag{1.3}$$

A precise large deviation is an important study task in applied probability, and it is usually used to quantitatively characterize the property of extremal events. As is well known, there is a vast amount of literature studying the asymptotic behavior of the large deviation of the risk models in the presence of heavy-tailed claim sizes. See, *e.g.*, Klüppelberg and Mikosch [6], Ng *et al.* [7], Leipus and Šiaulys [8], Asmussen [9], Liu [10], Yang *et al.* [11], Chen and Yuen [12], among many others. On the other hand, the precise large deviations to the risk model with the claim number being a Poisson ARCH process has not been considered in the literature.

We now comment on the approaches used in this work. Our method is much more elementary and does not use the classical treatment in the area of precise large deviation. For convenience in application, we first show that the accumulated aggregate claim S_n in (1.1) has the same distribution with another random walk. Second, we establish the moderate deviation principle (MDP) for the partial sum $\sum_{t=1}^{n} N_t$ generated by the Poisson ARCH(1) process N_t defined in (1.2). As a consequence of MDP, we are able to claim the genuine exponential decay for the probability that the sample average deviates away from its mathematical equilibrium value. This property is a crucial step of our proof of Theorem 1.1. Finally, we would like to point out that equation (1.3) agrees with existing ones in the literature. This indicates that the dependence structure of N_t defined by (1.2) does not affect the asymptotic behavior of the large deviations of $\{S_n, n \geq 0\}$.

The rest of the paper is organized as follows. Section 2 recalls various preliminaries and prepares a few lemmas. Section 3 presents the proof of the main result by establishing the corresponding asymptotic lower and upper bounds.

2 Preliminaries

Throughout this paper, for two positive functions $f(x)$ and $g(x)$, we write

$$\begin{aligned} &f(x) \sim g(x) \quad \text{if } \lim_{x\to\infty} f(x)/g(x) = 1; \\ &f(x) \lesssim g(x) \quad \text{if } \limsup_{x\to\infty} f(x)/g(x) \leq 1 \quad \text{and} \\ &f(x) \gtrsim g(x) \quad \text{if } \liminf_{x\to\infty} f(x)/g(x) \geq 1. \end{aligned}$$

For two positive bivariate functions $f(x,n)$ and $g(x,n)$, we say that the asymptotic relation $f(x,n) \lesssim g(x,n)$ holds uniformly for x in a nonempty set Δ_n if

$$\limsup_{n\to\infty} \sup_{x\in\Delta_n} \frac{f(x;n)}{g(x;n)} \leq 1.$$

First, we show that the accumulated aggregate claim S_n in (1.1) has the same distribution with another random walk.

Lemma 2.1 *Let $\{Y, Y_j, j \geq 1\}$ be a sequence of i.i.d. non-negative random variables such that Y and X in* (1.1) *are identically distributed, then*

$$S_n \stackrel{d}{=} \sum_{j=1}^{N_1+N_2+\cdots+N_n} Y_j,$$

where $\stackrel{d}{=}$ denotes the identical distribution.

Proof For any real r, the moment generating function of S_n is expressed as

$$\begin{aligned} M_s &= E\bigl\{\exp\{rS_n\}\bigr\} = E\left\{\exp\left\{r\sum_{i=1}^{n}\sum_{j=1}^{N_i} X_{i,j}\right\}\right\} \\ &= E\left\{e^{r\sum_{j=1}^{N_1} X_{1,j}} e^{r\sum_{j=1}^{N_2} X_{2,j}} \cdots e^{r\sum_{j=1}^{N_n} X_{k,j}} \cdot \sum_{n_1,n_2,\ldots,n_n} I_{(N_1=n_1,N_2=n_2,\ldots,N_n=n_n)}\right\} \\ &= \sum_{n_1,n_2,\ldots,n_n} E\bigl\{e^{r\sum_{j=1}^{n_1} X_{1,j}} e^{r\sum_{j=1}^{n_2} X_{2,j}} \cdots e^{r\sum_{j=1}^{n_n} X_{n,j}} \cdot I_{(N_1=n_1,N_2=n_2,\ldots,N_n=n_n)}\bigr\} \\ &= \sum_{n_1,n_2,\ldots,n_n} \bigl(Ee^{rX}\bigr)^{\sum_{i=1}^{n} n_i} \cdot P\{N_1 = n_1, N_2 = n_2, \ldots, N_n = n_n\} \\ &= E\bigl\{(M_X)^{N_1+N_2+\cdots+N_n}\bigr\}. \end{aligned}$$

