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Full Length Research Paper

A new approach of the radial basis functions method for telegraph equations

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This paper represents a new approach of the radial basis functions method for handling a class of the second-order hyperbolic telegraph equation. In this approach, we decompose the domain of the problem to a few sub domains as vertically or horizontally. The proposed approach is capable of reducing the size of calculations and easily overcomes the difficulty of solving complicated algebraic systems in large scale problems. To confirm the accuracy of the proposed approach, several examples are presented. The results of numerical experiments are presented with and without decomposition method and, will be compared with analytical solutions to confirm the convergence of the proposed method, good accuracy and the low computational time.

Key words: Radial basis functions, telegraph equations, decomposition domain.

INTRODUCTION

We consider the second-order linear hyperbolic telegraph equation in one-space dimension, given by

$$u_{tt}(x,t) + 2\alpha u_t(x,t) + \beta^2 u(x,t) = u_{xx}(x,t) + f(x,t), \quad a \leq x \leq b, \quad t \geq 0 \qquad (1)$$

Both the electric voltage and the current in a double conductor, satisfy the telegraph equation, where x is distance and t is time. Equations of the form (1) arise in the study of propagation of electrical signals in a cable of transmission line and wave phenomena. Interaction between convection and diffusion or reciprocal action of reaction and diffusion describes a number of nonlinear phenomena in physical, chemical and biological process (Mohebbi and Dehghan, 2008). In fact the telegraph equation is more suitable than ordinary diffusion equation in modeling reaction–diffusion for such branches of sciences (Dehghan and Ghesmati, 2010). For example, biologists encounter these equations in the study of pulsate blood flow in arteries and in one-dimensional random motion of bugs along a hedge (Mohanty, 2009). Also, the propagation of acoustic waves in Darcy-type porous media (Pascal, 1986); and parallel flows of viscous Maxwell fluids (Bohme, 1987) are just some of the phenomena governed (Evans and Bulut, 2003) by Equation (1). Some discussions about derivation of the telegraph equation are described in Mohebbi and Dehghan (2008).

In the last 20 years, the radial basis functions (RBFs) method is known as a powerful tool for scattered data interpolation problem. The use of RBFs as a meshless procedure for numerical solution of partial differential equations (PDEs) is based on the collocation scheme. Because of the collocation technique, this method does not need to evaluate any integral. The main advantage of numerical procedures which use RBFs over traditional techniques is meshless property of these methods. The RBFs were used actively for solving PDEs (Dehghan and Shokri, 2008). Also, Dehghan and Tatari (2006) used RBFs for finding the solution of an inverse problem with source control parameter.

In the last decade, the development of the RBFs as a truly meshless method for approximating the solutions of PDEs has drawn the attention of many researchers in science and engineering. One of the domain-type meshless methods has been obtained by directly collocating the RBFs, particularly the multiquadric (MQ), for the numerical approximation of the solution (Adibi and

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Table 1. Some well-known functions that generate RBFs.

Name of function	Definition
Gaussian (GA)	$\phi(r) = exp(-c^2r^2)$
Hardy Multiquadric (MQ)	$\phi(r) = \sqrt{r^2 + c^2}$
Inverse Multiquadric (IMQ)	$\phi(r) = \left(\sqrt{r^2 + c^2}\right)^{-1}$
Inverse Quadric (IQ)	$\phi(r) = (r^2 + c^2)^{-1}$

Es'haghi, 2007).

Dehghan and Shokri (2008) studied the numerical scheme to solve one and two-dimensional hyperbolic equations using collocation points and approximating directly the solution using the thin-plate-spline RBFs. The scheme works in a similar fashion as finite difference methods. In addition, Dehghan and Ghesmati (2010) used a numerical method based on the boundary integral equation (BIE) and an application of the dual reciprocity method (DRM) to solve the second-order one space-dimensional hyperbolic telegraph equation. In Dehghan and Ghesmati (2010) three different types of RBFs (cubic, thin plate spline and linear RBFs), were used to approximate functions in the dual reciprocity method (DRM). The traditional RBFs are globally defined functions which result in a full resultant coefficient matrix. This hinders the application of the RBFs to solve large scale problems due to severe ill-conditioning of the coefficient matrix (Dehghan and Tatari, 2006).

In this article, to overcome this ill-conditioning problem in large scale problem and to get good accuracy in short computational time, a new approach of the RBFs method is constructed based on decomposition domain idea. In this method, we decompose the domain of the problem into a few sub domains as vertically or horizontally.

Radial basis function approximation

Here, the RBFs method is defined as a technique for interpolation of the scattered data. Some well-known RBFs are listed in Table 1. Let r be the Euclidean distance between a fixed point $x^* \in \mathbb{R}^d$ and any $x \in \mathbb{R}^d$ that is $\|x - x^*\|_2$. A radial function $\phi^* = \phi(\|x - x^*\|_2)$ depends only on the distance between $x \in \mathbb{R}^d$ and fixed point $x^* \in \mathbb{R}^d$. This property concludes that, the RBF ϕ^* is radially symmetric about x^*. It is clear that the functions in Table 1 are globally supported, infinitely differentiable and depend on a free parameter c.

