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Numerical solution for a class of singular integral equations

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This paper is concerned with finding approximate solution for the singular integral equations. Relating the singular integrals to Cauchy principal-value integrals, we expand the kernel and the density function of singular integral equation by the sum of the chebyshev polynomials of the first, second, third and fourth kinds. Some numerical examples are presented to illustrate the accuracy and effectiveness of the present work. Numerical results show that the errors of approximate solutions of examples in different cases with small value of n are very small. These show that the methods developed are very accurate and in fact for a linear function give the exact solution.

Key words: Singular integral equations, Cauchy kernel, chebyshev polynomials, weight functions.

INTRODUCTION

During the last three decades, the singular integral equation methods with applications to several basic fields of engineering mechanics, like elasticity, plasticity, aerodynamics and fracture mechanics have been studied and improved by several scientists (Chakrabarti, 1989; Ladopoulous, 2000, 1987; Zabreyko, 1975; Prossdorf, 1977; Zisis and Ladopoulos, 1989). Hence, it is of interest to solve numerically this type of integral equations (Chakrabarti and Berghe, 2004; Abdou and Naser, 2003). Chebyshev polynomials are of great importance in many areas of mathematics particularly approximation theory (Abdulkawi et al., 2009; Eshkuvatov et al., 2009).

In this paper, we analyze the numerical solution of singular integral equations by using Chebyshev polynomials of first, second, third and fourth kind to obtain systems of linear algebraic equations; these systems are solved numerically. The methodology of the present work is expected to be useful for solving singular integral equations of the first kind, involving partly singular and partly regular kernels. The singularity is assumed to be of the Cauchy type. The method is illustrated by considering some examples.

Singular integral equation of first kind, with a Cauchy type singular kernel, over a finite interval can be represented by:

$$\int_{-1}^{1} \frac{k(t,x)\varphi(t)}{t-x}\,dt + \int_{-1}^{1} L(t,x)\varphi(t)\,dt = f(x), \qquad -1 < x < 1 \tag{1}$$

where $k(t,x), L(t,x)$ and $f(x)$ are given real-valued continuous functions belonging to the class Holder of continues functions and $k(t,t) \neq 0$. In Equation (1) the singular kernel is interpreted as Cauchy principle value. Integral equation of form 1 and other different forms have many applications (Chakrabarti, 1989; Ladopoulous, 2000; Ladopoulous, 1987; Gakhov, 1966; Martin and Rizzo, 1989; Zisis and Ladopoulos, 1989). The theory of this equation is well known and it is presented in Sheshko (2003) and Muskhelishvili (1977). An approximate method for solving Equation (1) using a polynomial approximation of degree n has been proposed by Chakrabarti and Berghe (2004).

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It is well known that the analytical solutions of the simple singular integral equation

$$\int_{-1}^{1}\frac{\varphi(t)}{t-x}dt = f(x), \qquad -1<x<1 \tag{2}$$

at $k(t,x)=1$ and $L(t,x)=0$, for the following four cases,

(I) The solution is unbounded at both end-points $x=\pm 1,$

(II) The solution is bounded at both end-points $x=\pm 1,$

(III) The solution is bounded at end $x=-1,$ but unbounded at end $x=+1$,

(IV) The solution is unbounded at end $x=-1,$ but bounded at end $x=+1$,

are given by Lifanov (1996). In this paper, the used approximate method for solving Equation 1 stems from recent work (Eshkuvatov et al., 2009) wherein an approximate method has been developed to solve the simple Equation (2). The approximate method developed below appears to be quite appropriate for solving the most general type Equations (1). Some examples are presented to illustrate the method.

THE APPROXIMATE SOLUTION

In this section, we present the method of the approximate solution of Equation (1) in four cases. Let the unknown function φ in Equation (1) be approximated by the polynomial function

$$\varphi_n(x) = W^{(j)}(x)\sum_{i=0}^{n} c_i^{(j)}\Psi_i^{(j)}(x) \qquad (j=1,2,3,4) \tag{3}$$

Where $c_i^{(j)},\ i=0,1,2,...,n$ are unknown coefficients and in case (I): $\Psi_i^{(1)}(x)=T_i(x)$, in case (II): $\Psi_i^{(2)}(x)=U_i(x)$, in case (III): $\Psi_i^{(3)}(x)=V_i(x)$ and in case (VI): $\Psi_i^{(4)}(x)=W_i(x)$, where T_i, U_i, V_i and $W_i,\ i=0,1,...n,$ are the Chebyshev polynomials of the first, second, third and fourth kinds respectively can be defined by the recurrence relations (Prem and Michael, 2005; Abdulkawi et al., 2009).