On the other hand, we have

$$\begin{aligned}
&E\left\{\exp\left\{r\sum_{j=1}^{N_1+N_2+\cdots+N_n} Y_j\right\}\right\} \\
&\quad = E\left\{\exp\left\{r\sum_{j=1}^{N_1+N_2+\cdots+N_n} Y_j\right\} \cdot \sum_m I_{(N_1+N_2+\cdots+N_n=m)}\right\} \\
&\quad = \sum_m E\left\{\exp\left\{r\sum_{j=1}^{m} Y_j\right\} \cdot I_{(N_1+N_2+\cdots+N_n=m)}\right\} \\
&\quad = \sum_m \left(Ee^{rY}\right)^m \cdot P\{N_1+N_2+\cdots+N_n=m\} = E\left\{(M_Y)^{N_1+N_2+\cdots+N_n}\right\},
\end{aligned}$$

where $m = n_1 + n_2 + \cdots + n_n$. Hence, by the uniqueness of the moment generating function, we know that S_k and $\sum_{j=1}^{N_1+\cdots+N_n} Y_j$ have the same distribution. □

Next, the following lemma establishes the MDP for $\{N_t, t \geq 0\}$.

Lemma 2.2 *Assume* $\{N_t, t \geq 1\}$ *defined by* (1.2), *and* $N_0 \geq 0$ *be a deterministic integer. Let* b_n *be a sequence of positive numbers satisfying* $b_n \to \infty$ *and* $b_n/n \to 0$, *we have*

$$\limsup_{n\to\infty} \frac{1}{b_n} \log \mathbb{P}\left\{\frac{1}{\sqrt{nb_n}}\sum_{t=1}^{n}\left(N_t - \frac{a_0}{1-a_1}\right) \in H\right\} \leq -\inf_{x\in H} I_M(x) \tag{2.1}$$

for each closed set $H \subset \mathbb{R}$; *and*

$$\liminf_{n\to\infty} \frac{1}{b_n} \log \mathbb{P}\left\{\frac{1}{\sqrt{nb_n}}\sum_{t=1}^{n}\left(N_t - \frac{a_0}{1-a_1}\right) \in G\right\} \geq -\inf_{x\in G} I_M(x) \tag{2.2}$$

for each open set $G \in \mathbb{R}$, *where the rate function* $I_M(\cdot)$ *is given as*

$$I_M(x) = \frac{x^2}{2\sigma^2}, \quad x \in \mathbb{R}, \textit{where } \sigma^2 = \mathbb{E}\left(\operatorname{Var}(N_t|\mathscr{F}_{t-1})\right) = \frac{a_0}{1-a_1}.$$

Proof By the Gärtner-Ellis theorem (Theorem 2.3.6, p.44, Dembo and Zeitouni [13]), all we need to show is that

$$\lim_{n\to\infty} \frac{1}{b_n} \log \mathbb{E}\exp\left\{\beta\sqrt{\frac{b_n}{n}}\sum_{t=1}^{n}(N_t - \mathbb{E}N_1)\right\} = \frac{1}{2}\sigma^2\beta^2, \quad \beta \in \mathbb{R}. \tag{2.3}$$

Let $\beta \in \mathbb{R}$ be fixed but arbitrary and write

$$l_n = a_1\left(e^{\theta_n} - 1\right),$$

where $\theta_n = \beta\sqrt{\frac{b_n}{n}}$. Observe that for any $t \geq 1$,

$$\begin{aligned}
&\mathbb{E}\left[\exp\left\{\theta_n(N_t - \mathbb{E}N_1) - l_n N_{t-1}\right\}|\mathcal{F}_{t-1}\right] \\
&\quad = \exp\{-l_n N_{t-1} - \theta_n \mathbb{E}N_1\}\mathbb{E}\left[\exp\{\theta_n N_t\}|\mathcal{F}_{t-1}\right]
\end{aligned}$$

$$
\begin{aligned}
&= \exp\{-l_n N_{t-1} - \theta_n \mathbb{E}N_1\} \exp\{(a_0 + a_1 N_{t-1})(e^{\theta_n} - 1)\} \\
&= \exp\left\{\frac{a_0}{1-a_1}\left[(1-a_1)(e^{\theta_n} - 1) - \theta_n\right]\right\}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\mathbb{E}\exp\left\{\sum_{t=1}^{n+1}\{\theta_n(N_t - \mathbb{E}N_1) - l_n N_{t-1}\}\right\} \\
&\quad = \left(\exp\left\{\frac{a_0}{1-a_1}\left[(1-a_1)(e^{\theta_n} - 1) - \theta_n\right]\right\}\right)^{n+1}.
\end{aligned}
\tag{2.4}
$$