Let $x_1, x_2, \ldots, x_N$ be a given set of distinct points in $\mathbb{R}^d$. The main idea behind the use of RBFs is interpolation by translating a single function, that is, the RBFs interpolation is considered as

$$F(x) = \sum_{i=0}^{N} L_i R_i(x) \quad (2)$$

where $\phi_i(x) = \phi(\|x - x_i\|)$ and λ_i, $i = 1,\ldots,N$, are unknown scalars. For instance, consider the given values $f_i = f(x_i), i = 1,\ldots,N$. To compute the unknown scalars λ_i, we impose $F(x_j) = f_j$ for $j = 1,\ldots,N$ which can be written as the following linear system equations.

$$Az = f \quad (3)$$

where $A_{i,j} = \phi_i(x_j), z = [\lambda_1, \ldots, \lambda_N]\ and\ f = [f_1, \ldots, f_N]$.

Since all applicable ϕ have global support, this method produces a dense matrix A. The matrix A can be shown to be positive definite (and therefore nonsingular) for distinct interpolation points for GA, IMQ and IQ by Schoenberg's Theorem (Dehghan and Tatari, 2006). Also, using the Micchelli Theorem, we can show that A is invertible for distinct sets of the scattered points in the case of MQ (Micchelli, 1986).

Although the matrix A is nonsingular in the aforementioned cases, usually it is very ill-conditioned, that is, the condition number of A

$$\kappa_s(A) = \|A\|_s \|A^{-1}\|_s, \qquad s = 1, 2, \infty \quad (4)$$

is a very large number (Dehghan and Tatari, 2006). Therefore, a small perturbation in initial data may produce a large amount of perturbation in the solution. Thus, we have to use more precision arithmetic than the standard floating point arithmetic in our computation. For a fixed number of interpolation points, the condition number of A depends on the shape parameter c, support of the RBFs and minimum separation distance of interpolation points. Also, the condition number grows with N for fixed values of shape parameter c. In practice, the shape parameter c must be adjusted with the number of interpolating points in order to produce an interpolation matrix which is well conditioned enough to be inverted in finite precision arithmetic (Sarra, 2005).

Despite research done by many scientists to develop algorithms for selecting the values of c which produce the most accurate interpolation (Rippa, 1999), the optimal choice of shape parameter is still an open question.

Generally, for a fixed number of collocation points N, larger values of c produce better approximations, but the matrix A will be more ill-conditioned. Spectral accuracy is obtained in interpolating smooth data using global, infinitely differentiable RBFs (Shu et al., 2004). We suppose that, the second-order hyperbolic telegraph equation $L[u](x) = f(x)$ is a well-pose equation and has a unique solution.

If $f(x) \in L^2(\mathbb{R}^2)$ then the solution of $L[u](x) = f(x)$ belongs to $L^2(\mathbb{R}^2)$ and $L^2(\mathbb{R}^2)$ is a Hilbert space. If $\{p_i(x)\}$ is an orthonormal basis for $L^2(\mathbb{R}^2)$, then one can prove that each $u(x) \in L^2(\mathbb{R}^2)$ has the representation

$$u(x) = \sum_{i=1}^{\infty} \lambda_i p_i(x). \quad (5)$$

That is, (5) means by definition that

$$\|u(x) - \sum_{i=1}^{N} \lambda_i p_i(x)\| \to 0 \qquad as \qquad N \to \infty. \qquad (6)$$

Concerning the RBFs, however, this is not generally the case, for example Gaussian RBFs does not form an orthonormal basis for $L^2(\mathbb{R}^2)$. We now investigate about expansion property for RBFs. The family of RBF considered here consists of functions $q: \mathbb{R}^r \to \mathbb{R}$ represented by

$$q(x) = \sum_{i=1}^{N} w_i \, . \, K\left(\frac{x-z_i}{\sigma}\right) \qquad (7)$$

Where $N \in \mathbb{N}, \sigma > 0, \; w_i \in \mathbb{R}$, and $z_i \in \mathbb{R}^r$ for $i = 1,\ldots,M$. We call this family S_K.

Theorem

Let $K: \mathbb{R}^r \to \mathbb{R}$ be an integrable bounded function such that, K is continuous almost everywhere and $\int_{\mathbb{R}^r} K(x)dx \neq 0$. Then, the family S_K is dense in $L^p(\mathbb{R}^r)$ for every $p \in [1, \infty)$.

Proof: See Park and Sandberg, 1991

Note that there is no requirement of radial symmetry of the kernel function *K* in the aforestated theorem. Thus, the theorem is stronger than necessary for RBFs, and might be useful for other purposes. By *K* radially symmetric, we mean that $\|x\|_2 = \|y\|_2$ implies $K(x) = K(y)$. In this case, the function $g: [0, \infty) \to \mathbb{R}$ is obtained by defining $g(d) = K(z)$, where *z* is any element of $\mathbb{R}^r$ such that $\|z\|_2 = d$.

Therefore, in the case of radial symmetry, Equation (7) can be written as

$$q(x) = \sum_{i=1}^{N} w_i \, . \, K\left(\frac{x-z_i}{\sigma}\right) = \sum_{i=1}^{N} w_i \, . \, g\left(\frac{\|x-z_i\|_2}{\sigma}\right) \qquad (8)$$

If we choose the function K such that

$$K\left(\frac{x-z_i}{\sigma}\right) = g\left(\frac{\|x-z_i\|_2}{\sigma}\right) = e^{-\left(\frac{\|x-z_i\|_2}{\sigma}\right)^2}, \qquad (9)$$

The Gaussian RBFs have been introduced and all conditions in theorem have been satisfied, thus, the family S_K is dense in $L^2(\mathbb{R}^2)$.