$$\left.\begin{array}{ll} T_0(x)=1, & T_1(x)=x \\ T_n(x)=2xT_{n-1}(x)-T_{n-2}(x) & n\geq 2 \end{array}\right\} \tag{4}$$

$$\left.\begin{array}{ll} U_0(x)=1, & U_1(x)=2x \\ U_n(x)=2xU_{n-1}(x)-U_{n-2}(x) & n\geq 2 \end{array}\right\} \tag{5}$$

$$\left.\begin{array}{ll} V_0(x)=1, & V_1(x)=2x-1 \\ V_n(x)=2xV_{n-1}(x)-V_{n-2}(x) & n\geq 2 \end{array}\right\} \tag{6}$$

$$\left.\begin{array}{ll} W_0(x)=1, & W_1(x)=2x+1 \\ W_n(x)=2xW_{n-1}(x)-W_{n-2}(x) & n\geq 2 \end{array}\right\} \tag{7}$$

and $W^i,\ i=0,1,...n,$ are the corresponding weight functions. Substituting the approximate solution Equation (3) for the unknown function into Equation (1) yields:

$$\sum_{i=0}^{n} c_i^{(j)}\left[\int_{-1}^{1}\frac{k(t,x)W^{(j)}(t)\Psi_i^{(j)}(t)}{t-x}dt+\int_{-1}^{1}L(t,x)W^{(j)}(t)\Psi_i^{(j)}(t)\,dt\right]=f(x),\ -1<x<1 \tag{8}$$

In above Equation (8), we next use the following chebyshev approximation to the kernels $k(t,x)$ and $L(t,x)$, given by (for fixed x,cf.(Chakrabarti and Berghe, 2004)

$$k(t,x)\cong\sum_{p=0}^{m}k_p(x)t^p\ , \qquad L(t,x)\cong\sum_{q=0}^{s}L_q(x)t^q \tag{9}$$

with known expressions for $K_p(x)$ and $L_q(x)$. Then Equation 8 gives

$$\sum_{i=0}^{n}c_i^{(j)}\alpha_i^{(j)}(x)=f(x), \qquad -1<x<1,\ (j=1,2,3,4) \tag{10}$$

Where

$$\alpha_i^{(j)}(x)=\sum_{p=0}^{m}k_p(x)u_{p,i}^{(j)}(x)+\sum_{q=0}^{s}L_q(x)\gamma_{q,i}^{(j)} \tag{11}$$

With

$$u_{p,i}^{(j)}(x)=\int_{-1}^{1}\frac{t^pW^{(j)}(t)\Psi_i^{(j)}(t)}{t-x}dt \qquad -1<x<1, \quad (j=1,2,3,4) \tag{12}$$

And

$$\gamma_{q,i}^{(j)}=\int_{-1}^{1}t^qW^{(j)}(t)\Psi_i^{(j)}(t)\,dt \tag{13}$$

Let $x_k^{(j)},\ j=1,2,3,4$, be the zeros of $U_n(x), T_{n+2}(x), W_{n+1}(x)$ and $V_{n+1}(x)$, respectively.

Substituting the collocation points $x_k^{(j)},\ j=1,2,3,4$ into Equation 10, we obtain the following systems of linear equations:

$$\sum_{i=0}^{n} c_i^{(j)} \alpha_i^{(j)}(x_k^{(j)}) = f(x_k^{(j)}), \quad (k=1,2,...,n+1),\ (j=1,2,3,4) \tag{14}$$

where

$$\alpha_i^{(j)}(x_k^{(j)}) = \sum_{p=0}^{m} k_p(x_k^{(j)}) u_{p,i}^{(j)}(x_k^{(j)}) + \sum_{q=0}^{s} L_q(x_k^{(j)}) \gamma_{q,i}^{(j)},\ (k=1,2,...,n+1),(j=1,2,3,4) \tag{15}$$

Solving the system of Equation 14 for the unknown coefficients $c_i^{(j)},\ j=1,2,3,4$, and substituting the values of $c_i^{(j)}$ into Equation 3 we obtain the approximate solutions of Equation 1 in the form of

$$\varphi_n(x) \cong W^{(j)}(x) \sum_{i=0}^{n} c_i^{(j)} \Psi_i^{(j)}(x) \qquad (j=1,2,3,4) \tag{16}$$

NUMERICAL EXAMPLES

In this section, we consider some problems to illustrate the above method. All results were computed using FORTRAN code.

Example 1. Consider the following singular integral equation

$$\int_{-1}^{1} \frac{(x+t^2)\varphi(t)}{t-x} + \int_{-1}^{1}(x^2+t^3)\varphi(t)\,dt = 2x^4 - 2x^2 - \frac{3}{8}, \qquad -1<x<1 \tag{17}$$

Where

$k(x,t) = x + t^2, \quad L(x,t) = x^2 + t^3, \quad f(t) = 2x^4 - 2x^2 - \frac{3}{8}$.