On the other hand,

$$
\begin{aligned}
&\mathbb{E}\exp\left\{\sum_{t=1}^{n+1}\{\theta_n(N_t - \mathbb{E}N_1) - l_n N_{t-1}\}\right\} \\
&\quad = \exp\{-(n+1)l_n \mathbb{E}N_1\}\mathbb{E}\left\{\exp\{\theta_n N_{n+1} - l_n N_0\}\exp\left\{(\theta_n - l_n)\sum_{t=1}^{n}(N_t - \mathbb{E}N_1)\right\}\right\} \\
&\quad = \exp\{-(n+1)l_n \mathbb{E}N_1\}\mathbb{E}\exp\left\{\theta_n N_{n+1} + (\theta_n - l_n)\sum_{t=1}^{n}(N_t - \mathbb{E}N_1)\right\}.
\end{aligned}
\tag{2.5}
$$

Combining (2.4) and (2.5) and by the definition of l_n,

$$
\begin{aligned}
&\left(\exp\left\{\frac{a_0}{1-a_1}\left[(1-a_1)(e^{\theta_n} - 1) - \theta_n\right]\right\}\right)^{n+1}\left(\exp\{a_1(e^{\theta_n} - 1)\mathbb{E}N_1\}\right)^{n+1} \\
&\quad = \left(\exp\left\{\frac{a_0}{1-a_1}\left[e^{\theta_n} - 1 - \theta_n\right]\right\}\right)^{n+1} \\
&\quad = \mathbb{E}\exp\left\{\theta_n N_{n+1} + (\theta_n - l_n)\sum_{t=1}^{n}(N_t - \mathbb{E}N_1)\right\}.
\end{aligned}
$$

By the Taylor expansion $e^{\theta_n} = 1 + \theta_n + \frac{1}{2}\theta_n^2 + o(\theta_n^2)$, the right-hand side is asymptotically equivalent to

$$
\exp\left\{\frac{1}{2}\sigma^2\beta^2 b_n\right\}.
$$

Thus

$$
\lim_{n\to\infty}\frac{1}{b_n}\log\mathbb{E}\exp\left\{\theta_n N_{n+1} + (\theta_n - l_n)\sum_{t=1}^{n}(N_t - \mathbb{E}N_1)\right\} = \frac{1}{2}\sigma^2\beta^2.
$$

By the fact that $\sup_{t\geq 1}\mathbb{E}\exp\{\theta N_t\} < \infty$ ($\forall\theta > 0$) (see Li [2]) and $\theta_n \to 0$, a standard argument of an exponential approximation by the Hölder inequality enables us to remove the term $\theta_n N_{n+1}$ from the above equation. So we have

$$
\lim_{n\to\infty}\frac{1}{b_n}\log\mathbb{E}\exp\left\{(\theta_n - l_n)\sum_{t=1}^{n}(N_t - \mathbb{E}N_1)\right\} = \frac{1}{2}\sigma^2\beta^2.
\tag{2.6}
$$

By the Hölder inequality, therefore,

$$\begin{aligned}
&\mathbb{E}\exp\left\{(\theta_n - l_n)\sum_{t=1}^{n}(N_t - \mathbb{E}N_1)\right\} \\
&\quad\le \left(\mathbb{E}\exp\left\{\theta_n\sum_{t=1}^{n}(N_t - \mathbb{E}N_1)\right\}\right)^{\frac{\theta_n - l_n}{\theta_n}} \le \mathbb{E}\exp\left\{\theta_n\sum_{t=1}^{n}(N_t - \mathbb{E}N_1)\right\},
\end{aligned}$$

where the second step follows from the fact that

$$\mathbb{E}\exp\left\{\theta_n\sum_{t=1}^{n}(N_t - \mathbb{E}N_1)\right\} \ge 1,$$

which can be proved by Jensen's inequality.