We are now ready to discuss Kansa's collocation method. Assume that the domain $\Omega \subset \mathbb{R}^d$ and the second-order hyperbolic telegraph equation of the form

$$L[u](x) = f(x), \quad x \; in \; \Omega \qquad (10)$$

are given with (for simplicity of description) Dirichlet boundary conditions

$$u(x) = g(x), \quad x \; on \; \partial\Omega \qquad (11)$$

We expand u by RBFs, that is,

$$u(x) = \sum_{i=1}^{N} \lambda_i \, \phi(\|x - \xi_i\|) \qquad (12)$$

where the points $\xi_1, \ldots, \xi_N$ are a set of centers for the RBFs which are usually selected to coincide with the collocation points $\chi = \{x_1, \ldots, x_N\} \subset \Omega$. We assume the simplest possible setting here, that is, no polynomial terms is added to the expansion (12). The collocation matrix which arises when matching the differential Equation (10) and the boundary conditions (11) at the collocation points χ has below form Adibi and Es'haghi (2007);

$$A - \begin{pmatrix} \Phi \\ L[\Phi] \end{pmatrix}, \qquad (13)$$

where the two blocks are generated as follows:

$$\Phi_{j,k} = \phi(\|x_j - \xi_k\|), \qquad x_j \in B, \xi_k \in \chi, \qquad (14)$$

And

$$L[\Phi]_{j,k} = L[\phi](\|x_j - \xi_k\|), \qquad x_j \in I, \xi_k \in \chi, \qquad (15)$$

The set χ is split into a set I of interior points, and B of boundary points. The problem is well-posed if the linear system Ac = y, with y as a vector consisting of entries $g(x_j), x_j \in B$, followed by $f(x_j), x_j \in I$, has a unique solution. The collocation matrix A has not been proven to be non-singular but in Hon and Schaback (2001), it was shown that finding a numerically singular matrix was very rare.

Implementation of the domain decomposition method

Let us consider the following hyperbolic telegraph equation:

$$\frac{\partial^2 u}{\partial t^2} + 2\alpha\frac{\partial u}{\partial t} + \beta^2 u = \frac{\partial^2 u}{\partial x^2} + f(x,t), \quad x \in \Omega = [a,b] \subset \mathbb{R}, \quad 0 < t < T \qquad (16)$$

With the initial conditions

$$\begin{cases} u(x,0) = g_1(x), & x \in \Omega \\ u_t(x,0) = g_2(x), & x \in \Omega \end{cases} \qquad (17)$$

and Neumann boundary condition

$$\begin{cases} u_x(a,t) = K_1(x) & 0 < t \leq T \\ u_x(b,t) = K_2(x) & 0 < t \leq T \end{cases} \qquad (18)$$

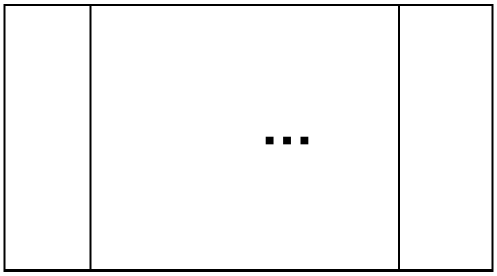

Figure 1. Vertically decomposition.

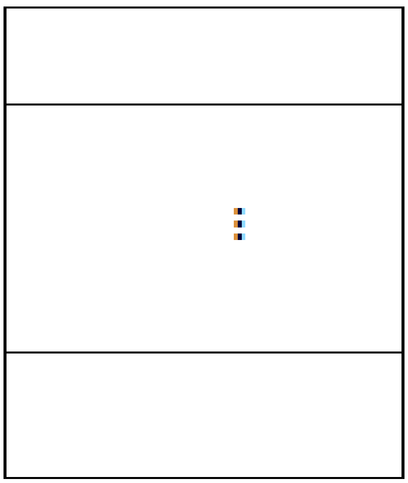

Figure 2. Horizontally decomposition.

In this method, we decompose Ω into a few sub domains as vertically, Figure 1, or horizontally, Figure 2.

We explain this method in two parts: vertically decomposition and horizontally decomposition. In the horizontal approach of domain decomposition, the Ω is assumed to be decomposed vertically into $\Omega_1, \ldots, \Omega_M$ with the same shape and area. Now, we use the RBFs for discretization of both time and space variables. Let $\Omega_k = \left\{\left(x_i^{(k)}, t_i^{(k)}\right), a \le x_i^{(k)} \le b, c_k \le t_i^{(k)} \le e_k, i = 1, \ldots, N\right\}, k = 1, \ldots, M$ be a set of scatted nodes. Where for getting better results $x_i^{(k)}\, and\; t_i^{(k)}$ are shifted to Chebyshev-Gauss-Lobatto nodes on $[a,b]\;\; and\;\; [c_k, e_k]$ respectively as follows:

$$x_i^{(k)} = \frac{b-a}{2}\left(\cos\left(\frac{(i-1)\pi}{N1-1}\right)\right) + \frac{b+a}{2} \tag{19}$$

$$t_i^{(k)} = \frac{e_k - c_k}{2}\left(\cos\left(\frac{(i-1)\pi}{N1-1}\right)\right) + \frac{e_k + c_k}{2} \tag{20}$$

We use the same discretization in all sub domains. We have a net with $N1 \times N1 = N$ nodes in each sub domain. Then the solution of the problem (16) to (18) on Ω_k is considered as follows:

$$\tilde{u}^{(k)}(x,t) = \sum_{i=1}^{N} \lambda_i^{(k)} \phi_i^{(k)}(x,t) \tag{21}$$

Where $\phi_i^{(k)}(x,t) = \phi\left(\left\|(x,t) - \left(x_i^{(k)}, t_i^{(k)}\right)\right\|_2\right)$ for the Gaussian RBF $\phi(d) = e^{-\frac{d^2}{c^2}}$ with $c = 3$ and $\lambda_i^{(k)}, i = 1, \ldots, N$ are unknown constants that must be found. The nodes $\left(x_i^{(k)}, t_i^{(k)}\right), i = 1, \ldots, N$ are centers for $\phi_i^{(k)}(x,t), i = 1, \ldots, N$.