So, one gets

$$k_0(x) = x,\ k_1(x) = 0,\ k_2(x) = 1 \quad k_p(x) = 0, \qquad (p>2)$$
$$L_0(x) = x^2,\ L_1(x) = 0,\ L_2(x) = 0,\ L_3(x) = 1,\ L_q(x) = 0 \quad (q>3)$$

Hence we find that relation (10) produces

$$\sum_{i=0}^{n} c_i^{(j)} \alpha_i^{(j)}(x) = 2x^4 - 2x^2 - \frac{3}{8} \qquad -1<x<1,\ (j=1,2,3,4)$$

Thus Equation 11 gives

$$\alpha_i^{(j)}(x) = x u_{0,i}^{(j)}(x) + u_{2,i}^{(j)}(x) + x^2 \gamma_{0,i}^{(j)} + \gamma_{3,i}^{(j)},\ (j=1,2,3,4),\ (i=0,1,2,...)$$

Firstly, let us consider in detail the case (I) , $j=1$, for $n=3$. This result in

$$u_{0,i}^{(1)}(x) = \int_{-1}^{1} \frac{T_i(t)}{\sqrt{1-t^2}\,t-x}\,dt, \quad u_{2,i}^{(1)}(x) = \int_{-1}^{1} \frac{t^2 T_i(t)}{\sqrt{1-t^2}\,t-x}\,dt,\ -1<t<1, \tag{18}$$

$$\gamma_{0,i}^{(1)} = \int_{-1}^{1} \frac{T_i(t)}{\sqrt{1-t^2}}\,dt, \qquad \gamma_{3,i}^{(1)} = \int_{-1}^{1} \frac{t^3 T_i(t)}{\sqrt{1-t^2}}\,dt, \tag{19}$$

By applying the following relations

$$\int_{-1}^{1} \frac{T_i(t)}{\sqrt{1-t^2}(t-x)}\,dt = \pi U_{i-1}(x), \qquad \int_{-1}^{1} \frac{1}{\sqrt{1-t^2}(t-x)}\,dt = 0 \tag{20}$$

$$\int_{-1}^{1} \frac{T_i(t)T_j(t)}{\sqrt{1-t^2}}\,dt = \begin{cases} 0 & i \neq j \\ \pi & i = j = 0 \\ \pi/2 & i = j \neq 0 \end{cases} \tag{21}$$

It is easy to estimate the values $u_{0,i}^{(1)}, u_{2,i}^{(1)}, \gamma_{0,i}^{(1)}$ and $\gamma_{3,i}^{(1)}$. From Equations 10 and 18-21 we get

$$\alpha_i^{(1)}(x) = \begin{cases} \pi(x^2+x); & i=0 \\ \pi\left(x^2+x+\frac{7}{8}\right); & i=1 \\ \pi(2x^3+2x^2); & i=2 \\ \pi\left(4x^4+4x^3-x^2-x+\frac{1}{8}\right); & i=3 \end{cases} \tag{22}$$

By choosing the collocation points $x_k = \cos\left(\frac{(2k-1)\pi}{2(n+2)}\right), (k=1,2,3,4)$, for $n=3,$ we obtain the following system of linear equations:

$$\sum_{i=0}^{3} c_i^{(1)} \alpha_i^{(1)}(x_k) = f(x_k), \quad k=1,2,3,4$$

By solving this system for the unknown coefficients $c_i^{(1)},\ i=0,1,2,3$ that produces

$$\left.\begin{matrix} c_0^{(1)} = 0.3183098, & c_1^{(1)} = -0.1591549 \\ c_2^{(1)} = -0.3183098, & c_3^{(1)} = 0.1591549 \end{matrix}\right\} \tag{23}$$

From Equation (23) we obtain the approximate solution of Equation (17) in the form

$$\varphi_n(x) \cong \frac{2}{\pi\sqrt{1-x^2}}(x^3 - x^2 - x + 1) \tag{24}$$

Which coincides with the exact solution. The error of approximate solution (24) of Equation 17 at $n=20$ is given in Table 1.

Table 1. Errors of approximate solutions of Equation 17 in Cases (I)-(IV) at n=20.