By the fact that $\theta_n = \beta\sqrt{\frac{b_n}{n}}$ and by (2.6), we obtain the lower bound

$$\liminf_{n\to\infty}\frac{1}{b_n}\log\mathbb{E}\exp\left\{\beta\sqrt{\frac{b_n}{n}}\sum_{t=1}^{n}(N_t - \mathbb{E}N_1)\right\} \ge \frac{1}{2}\sigma^2\beta^2. \tag{2.7}$$

On the other hand, given a small number $0 < \delta < 1$, $\theta_n - l_n > (1-\delta)\theta_n = (1-\delta)\beta\sqrt{\frac{b_n}{n}}$ as n is sufficiently large. By the Hölder inequality

$$\begin{aligned}
&\mathbb{E}\exp\left\{(1-\delta)\beta\sqrt{\frac{b_n}{n}}\sum_{t=1}^{n}(N_t - \mathbb{E}N_1)\right\} \\
&\quad\le \left(\mathbb{E}\exp\left\{(\theta_n - l_n)\sum_{t=1}^{n}(N_t - \mathbb{E}N_1)\right\}\right)^{\frac{(1-\delta)\theta_n}{\theta_n - l_n}} \le \mathbb{E}\exp\left\{(\theta_n - l_n)\sum_{t=1}^{n}(N_t - \mathbb{E}N_1)\right\}.
\end{aligned}$$

By (2.6), therefore,

$$\limsup_{n\to\infty}\frac{1}{b_n}\log\mathbb{E}\exp\left\{(1-\delta)\beta\sqrt{\frac{b_n}{n}}\sum_{t=1}^{n}(N_t - \mathbb{E}N_1)\right\} \le \frac{1}{2}\sigma^2\beta^2.$$

Since $\beta \in \mathbb{R}$ can be arbitrary, replacing it by $(1-\delta)^{-1}\beta$ in the above leads to

$$\limsup_{n\to\infty}\frac{1}{b_n}\log\mathbb{E}\exp\left\{\beta\sqrt{\frac{b_n}{n}}\sum_{t=1}^{n}(N_t - \mathbb{E}N_1)\right\} \le \frac{1}{2}\sigma^2\left(\frac{\beta}{1-\delta}\right)^2.$$

Letting $\delta \to 0^+$ on the right-hand side yields the desired upper bound, which, together with the lower bound (2.7), leads to (2.3). □

As a consequence of Lemma 2.2, for every $\eta > 0$, considering the closed set $H = \{x, |x| \ge \eta\}$ and $b_n = \sqrt{n}$, we have

$$\mathbb{P}\left(\frac{1}{n}\left|\sum_{t=1}^{n}(N_t - \mathbb{E}N_1)\right| \ge \eta\right) \le \mathbb{P}\left(\frac{1}{n^{3/4}}\left|\sum_{t=1}^{n}(N_t - \mathbb{E}N_1)\right| \ge \eta\right) \le \exp\{-c_\eta\sqrt{n}\}, \tag{2.8}$$

where $c_\eta = \frac{\eta^2}{2\sigma^2} > 0$ is independent of n. This gives the genuine exponential decay for the probability that the sample average deviates from its expectation.

The last lemma below is a restatement of Theorem 3.1 of Ng *et al.* [7].

Lemma 2.3 *Let $\{Y, Y_j, j \geq 1\}$ be a sequence of i.i.d. non-negative random variables with common distribution function $F_Y \in \mathscr{C}$ and finite expectation μ, let $Q_n = \sum_{j=1}^n Y_j$. Then, for any fixed $\gamma > 0$,*

$$P(Q_n - n\mu > y) \sim n\overline{F}_Y(y) \quad (n \to \infty) \text{ uniformly for } y \geq \gamma n.$$

3 Proof of Theorem 1.1

Throughout this section, unless otherwise stated, every limit relation is understood as valid uniformly for all $x \geq \gamma n$ as $n \to \infty$. Trivially, equation (1.3) amounts to the conjunction of

$$\begin{aligned} &\mathbb{P}\left(S_n - \frac{a_0}{1-a_1} n\mu > x\right) \lesssim \frac{na_0}{1-a_1} \overline{F}_X(x) \quad \text{and} \\ &\mathbb{P}\left(S_n - \frac{a_0}{1-a_1} n\mu > x\right) \gtrsim \frac{na_0}{1-a_1} \overline{F}_X(x), \end{aligned} \tag{3.1}$$

which will be proven separately in the following two subsections.