If we consider collocation points the same as centers and (16) to (18) are approximated using (21) with these collocation points, we have

$$(2N1) + (N1-1) + (N1-1) + (N1-2)(N1-1) = (N1)^2 + N1$$

equations with $N = N1 \times N1$ unknowns.

To cope with this difficulty, we use of $(N1)(N1-1)$ collocation points on each sub domain. Again these collocation points are shifted to Chebyshev-Gauss-Lobatto nodes on $[a,b]$ and $[c_k, e_k]$ respectively as follows:

$$xx_i^{(k)} = \frac{b-a}{2}\left(\cos\left(\frac{(i-1)\pi}{N1-1}\right)\right) + \frac{b+a}{2} \tag{22}$$

$$tt_i^{(k)} = \frac{e_k - c_k}{2}\left(\cos\left(\frac{(i-1)\pi}{N1-2}\right)\right) + \frac{e_k + c_k}{2} \tag{23}$$

Now the collocation technique is used for finding unknowns $\lambda_i^{(k)}, i = 1, \ldots, N$. Let

$$\Omega_k = \Omega_1^{(k)} \cup \Omega_2^{(k)} \cup \Omega_3^{(k)} \cup \Omega_4^{(k)}, k = 1, \ldots, M$$

Where

$$\Omega_1^{(k)} = \left\{\left(xx_i^{(k)}, tt_i^{(k)}\right), a \le xx_i^{(k)} \le b, tt_i^{(k)} = c_k, i = 1, \ldots, N\right\}, k = 1, \ldots, M$$

$$\Omega_2^{(k)} = \left\{\left(xx_i^{(k)}, tt_i^{(k)}\right), xx_i^{(k)} = a, c_k < tt_i^{(k)} \le e_k, i = 1, \ldots, N\right\}, k = 1, \ldots, M$$

$$\Omega_3^{(k)} = \left\{\left(xx_i^{(k)}, tt_i^{(k)}\right), xx_i^{(k)} = b, c_k < tt_i^{(k)} \le e_k, i = 1, \ldots, N\right\}, k = 1, \ldots, M$$

$$\Omega_4^{(k)} = \left\{\left(xx_i^{(k)}, tt_i^{(k)}\right), a < xx_i^{(k)} < b, c_k < tt_i^{(k)} \le e_k, i = 1, \ldots, N\right\}, k = 1, \ldots, M$$

Also we assume $\Omega_i^{(k)} \neq \emptyset, for\; 1 \le i \le 4$.

Now (16) to (18) are approximated using (21), thus we have

$(2N1) + (N1-2) + (N1-2) + (N1-2)(N1-2) = (N1)^2$ equations with $N = N1 \times N1$ unknowns such that:

$$\sum_{i=1}^{N} \lambda_i^{(k)} \phi_i^{(k)}\left(xx_j^{(k)}, tt_j^{(k)}\right) = g_1\left(xx_j^{(k)}\right) \qquad \left(xx_j^{(k)}, tt_j^{(k)}\right) \in \Omega_1^{(k)}$$

$$\sum_{i=1}^{N} \lambda_i^{(k)} \frac{\partial}{\partial t} \phi_i^{(k)}\left(xx_j^{(k)}, tt_j^{(k)}\right) = g_2\left(xx_j^{(k)}\right) \qquad \left(xx_j^{(k)}, tt_j^{(k)}\right) \in \Omega_1^{(k)}$$

$$\sum_{i=1}^{N} \lambda_i^{(k)} \frac{\partial}{\partial x} \phi_i^{(k)}\left(xx_j^{(k)}, tt_j^{(k)}\right) = K_1\left(tt_j^{(k)}\right) \qquad \left(xx_j^{(k)}, tt_j^{(k)}\right) \in \Omega_2^{(k)}$$

$$\sum_{i=1}^{N} \lambda_i^{(k)} \frac{\partial}{\partial x} \phi_i^{(k)}\left(xx_j^{(k)}, tt_j^{(k)}\right) = K_2\left(tt_j^{(k)}\right) \qquad \left(xx_j^{(k)}, tt_j^{(k)}\right) \in \Omega_3^{(k)}$$

$$\sum_{i=1}^{N} \lambda_i^{(k)} \left[\frac{\partial^2}{\partial t^2} \phi_i^{(k)}\left(xx_j^{(k)}, tt_j^{(k)}\right) + 2\alpha \frac{\partial}{\partial t} \phi_i^{(k)}\left(xx_j^{(k)}, tt_j^{(k)}\right) + \beta^2 \phi_i^{(k)}\left(xx_j^{(k)}, tt_j^{(k)}\right) - \frac{\partial^2}{\partial x^2} \phi_i^{(k)}\left(xx_j^{(k)}, tt_j^{(k)}\right)\right] = f\left(xx_j^{(k)}, tt_j^{(k)}\right) \qquad \left(xx_j^{(k)}, tt_j^{(k)}\right) \in \Omega_4^{(k)}$$

These result to a linear system of equations in each sub domains. For the first sub domain we have $g_1(x)$ and $g_2(x)$ as initial conditions but for k-th sub domain we need to use solution and derivative of solution in (k-1)-th sub domain as initial conditions. These linear systems are solved one by one and the approximate solution of the problem (16) to (18) is obtained on each sub domain. Due to the radial property of RBFs and because of the identical distribution of centers and collocation points in each sub domain, the coefficient matrix remains unchanged in each sub domain. Concerning coefficient matrix, the LU factorization is applied only once and it will be utilized in our algorithm. This leads to saving a lot of time in computations.