X	error (j=1)	error (j=2)	error (j=3)	error (j=4)
-9.500000E-01	0.000000E+00	0.000000E+00	5.960464E-08	5.960464E-08
-9.000000E-01	0.000000E+00	0.000000E+00	1.192093E-07	1.192093E-07
-7.000000E-01	0.000000E+00	0.000000E+00	1.192093E-07	1.192093E-07
-5.000000E-01	5.960464E-08	5.960464E-08	1.788139E-07	1.788139E-07
-3.000000E-01	0.000000E+00	5.960464E-08	1.788139E-07	1.788139E-07
-1.000000E-01	5.960464E-08	5.960464E-08	1.192093E-07	1.192093E-07
0.000000E+00	5.960464E-08	5.960464E-08	1.192093E-07	1.192093E-07
1.000000E-01	1.192093E-07	5.960464E-08	5.960464E-08	5.960464E-08
3.000000E-01	8.940697E-08	8.940697E-08	8.940697E-08	8.940697E-08
5.000000E-01	8.940697E-08	8.940697E-08	5.960464E-08	5.960464E-08
7.000000E-01	1.043081E-07	7.450581E-08	1.490116E-08	1.490116E-08
9.000000E-01	9.313226E-08	5.029142E-08	4.656613E-08	1.303852E-08
9.500000E-01	5.774200E-08	3.632158E-08	6.705523E-08	4.656613E-09

Secondly, let us consider in detail the case (II) , $j=2$, for $n=3$. This result in

$$u_{0,i}^{(2)}(x)=\int_{-1}^{1}\frac{\sqrt{1-t^2}\,U_i(t)}{t-x}dt,\qquad u_{2,i}^{(2)}(x)=\int_{-1}^{1}\frac{t^2\sqrt{1-t^2}\,U_i(t)}{t-x}dt,\quad -1<t<1, \tag{25}$$

$$\gamma_{0,i}^{(2)}=\int_{-1}^{1}\sqrt{1-t^2}U_i(t)\,dt,\qquad \gamma_{3,i}^{(2)}=\int_{-1}^{1}\sqrt{1-t^2}\;t^3U_i(t)\,dt, \tag{26}$$

By applying the following relations

$$\int_{-1}^{1}\frac{\sqrt{1-t^2}U_i(t)}{t-x}dt=-\pi T_{i+1}(x) \tag{27}$$

$$\int_{-1}^{1}\sqrt{1-t^2}U_i(t)U_j(t)\;dt=\begin{cases}0 & i\neq j\\ \dfrac{\pi}{2} & i=j\end{cases} \tag{28}$$

It is easy to estimate the values $u_{0,i}^{(2)}, u_{2,i}^{(2)}, \gamma_{0,i}^{(2)}$ and $\gamma_{3,i}^{(2)}$. From the relations 11 and Equations (25) to (28) we get

$$\alpha_i^{(2)}(x)=\begin{cases}-\dfrac{\pi}{2}\left(2x^3+x^2-x\right) & i=0\\ -\pi\left(2x^4+2x^3-x^2-x-\dfrac{3}{8}\right) & i=1\\ -\pi\left(4x^5+4x^4-3x^3-3x^2+\dfrac{1}{4}x\right) & i=2\\ -\pi\left(8x^6+8x^5-8x^4-8x^3+x^2+x-\dfrac{1}{16}\right) & i=3\end{cases} \tag{29}$$

By choosing the collocation points $x_k^{(2)}=\cos\left(\dfrac{(2k-1)\pi}{2(n+2)}\right), (k=1,2,3,4)$, for $n=3$, we obtain the following system of linear equations:

$$\sum_{i=0}^{3}c_i^{(2)}\,\alpha_i^{(2)}(x_k^{(2)})=f(x_k^{(2)}),\quad k=1,2,3,4$$

By solving this system for the unknown coefficients $c_i^{(2)},\ i=0,1,2,3$ that produces

$$\left.\begin{aligned}c_0^{(2)}&=0.6366197, & c_1^{(2)}&=-0.3183099\\ c_2^{(2)}&=2.279989\times10^{-8}, & c_3^{(2)}&=-7.819254\times10^{-9}\end{aligned}\right\} \tag{30}$$

From Equation (30) we obtain the approximate solution of Equation (17) in the form

$$\varphi_n(x)\cong\frac{2\sqrt{1-x^2}}{\pi}(1-x) \tag{31}$$

Which coincides with the exact solution. The error of approximate Solution (31) of Equation (17) at $n=20$ is given in Table 1.

Thirdly, let us consider in detail the case (III) , $j=3$, for $n=3$. This result in

$$u_{0,i}^{(3)}(x)=\int_{-1}^{1}\sqrt{\frac{1+t}{1-t}}\frac{V_i(t)}{t-x}dt,\qquad u_{2,i}^{(3)}(x)=\int_{-1}^{1}\sqrt{\frac{1+t}{1-t}}\;\frac{t^2V_i(t)}{t-x}dt,\quad -1<t<1, \tag{32}$$

$$\gamma_{0,i}^{(3)}=\int_{-1}^{1}\sqrt{\frac{1+t}{1-t}}V_i(t)\,dt,\qquad \gamma_{3,i}^{(3)}=\int_{-1}^{1}\sqrt{\frac{1+t}{1-t}}\;t^3V_i(t)\,dt, \tag{33}$$