3.1 Proof of the first relation (3.1)

Write $\nu = \mathbb{E}N_1 = \frac{a_0}{1-a_1}$. For arbitrarily fixed, but small η, $0 < \eta < 1$, by Lemma 2.1, we derive

$$\begin{aligned} &\mathbb{P}(S_n - \nu n\mu > x) \\ &\quad = \mathbb{P}\left(\sum_{j=1}^{N_1+\cdots+N_n} Y_j - \nu n\mu > x\right) \\ &\quad = \mathbb{P}\left\{\sum_{t=1}^n N_t < n(\nu+\eta), \sum_{j=1}^{N_1+\cdots+N_n} Y_j - \nu n\mu > x\right\} \\ &\qquad + \mathbb{P}\left\{\sum_{t=1}^n N_t \geq n(\nu+\eta), \sum_{j=1}^{N_1+\cdots+N_n} Y_j - \nu n\mu > x\right\} \\ &\quad \leq \mathbb{P}\left(\sum_{j=1}^{[n(\nu+\eta)]} Y_j - \nu n\mu > x\right) + \mathbb{P}\left\{\sum_{t=1}^n N_t \geq n(\nu+\eta), \sum_{j=1}^{N_1+\cdots+N_n} Y_j - \nu n\mu > x\right\} \\ &\quad = \Delta_1 + \Delta_2, \end{aligned} \tag{3.2}$$

where $[\cdot]$ denotes the integral part of $\cdot$, throughout this paper.

From Lemma 2.3, we see that, for some small η such that $\gamma - \eta\mu > 0$,

$$\begin{aligned} \Delta_1 &= \mathbb{P}\left(\sum_{j=1}^{[n(\nu+\eta)]} Y_j - \big[n(\nu+\eta)\big]\mu > x + n\nu\mu - \big[n(\nu+\eta)\big]\mu\right) \\ &\sim \big[n(\nu+\eta)\big]\overline{F}_Y\big(x + n\nu\mu - \big[n(\nu+\eta)\big]\mu\big) \\ &\leq n(\nu+\eta)\overline{F}_X\big(x(1-\eta\mu/\gamma)\big). \end{aligned} \tag{3.3}$$

As for the second term,

$$\begin{aligned}
\Delta_2 &= \mathbb{P}\left\{\sum_{t=1}^{n} N_t \geq n(\nu+\eta), \sum_{j=1}^{N_1+\cdots+N_n} Y_j - n\nu\mu > x\right\} \\
&= \sum_{m \geq n(\nu+\eta)} \mathbb{P}\left\{\sum_{t=1}^{n} N_t = m\right\} \mathbb{P}\left\{\sum_{j=1}^{m} Y_j - n\nu\mu > x\right\} \\
&\leq \sum_{m \geq n(\nu+\eta)} \mathbb{P}\left\{\sum_{t=1}^{n} N_t = m\right\} \mathbb{P}\left\{\sum_{j=1}^{m} Y_j > x\right\} \\
&\leq \sum_{m \geq n(\nu+\eta)} \mathbb{P}\left\{\sum_{t=1}^{n} N_t = m\right\} \sum_{j=1}^{m} \mathbb{P}\left\{Y_j > \frac{x}{m}\right\} \\
&\leq \sum_{m \geq n(\nu+\eta)} \mathbb{P}\left\{\sum_{t=1}^{n} N_t = m\right\} m\overline{F}_Y\left(\frac{x}{m}\right).
\end{aligned}$$

By the assumption that $F_Y = F_X \in \mathscr{C}$ there is a constant $C > 0$ independent of m, and for every $p > J_F^+$ such that $\overline{F}_X(\frac{x}{m}) \leq Cm^p\overline{F}_X(x)$ for all m and sufficiently large x (see Lemma 2.2, Ng *et al.* [7]), where J_F^+ is called the upper Matuszewska index. Hence

$$\begin{aligned}
\Delta_2 &\leq C\overline{F}_X(x) \sum_{m \geq n(\nu+\eta)} m^{p+1}\mathbb{P}\left\{\sum_{t=1}^{n} N_t = m\right\} \\
&\leq C\overline{F}_X(x)\mathbb{E}\left\{\left(\sum_{t=1}^{n} N_t\right)^{p+1} \mathbb{1}_{\{\sum_{t=1}^{n} N_t \geq n(\nu+\eta)\}}\right\}.
\end{aligned} \tag{3.4}$$