In vertical approach of domain decomposition, as before processing, the collocation technique will be used for finding unknown $\lambda_i^{(k)}, i = 1, \dots, N$.

Let $\Omega_k = \Omega_1^{(k)} \cup \Omega_2^{(k)} \cup \Omega_3^{(k)} \cup \Omega_4^{(k)}, k = 1, \dots, M$

where

$$\Omega_1^{(k)} = \left\{\left(xx_i^{(k)}, tt_i^{(k)}\right), a_k \le xx_i^{(k)} \le b_k, tt_i^{(k)} = 0, i = 1, \dots, N\right\}, k = 1, \dots, M$$

$$\Omega_2^{(k)} = \left\{\left(xx_i^{(k)}, tt_i^{(k)}\right), xx_i^{(k)} = a_k, 0 < tt_i^{(k)} \le T, i = 1, \dots, N\right\}, k = 1, \dots, M$$

$$\Omega_3^{(k)} = \left\{\left(xx_i^{(k)}, tt_i^{(k)}\right), xx_i^{(k)} = b_k, 0 < tt_i^{(k)} \le T, i = 1, \dots, N\right\}, k = 1, \dots, M$$

$$\Omega_4^{(k)} = \left\{\left(xx_i^{(k)}, tt_i^{(k)}\right), a_k < xx_i^{(k)} < b_k, 0 < tt_i^{(k)} \le T, i = 1, \dots, N\right\}, k = 1, \dots, M$$

Also, we assume $\Omega_i^{(k)} \neq \emptyset, for\ 1 \le i \le 4$.

Now (16) to (18) are approximated using (21), thus we have

$$(2N1) + (N1-2) + (N1-2) + (N1-2)(N1-2) = (N1)^2$$

equations with $N = N1 \times N1$ unknowns such that:

$$\sum_{i=1}^{N} \lambda_i^{(k)} \phi_i^{(k)}\left(xx_j^{(k)}, tt_j^{(k)}\right) = g_1\left(xx_j^{(k)}\right) \qquad \left(xx_j^{(k)}, tt_j^{(k)}\right) \in \Omega_1^{(k)}$$

$$\sum_{i=1}^{N} \lambda_i^{(k)} \frac{\partial}{\partial t} \phi_i^{(k)}\left(xx_j^{(k)}, tt_j^{(k)}\right) = g_2\left(xx_j^{(k)}\right) \qquad \left(xx_j^{(k)}, tt_j^{(k)}\right) \in \Omega_1^{(k)}$$

$$\sum_{i=1}^{N} \lambda_i^{(k)} \frac{\partial}{\partial x} \phi_i^{(k)}\left(xx_j^{(k)}, tt_j^{(k)}\right) = \begin{cases} K_1\left(tt_j^{(k)}\right) & k = 1 \\ o[(k-2)(N1-2)+s] & k \neq 1 \end{cases} \qquad \left(xx_j^{(k)}, tt_j^{(k)}\right) \in \Omega_2^{(k)}$$

$$\sum_{i=1}^{N} \lambda_i^{(k)} \frac{\partial}{\partial x} \phi_i^{(k)}\left(xx_j^{(k)}, tt_j^{(k)}\right) = \begin{cases} K_2\left(tt_j^{(k)}\right) & k = M \\ z[(k-1)(N1-2)+r] & k \neq M \end{cases} \qquad \left(xx_j^{(k)}, tt_j^{(k)}\right) \in \Omega_3^{(k)}$$

That s and r vary from 1 to N1-2 and for every $\left(xx_j^{(k)}, tt_j^{(k)}\right) \in \Omega_2^{(k)}$ one unit will be added to s and for every $\left(xx_j^{(k)}, tt_j^{(k)}\right) \in \Omega_3^{(k)}$ one unit will be added to r and finally we have:

$$\sum_{i=1}^{N} \lambda_i^{(k)} \left[\frac{\partial^2}{\partial t^2} \phi_i^{(k)}\left(xx_j^{(k)}, tt_j^{(k)}\right) + 2\alpha \frac{\partial}{\partial t} \phi_i^{(k)}\left(xx_j^{(k)}, tt_j^{(k)}\right) + \beta^2 \phi_i^{(k)}\left(xx_j^{(k)}, tt_j^{(k)}\right) - \frac{\partial^2}{\partial x^2} \phi_i^{(k)}\left(xx_j^{(k)}, tt_j^{(k)}\right)\right] = f\left(xx_j^{(k)}, tt_j^{(k)}\right) \qquad \left(xx_j^{(k)}, tt_j^{(k)}\right) \in \Omega_4^{(k)}$$

In each sub domain we have a linear system of equations. Due to the radial property of RBFs and because of the identical distribution of centers and collocation points in each sub domain, the coefficient matrix remains unchanged in each sub domain. Concerning coefficient matrix, the LU factorization is applied only once and it will be utilized in our algorithm. The results have been achieved; however, a few unknown elements exist in the solution on each sub-domain. To find these elements, we impose the solutions in the sub-domains to get the same amount and the same derivative in the collocation points of their common boundaries (except for the collocation points in $t = 0$). This way, there will be a small linear system that can be solved more easily. Eventually, the solutions will be obtained without any unknown in each sub domains.