By applying the following relations

$$\int_{-1}^{1}\sqrt{\frac{1+t}{1-t}}V_i(t)V_j(t)dt=\begin{cases}0 & i\neq j\\ \pi & i=j\end{cases} \qquad (34)$$

$$\int_{-1}^{1}\sqrt{\frac{1+t}{1-t}}\frac{V_i(t)}{t-x}dt=\pi W_i(x) \qquad (35)$$

It is easy to estimate the values $u_{0,i}^{(3)}, u_{2,i}^{(3)}, \gamma_{0,i}^{(3)}$ and $\gamma_{3,i}^{(3)}$. From the relations (6) and Equations (32) to (35) we get

$$\alpha_i^{(3)}(t)=\begin{cases}\pi\left(2x^2+2x+\frac{7}{8}\right) & i=0\\ \pi\left(2x^3+3x^2+x+\frac{7}{8}\right) & i=1\\ \pi\left(4x^4+6x^3+x^2-x+\frac{1}{8}\right) & i=2\\ \pi\left(8x^5+12x^4-5x^2+x+\frac{1}{8}\right) & i=3\end{cases} \qquad (36)$$

By choosing the collocation points $x_k^{(3)}=\cos\left(\frac{2k\pi}{(2n+3)}\right),(k=1,2,3,4)$, for $n=3,$ we obtain the following system of linear equations:

$$\sum_{i=0}^{3}c_i^{(3)}\,\alpha_i^{(3)}(x_k^{(3)})=f(x_k^{(3)}),\quad k=1,2,3,4$$

By solving this system for the unknown coefficients $c_i^{(3)},\ i=0,1,2,3$ that produces

$$\left.\begin{matrix}c_0^{(3)}=0.3183098, & c_1^{(3)}=-0.4774647,\\ c_2^{(3)}=0.1591549, & c_3^{(3)}=1.330901\times10^{-8}\end{matrix}\right\} \qquad (37)$$

From Equation (37) we obtain the approximate solution of Equation (17) in the form of

$$\varphi_n(x)\cong\frac{2}{\pi}\sqrt{\frac{1+x}{1-x}}\left(x^2-2x+1\right) \qquad (38)$$

Which coincides with the exact solution. The error of approximate Solution (38) of Equation (17) at $n=20$ is given in Table 1.

Fourthly, In case (IV) , $j=4$, for n=3. This result in

$$u_{0,i}^{(4)}(x)=\int_{-1}^{1}\sqrt{\frac{1-t}{1+t}}\frac{W_i(t)}{t-x}dt,\quad u_{2,i}^{(4)}(x)=\int_{-1}^{1}\sqrt{\frac{1-t}{1+t}}\frac{t^2W_i(t)}{t-x}dt,\quad -1<t<1, \qquad (39)$$

$$\gamma_{0,i}^{(4)}=\int_{-1}^{1}\sqrt{\frac{1-t}{1+t}}W_i(t)\,dt,\qquad \gamma_{3,i}^{(4)}=\int_{-1}^{1}\sqrt{\frac{1-t}{1+t}}\,t^3W_i(t)\,dt, \qquad (40)$$

By applying the relations

$$\int_{-1}^{1}\sqrt{\frac{1-t}{1+t}}W_i(t)W_j(t)dt=\begin{cases}0 & i\neq j\\ \pi & i=j\end{cases} \qquad (41)$$

$$\int_{-1}^{1}\sqrt{\frac{1-t}{1+t}}\frac{W_i(t)}{t-x}dt=-\pi V_i(x) \qquad (42)$$

It is easy to estimate the values $u_{0,i}^{(4)}, u_{2,i}^{(4)}, \gamma_{0,i}^{(4)}$ and $\gamma_{3,i}^{(4)}$. From the relations (7) and Equations (39) to (42) we get

$$\alpha_i^{(4)}(x)=\begin{cases}\frac{-7\pi}{8}; & i=0\\ -\pi\left(2x^3+x^2-x-\frac{7}{8}\right); & i=1\\ -\pi\left(4x^4+2x^3-3x^2-x+\frac{1}{8}\right); & i=2\\ -\pi\left(8x^5+4x^4-8x^3-3x^2+x-\frac{1}{8}\right); & i=3\end{cases} \qquad (43)$$

By choosing the collocation points $x_k^{(4)}=\cos\left(\frac{(2k-1)\pi}{(2n+3)}\right),(k=1,2,3,4)$, for $n=3,$ we obtain the following system of linear equations:

$$\sum_{i=0}^{3}c_i^{(4)}\,\alpha_i^{(4)}(x_k^{(4)})=f(x_k^{(4)}),\quad k=1,2,3,4$$

By solving this system for the unknown coefficients $c_i^{(4)},\ i=0,1,2,3$ that produces

$$\left.\begin{matrix}c_0^{(4)}=0.3183098, & c_1^{(4)}=0.1591549\\ c_2^{(4)}=-0.1591549, & c_3^{(4)}=2.358931\times10^{-8}\end{matrix}\right\} \qquad (44)$$

From Equation (44) we obtain the approximate solution of Equation (17) in the form of

$$\varphi_n(x)\cong\frac{-2}{\pi}\sqrt{\frac{1-x}{1+x}}(x^2-1) \qquad (45)$$

Which coincides with the exact solution. The error of approximate Solution (45) of Equation 46 at $n=20$ is given in Table 1.