We now claim that

$$\lim_{n\to\infty} \mathbb{E}\left\{\left(\sum_{t=1}^{n} N_t\right)^{p+1} \mathbb{1}_{\{\sum_{t=1}^{n} N_t \geq n(\nu+\eta)\}}\right\} = 0. \tag{3.5}$$

Indeed, by the Cauchy-Schwarz inequality

$$\begin{aligned}
&\mathbb{E}\left\{\left(\sum_{t=1}^{n} N_t\right)^{p+1} \mathbb{1}_{\{\sum_{t=1}^{n} N_t \geq n(\nu+\eta)\}}\right\} \\
&\quad\leq \left\{\mathbb{E}\left(\sum_{t=1}^{n} N_t\right)^{2(p+1)}\right\}^{1/2} \left\{\mathbb{P}\left(\sum_{t=1}^{n} N_t \geq n(\nu+\eta)\right)\right\}^{1/2}.
\end{aligned}$$

Using (2.8), we have

$$\mathbb{P}\left(\sum_{t=1}^{n} N_t \geq n(\nu+\eta)\right) \leq \mathbb{P}\left(\frac{1}{n}\left|\sum_{t=1}^{n}(N_t - \nu)\right| \geq \eta\right) \leq \exp\{-c_\eta\sqrt{n}\},$$

where $c_\eta = \frac{\eta^2}{2\sigma^2} > 0$ is independent of n. This together with fact that $\mathbb{E}(\sum_{t=1}^{n} N_t)^{2(p+1)} = O(n^{2(p+1)})$ proves (3.5).

Hence, by (3.4) and (3.5), we have

$$\Delta_2 \le o\big(\overline{F}_X(x)\big) = o\big(n\nu\overline{F}_X(x)\big). \tag{3.6}$$

Substituting (3.3) and (3.6) into (3.2) yields

$$\mathbb{P}(S_n - n\nu\mu > x) \lesssim n(\nu+\eta)\overline{F}_X\big(x(1-\eta\mu/\gamma)\big) + o\big(n\nu\overline{F}_X(x)\big).$$

By the arbitrariness of η and the condition $F_X \in \mathscr{C}$, we obtain the first relation (3.1).

3.2 Proof of the second relation (3.1)

Let $0 < \eta < 1$ be arbitrarily fixed with η small. Consider the decomposition

$$\begin{aligned}
\mathbb{P}(S_n - n\nu\mu > x) &= \mathbb{P}\left(\sum_{j=1}^{N_1+N_2+\cdots+N_n} Y_j - n\nu\mu > x\right) \\
&\ge \mathbb{P}\left(\sum_{t=1}^{n} N_t > n(\nu-\eta), \sum_{j=1}^{N_1+N_2+\cdots+N_n} Y_j - n\nu\mu > x\right) \\
&\ge \mathbb{P}\left(\sum_{t=1}^{n} N_t > n(\nu-\eta)\right)\cdot\mathbb{P}\left(\sum_{j=1}^{[n(\nu-\eta)]} Y_j - n\nu\mu > x\right).
\end{aligned} \tag{3.7}$$

Applying (2.8) we have

$$\mathbb{P}\left\{\sum_{t=1}^{n} N_t > n(\nu-\eta)\right\} \ge \mathbb{P}\left\{\frac{1}{n}\left|\sum_{t=1}^{n}(N_t-\nu)\right| < \eta\right\} \to 1, \quad \text{as } n\to\infty. \tag{3.8}$$

By Lemma 2.3, have

$$\begin{aligned}
\mathbb{P}\left(\sum_{j=1}^{[n(\nu-\eta)]} Y_j - n\nu\mu > x\right) &\sim \big[n(\nu-\eta)\big]\overline{F}_Y\big(x + n\nu\mu - \big[n(\nu-\eta)\big]\mu\big) \\
&\ge n(\nu-\eta)\frac{[n(\nu-\eta)]}{n(\nu-\eta)}\overline{F}_X\big(x(1+\eta\mu/\gamma)\big).
\end{aligned} \tag{3.9}$$

Finally, substituting (3.8) and (3.9) into (3.7) yields

$$\mathbb{P}(S_n - n\nu\mu > x) \gtrsim n(\nu-\eta)\overline{F}_X\big(x(1+\eta\mu/\gamma)\big).$$

By the arbitrariness of η and the condition $F_X \in \mathscr{C}$, we obtain the second relation (3.1).

Competing interests

The author declares that they have no competing interests.

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