NUMERICAL RESULTS

Here we present some numerical results to test the efficiency of the new scheme for solving the hyperbolic telegraph equation.

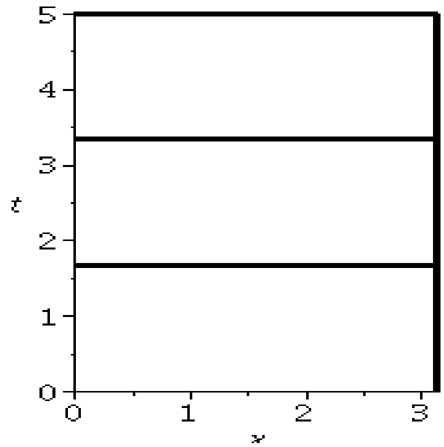

Figure 3. The shape of domain decomposition for example 1.

Table 2. Errors and computational time, without domain decomposition method, in example 1.

Computational domain	$L_\infty - error$	$L_2 - error$	RMS	Total time (s)
D	1.7×10^{-8}	5.5×10^{-5}	5.2×10^{-9}	265

Table 3. Errors and computational time, with domain decomposition method, in example 1.

Computational domain	$L_\infty - error$	$L_2 - error$	RMS	Total time (s)
D_1	2.3×10^{-9}	2.3×10^{-5}	7.2×10^{-10}	
D_2	9.5×10^{-10}	1.6×10^{-5}	3.7×10^{-10}	30
D_3	3.5×10^{-10}	1.0×10^{-5}	1.3×10^{-10}	

Example 1

In this example, we consider the hyperbolic telegraph Equation (1) with $\alpha = 4$ and $\beta = 2$ in the interval $0 \le x \le \pi$ and $0 \le t \le 5$. In this case we have $f(x,t) = -2\exp(-t)sin(x)$. The exact solution by Dehghan and Ghesmati (2010) is $u(x,t) = exp(-t)sin(x)$. The initial conditions are given by:

$$\begin{cases} u(x,0) = g_1(x) = \sin(x), & 0 \le x \le \pi \\ u_t(x,0) = g_2(x) = -\sin(x) & 0 \le x \le \pi \end{cases}$$

The Neumann boundary conditions obtained from the exact solution.

We decompose the domain

$D = \{(x,t) | 0 \le x \le \pi, 0 \le t \le 5\}$ into three sub domains

$$D_1 = \left\{(x,t) \middle| 0 \le x \le \pi, 0 \le t \le \frac{5}{3}\right\},$$
$$D_2 = \{(x,t) | 0 \le x \le \pi, \frac{5}{3} \le t \le \frac{10}{3}\} \quad ,$$
$$D_3 = \left\{(x,t) \middle| 0 \le x \le \pi, \frac{10}{3} \le t \le 5\right\}$$

The Figure 3 presents a view of domain decomposition.

We use GA-RBFs with $c = 3$, $\delta = 50$ (the number of floating point arithmetic) and $N = 256$ in the domain D without domain decomposition and use GA-RBFs with $c = 3$, $\delta = 50$ and $N = 121$ in three sub domains D_1, D_2 and D_3 with vertical domain decomposition.

The L_∞ and L_2 errors and Root-Mean-Square (RMS) of error and time of computations are presented and compared with and without domain decomposition in Tables 3 and 2 respectively. It should be mentioned that the total time required for getting the results in all sub domains is presented in Table 3.

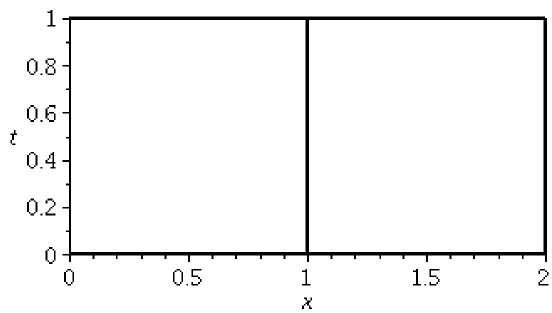

Figure 4. The shape of domain decomposition for example 2.

Table 4. Errors and computational time, without domain decomposition method, in example 2.

Computational domain	$L_\infty - error$	$L_2 - error$	RMS	Total time (s)
D	8.0×10^{-1}	4.6×10^{-1}	2.6×10^{-1}	272

Example 2

Consider the hyperbolic telegraph Equation (1) with $\alpha = 10$ and $\beta = 5$ in the interval $0 \le x \le 2$ and $0 \le t \le 1$. In this case we have

$$f(x,t) = 10\left(1 + tan^2\left(\frac{x+t}{2}\right)\right) + 25tan\left(\frac{x+t}{2}\right)$$

The exact solution by Dehghan and Ghesmati (2010) is $u(x,t) = tan\left(\frac{x+t}{2}\right)$. The initial conditions are given by

$$\begin{cases} u(x,0) = g_1(x) = tan\left(\frac{x}{2}\right), & 0 \le x \le 2 \\ u_t(x,0) = g_2(x) = \frac{1}{2}sec^2\left(\frac{x}{2}\right) & 0 \le x \le 2 \end{cases}$$

The Neumann boundary conditions obtained from the exact solution.
We decompose the domain

$D = \{(x,t)\,|0 \le x \le 2, 0 \le t \le 1\}$ into two sub domains
$D_1 = \{(x,t)|0 \le x \le 1, 0 \le t \le 1\}$,
$D_2 = \{(x,t)|1 \le x \le 2, 0 \le t \le 1\}$

The Figure 4 presents a view of domain decomposition.
We use GA-RBFs with $c = 3$, $\delta = 50$ and $N = 256$ in the domain D without domain decomposition and use GA-RBFs with $c = 3$, $\delta = 50$ and $N = 121$ in two sub domains D_1, D_2 with vertical domain decomposition. The L_∞ and L_2 errors and Root-Mean-Square (RMS) of error and time of computations are presented and compared without and with domain decomposition in Tables 4 and 5 respectively. It should be mentioned that the total time required for getting the results in all sub domains is presented in Table 5.