Example2 . Consider the following singular integral equation

$$\int_{-1}^{1}\frac{\varphi(t)}{t-x}\,dt+\int_{-1}^{1}(x^3+xt^2)\,\varphi(t)\,dt=x^3+x \tag{46}$$

Which corresponds with $k(t,x)=1$ and $L(t,x)=x^3+xt^2$. So one gets

$$k_0(x)=1,\quad k_p(x)=0,\qquad (p>0)$$
$$L_0(x)=x^3,\ L_1(x)=0,\ L_2(x)=x,\ L_q(x)=0\quad (q>2)$$

Hence we find that relation (10) produces

$$\sum_{i=0}^{n}c_i^{(j)}\alpha_i^{(j)}(x)=x^3+x \qquad -1<x<1,\ (j=1,2,3,4)$$

Thus (11) gives

$$\alpha_i^{(j)}(x)=u_{0,i}^{(j)}(x)+x^3\gamma_{0,i}^{(j)}+x\gamma_{2,i}^{(j)},\ \ (j=1,2,3,4),\ \ (i=0,1,2,\dots)$$

Firstly, let us consider in detail the case (I) , $j=1$, for $n=3$. This result in

$$\gamma_{2,i}^{(1)}=\int_{-1}^{1}\frac{t^2T_i(t)}{\sqrt{1-t^2}}\,dt, \tag{47}$$

From the relations (18) to (21) and (47) we obtain

$$\alpha_i^{(1)}(x)=\begin{cases}\pi\left(x^3+x/2\right) & i=0\\ \pi & i=1\\ 9\pi x/4 & i=2\\ \pi\left(4x^2-1\right) & i=3\end{cases} \tag{48}$$

By choosing the collocation points $x_k=\cos\left(\frac{(2k-1)\pi}{2(n+2)}\right),(k=1,2,3,4)$, for $n=3,$ we obtain the following system of linear equations:

$$\sum_{i=0}^{3}c_i^{(1)}\,\alpha_i^{(1)}(x_k)=f(x_k),\quad k=1,2,3,4$$

By solving this system for the unknown coefficients $c_i^{(1)},\ i=0,1,2,3$ that produces

$$\left.\begin{array}{ll}c_0^{(1)}=0.3183098, & c_1^{(1)}=1.090772\times10^{-8}\\ c_2^{(1)}=0.07073557, & c_3^{(1)}=1.830649\times10^{-8}\end{array}\right\} \tag{49}$$

From (49) we obtain the approximate solution of Equation 46 in the form of

$$\varphi_n(x)\cong\frac{1}{9\pi\sqrt{1-x^2}}\left(7+4x^2\right) \tag{50}$$

Which coincides with the exact solution. The error of approximate Solution (50) of Equation (46) at $n=20$ is given in Table 2.

Secondly, let us consider in detail the case (II) , $j=2$, for $n=3$. This result in

$$\gamma_{2,i}^{(2)}=\int_{-1}^{1}\sqrt{1-t^2}\,t^2\,U_i(t)\,dt, \tag{51}$$

By applying the relations (25)-(28) and (51) we get

$$\alpha_i^{(2)}(x)=\begin{cases}\frac{\pi}{2}\left(x^3-\frac{7x}{4}\right) & i=0\\ -\pi\left(2x^2-1\right) & i=1\\ -\pi\left(4x^3-\frac{25x}{8}\right) & i=2\\ -\pi\left(8x^4-8x^2+1\right) & i=3\end{cases} \tag{52}$$

By choosing the collocation points $x_k^{(2)}=\cos\left(\frac{(2k-1)\pi}{2(n+2)}\right),(k=1,2,3,4)$, for $n=3,$ we obtain the following system of linear equations:

$$\sum_{i=0}^{3}c_i^{(2)}\,\alpha_i^{(2)}(x_k^{(2)})=f(x_k^{(2)}),\quad k=1,2,3,4$$

By solving this system for the unknown coefficients $c_i^{(2)},\ i=0,1,2,3$ that produces

$$\left.\begin{array}{ll}c_0^{(2)}=-1.170559, & c_1^{(2)}=-1.331665\times10^{-9}\\ c_2^{(2)}=-0.2258973, & c_3^{(2)}=-1.644008\times10^{-8}\end{array}\right\} \tag{53}$$