Example 3

In this example, we consider the hyperbolic telegraph Equation (1) with $\alpha = 6$ and $\beta = 2$ in the interval $0 \le x \le 4$ and $0 \le t \le 2$. In this case we have

$$f(x,t) = -12sin(t)sin(x) + 4cos(t)sin(x)$$

The exact solution by Dehghan and Ghesmati (2010) is $u(x,t) = cos(t)sin(x)$. The initial conditions are given by:

$$\begin{cases} u(x,0) = g_1(x) = sin(x), & 0 \le x \le 4 \\ u_t(x,0) = g_2(x) = 0 & 0 \le x \le 4 \end{cases}$$

The Neumann boundary conditions obtained from the exact solution.
We decompose the domain

$D = \{(x,t)\,|0 \le x \le 4, 0 \le t \le 2\}$ into two sub domains
$D_1 = \{(x,t)\,|0 \le x \le 2, 0 \le t \le 2\}$

Table 5. Errors and computational time, with domain decomposition method, in example 2.

Computational domain	$L_\infty - error$	$L_2 - error$	RMS	Total time (s)
D_1	1.3×10^{-3}	6.0×10^{-3}	1.5×10^{-4}	32
D_2	2.0×10^{-2}	4.2×10^{-2}	4.6×10^{-3}	

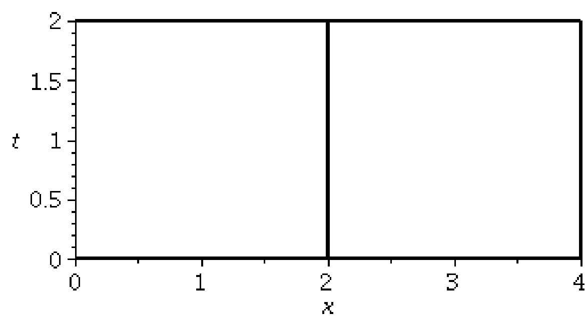

Figure 5. The shape of domain decomposition for example 3.

Table 6. Errors and computational time, without domain decomposition method, in example 3.

Computational domain	$L_\infty - error$	$L_2 - error$	RMS	Total time (s)
D	8.4×10^{-1}	4.4×10^{-1}	2.4×10^{-1}	266

Table 7. Errors and computational time, with domain decomposition method, in example 3.

Computational domain	$L_\infty - error$	$L_2 - error$	RMS	Total time (s)
D_1	2.7×10^{-9}	2.6×10^{-5}	9.4×10^{-10}	
D_2	2.7×10^{-9}	2.1×10^{-5}	7.0×10^{-10}	18

$$D_2 = \{(x,t) | 2 \leq x \leq 4, 0 \leq t \leq 2\}$$

The Figure 5 presents a view of domain decomposition. We use GA-RBFs with $c = 3$, $\delta = 40$ and $N = 256$ in the domain D without domain decomposition and use GA-RBFs with $c = 3$, $\delta = 40$ and $N = 100$ in two sub domains D_1, D_2 with vertical domain decomposition. The L_∞ and L_2 errors and Root-Mean-Square (RMS) of error and time of computations are presented and compared without and with domain decomposition in Tables 6 and 7 respectively. It should be mentioned that the total time required for getting the results in all sub domains is presented in Table 7.

Example 4

In this example, we consider the hyperbolic telegraph Equation (1) with $\alpha = \frac{1}{2}$ and $\beta = 1$ in the interval $1 \leq x \leq 4$ and $0 \leq t \leq 5$. In this case, we have

$$f(x,t) = (2 - 2t + t^2)(x - x^2)e^{-t} + 2t^2 e^{-t}$$

The exact solution by Dehghan and Ghesmati (2010) is $u(x,t) = (x - x^2)t^2 e^{-t}$. The initial conditions are given by

$$\begin{cases} u(x,0) = g_1(x) = 0, & 1 \leq x \leq 4 \\ u_t(x,0) - g_2(x) - 0 & 1 \leq x \leq 4 \end{cases}$$

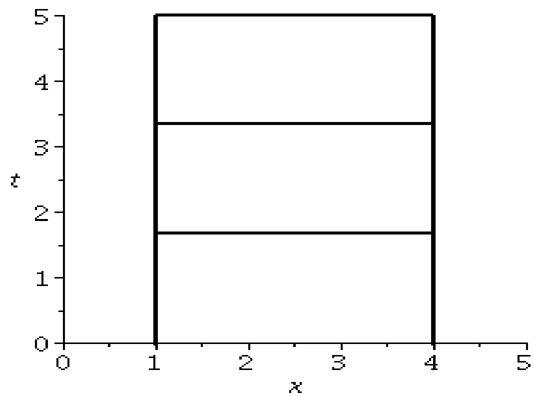

Figure 6. The shape of domain decomposition for example 4.