Table 2. Errors of approximate solutions of Equation 46 in Case (I), Case (II) and Case (IV) respectively at *n*=20.

x	error (j=1)	error (j=2)	error (j=4)
-9.500000E-01	0.000000E+00	0.000000E+00	0.000000E+00
-9.000000E-01	5.960464E- 08	5.960464E- 08	0.000000E+00
-7.000000E-01	8.940697E- 08	1.192093E- 07	5.960464E- 08
-5.000000E-01	8.940697E- 08	1.192093E- 07	1.192093E- 07
-3.000000E-01	8.940697E- 08	1.788139E- 07	1.192093E- 07
-1.000000E-01	1.192093E- 07	1.788139E- 07	1.788139E- 07
0.000000E+00	1.043081E- 07	1.788139E- 07	1.788139E- 07
1.000000E-01	1.192093E- 07	1.788139E- 07	1.192093E- 07
3.000000E-01	8. 940697E-08	1.788139E- 07	5.960464E- 08
5.000000E-01	8.940697E- 08	1.192093E- 07	1.192093E-07
7.000000E-01	8.940697E- 08	1.192093E- 07	0.00000E+ 00
9.000000E-01	5.9604641E-08	5.960464E- 08	5.960464E-08
9.500000E-01	0.000000E+00	0.000000E+00	0.000000E+00

From Equation (53) we obtain the approximate solution of Equation (46) in the form of

$$\varphi_n(x) \cong \frac{-\sqrt{1-x^2}}{31\pi}\left[92+88x^2\right] \quad (54)$$

which coincides with the exact solution. The error of approximate solution (54) of Equation 46 at $n=20$ is given in Table 2.

Thirdly, In case (IV), $j=4$, for n=3. This result in

$$\gamma_{2,i}^{(4)} = \int_{-1}^{1}\sqrt{\frac{1-t}{1+t}}\; t^2 W_i(t)\,dt, \quad (55)$$

By applying the relations (39) to (42) and (55) we get

$$\alpha_i^{(4)}(x) = \begin{cases} \pi\left(x^3+\dfrac{x}{2}-1\right) & i=0 \\ -\pi\left(\dfrac{9}{4}x-1\right) & i=1 \\ -\pi\left(4x^2-\dfrac{9x}{4}-1\right) & i=2 \\ -\pi\left(8x^3-4x^2-4x+1\right) & i=3 \end{cases} \quad (56)$$

By choosing the collocation points $x_k^{(4)} = \cos\left(\dfrac{(2k-1)\pi}{(2n+3)}\right), (k=1,2,3,4)$, for $n=3,$ we obtain the following system of linear equations:

$$\sum_{i=0}^{3} c_i^{(4)}\,\alpha_i^{(4)}(x_k^{(4)}) = f(x_k^{(4)}), \quad k=1,2,3,4$$

By solving this system for the unknown coefficients $c_i^{(4)},\ i=0,1,2,3$ that produces

$$c_0^{(4)} = c_1^{(4)} = -.5852794, \quad c_2^{(4)} = c_3^{(4)} = -0.1129487 \quad (57)$$

From Equation (57) we obtain the approximate solution of Equation (46) in the form of

$$\varphi_n(x) \cong \frac{-1}{31\pi}\sqrt{\frac{1-x}{1+x}}(1+x)(92+88x^2) \quad (58)$$

which coincides with the exact solution. The error of approximate Solution (58) of Equation (46) at $n=20$ is given in Table 2.

Similarly, doing the same operations as we did for Case (I), Case (II) and Case (IV), one can solve for Case (III).

Example 3. Consider the following singular integral equation

$$\int_{-1}^{1}\frac{\varphi(t)}{t-x}dt + \int_{-1}^{1}(x^2+t^2)\varphi(t)\,dt = \frac{-3}{2}x^2+2x, \quad (59)$$

which corresponds with $k(t,x)=1$ and $L(t,x)=x^2+t^2$. So, one gets

$$k_0(x)=1, \quad k_p(x)=0, \qquad (p>0)$$
$$L_0(x)=x^2,\ L_1(x)=0,\ L_2(x)=1,\ L_q(x)=0 \ \ (q>2)$$

Table 3. Errors of approximate solutions of Equation 59 in Case (II) and Case (III) at *n*=20.

x	error (j=2)	error (j=3)
-9.500000E-01	2.980232E-08	2.980232E-08
-9.000000E-01	2.980232E-08	5.960464E-08
-7.000000E-01	0.000000E+00	5.960464E-08
-5.000000E-01	0.000000E+00	1.192093E-07
-3.000000E-01	0.000000E+00	1.192093E-07
-1.000000E-01	5.960464E-08	1.192093E-07
0.000000E+00	5.960464E-08	1.192093E-07
1.000000E-01	5.960464E-08	1.192093E-07
3.000000E-01	1.192093E-07	1.192093E-07
5.000000E-01	1.192093E-07	8.940697E-08
7.000000E-01	1.192093E-07	0.000000E+00
9.000000E-01	8.940697E-08	1.788139E-07
9.500000E-01	5.960464E-08	3.278255E-07

Hence the relation (10) produces

$$\sum_{i=0}^{n} c_i^{(j)} \alpha_i^{(j)}(x) = \frac{-3}{2}x^2 + 2x, \qquad -1 < x < 1,\ j = 1,2,3,4 \tag{60}$$

where Equation (11) gives

$$\alpha_i^{(j)}(x) = u_{0,i}^{(j)}(x) + x^2 \gamma_{0,i}^{(j)} + \gamma_{2,i}^{(j)},\ \ (j = 1,2,3,4),\ \ (i = 0,1,2,\ldots)$$

Firstly, let us consider in detail the case (II), $j = 2$, for $n = 3$. From Equation (25) to (28) and (51) we get

$$\alpha_i^{(2)}(x) = \begin{cases} \frac{\pi}{8}\left(4x^2 - 8x + 1\right) & i = 0 \\ -\pi\left(2x^2 - 1\right) & i = 1 \\ \frac{-\pi}{8}\left(32x^3 - 24x - 1\right) & i = 2 \\ -\pi\left(8x^4 - 8x^2 + 1\right) & i = 3 \end{cases} \tag{61}$$

By solving the system (60), at the collocation points $x_k^{(2)} = \cos\left(\frac{(2k-1)\pi}{2(n+2)}\right), (k = 1,2,3,4)$, for the unknown coefficients $c_i^{(2)},\ i = 0,1,2,3$ we obtain

$$\left.\begin{matrix} c_0^{(2)} = -0.6366197, & c_1^{(2)} = 0.07957754 \\ c_2^{(2)} = 1.746461\times 10^{-8}, & c_3^{(2)} = 1.827517\times 10^{-8} \end{matrix}\right\} \tag{62}$$

So the approximate solution of Equation 59 is given by

$$\varphi_n(x) \cong \frac{-\sqrt{1-x^2}}{2\pi}(4 - x), \tag{63}$$

Which coincides with the exact solution, the error of the approximate solution (63) of Equation 59 at $n = 20$ is given in Table 3.

Secondly, In case (III), $j = 3$, for n=3. This result in

$$\gamma_{2,i}^{(3)} = \int_{-1}^{1} \sqrt{\frac{1+t}{1-t}}\ t^2 V_i(t)\, dt, \tag{64}$$

From (32)-(35) and (64) we get

$$\alpha_i^{(3)}(x) = \begin{cases} \pi\left(x^2 + \frac{3}{2}\right) & i = 0 \\ \pi\left(2x + \frac{5}{4}\right) & i = 1 \\ \pi\left(4x^2 + 2x - \frac{3}{4}\right) & i = 2 \\ \pi\left(8x^3 + 4x^2 - 4x + 1\right) & i = 3 \end{cases} \tag{65}$$

By solving the system (60), at the collocation points $x_k^{(3)} = \cos\left(\frac{2k\pi}{(2n+3)}\right), (k = 1,2,3,4)$, for the unknown coefficients $c_i^{(3)},\ i = 0,1,2,3$ we obtain

$$\begin{matrix} c_0^{(3)} = -0.3183099, & c_1^{(3)} = 0.3580987, \\ c_2^{(3)} = -0.03978872, & c_3^{(3)} = -8.155105\times 10^{-9} \end{matrix} \tag{66}$$

Hence, the approximate solution of Equation 59 is given by

$$\varphi_n(x) \cong \frac{-1}{2\pi}\sqrt{\frac{1+x}{1-x}}\,(x^2-5x+4) \tag{67}$$

Which coincides with the exact solution, the error of the approximate solution (67) of Equation (59) at $n=20$ is given in Table 3.

Similarly, doing the same operations as we did for Case (II) and Case (III), one can solve for Case (I) and Case (IV). Table 1 illustrates errors of approximate solutions of Equation 17 in Cases (I)-(IV) at $n=20$. Table 2 illustrates errors of approximate solutions of Equation 46 in Case (I), Case (II) and Case (IV) respectively at $n=20$. Table 3 illustrates errors of approximate solutions of Equation 59 in Case (II) and Case (III) at $n=20$.

Conclusion

Numerical results (Tables 1, 2 and 3) show that the errors of approximate solutions of Examples 1-3 in different Cases with small value of n are very small. These show that the methods developed are very accurate and in fact for a linear function give the exact solution.

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