Table 8. Errors and computational time, without domain decomposition method, in example 4.

Computational domain	$L_\infty - error$	$L_2 - error$	RMS	Total time (s)
D	4.4×10^{-5}	2.9×10^{-3}	1.3×10^{-5}	172

Table 9. Errors and computational time, with domain decomposition method, in example 4.

Computational domain	$L_\infty - error$	$L_2 - error$	RMS	Total time (s)
D_1	2.2×10^{-6}	9.1×10^{-4}	1.1×10^{-6}	
D_2	1.5×10^{-6}	8.3×10^{-4}	7.9×10^{-7}	30
D_3	6.9×10^{-7}	5.7×10^{-4}	3.6×10^{-7}	

The Neumann boundary conditions obtained from the exact solution.

We decompose the domain $D = \{(x,t)\,|1 \le x \le 4, 0 \le t \le 5\}$ into three sub domains

$$D_1 = \left\{(x,t)\,\middle|1 \le x \le 4, 0 \le t \le \frac{5}{3}\right\}$$

$$D_2 = \{(x,t)|1 \le x \le 4, \frac{5}{3} \le t \le \frac{10}{3}\}$$

$$D_3 = \{(x,t)|1 \le x \le 4, \frac{10}{3} \le t \le 5\}$$

The Figure 6 presents a view of domain decomposition.

We use GA-RBFs with $c = 3$, $\delta = 40$ and $N = 225$ in the domain D without domain decomposition and use GA-RBFs with $c = 3$, $\delta = 40$ and $N = 121$ in three sub domains D_1, D_2, D_3 with horizontal domain decomposition.

The L_∞ and L_2 errors and Root-Mean-Square (RMS) of error and time of computations are presented and compared without and with domain decomposition in Tables 8 and 9 respectively . It should be mentioned that the total time required for getting the results in all sub domains is presented in Table 9.

Example 5

In this example, we consider the hyperbolic telegraph Equation (1) with $\alpha = 2$ and $\beta = 1$ in the interval $0 \le x \le 2$and $0 \le t \le 1$. In this case we have

$$f(x,t) = 20\left(\frac{1}{1 + 25(x+t)^2}\right) + arctan(5(x+t))$$

The exact solution is $u(x,t) = arctan\left(5(x+t)\right)$. The

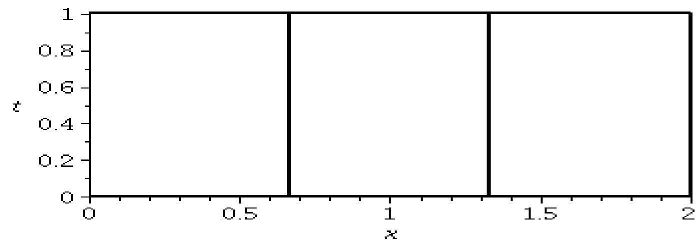

Figure 7. The shape of domain decomposition for example 5.

Table 10. Errors and computational time, without domain decomposition method, in example 5.

Computational domain	$L_\infty - error$	$L_2 - error$	RMS	Total time (s)
D	7.9×10^{-2}	1.5×10^{-1}	2.9×10^{-2}	274

Table 11. Errors and computational time, with domain decomposition method, in example 5.

Computational domain	$L_\infty - error$	$L_2 - error$	RMS	Total time (s)
D_1	9.3×10^{-4}	1.5×10^{-2}	3.4×10^{-4}	22
D_2	3.4×10^{-4}	6.3×10^{-3}	8.5×10^{-5}	
D_3	4.9×10^{-6}	7.3×10^{-4}	1.4×10^{-6}	

initial conditions are given by:

$$\begin{cases} u(x,0) = g_1(x) = arctan(5x), & 0 \le x \le 2 \\ u_t(x,0) = g_2(x) = \dfrac{5}{1+25x^2}, & 0 \le x \le 2 \end{cases}$$

The Neumann boundary conditions obtained from the exact solution.

We decompose the domain $D = \{(x,t)\,|1 \le x \le 2{,}0 \le t \le 1\}$ into three sub domains

$$D_1 = \left\{(x,t)\,\middle|0 \le x \le \frac{2}{3}, 0 \le t \le 1\right\}$$

$$D_2 = \{(x,t)|\frac{2}{3} \le x \le \frac{4}{3}, 0 \le t \le 1\}$$

$$D_3 = \{(x,t)|\frac{4}{3} \le x \le 2{,}0 \le t \le 1\}$$

The Figure 7 presents a view of domain decomposition. We use GA-RBFs with $c = 3$, $\delta = 50$ and $N = 256$ in the domain D without domain decomposition and use GA-RBFs with $c = 3$, $\delta = 50$ and $N = 100$ in three sub domains D_1, D_2, D_3 with vertical domain decomposition. The L_∞ and L_2 errors and Root-Mean-Square (RMS) of error and time of computations are presented and compared without and with domain decomposition in Tables 10 and 11 respectively. It should be mentioned that the total time required for getting the results in all sub domains is presented in Table 11.

Conclusion

The RBFs were used for solving a class of second-order hyperbolic telegraph equation. We proposed a numerical scheme to solve these hyperbolic equations using collocation points and estimated the solution directly using the GA RBFs. In this scheme, we decomposed the domain of problem into a few sub domains as vertically or horizontally. The numerical results demonstrate the high accuracy of the scheme proposed in this research in comparison with the classical methods lacking domain decomposition. Numerical results show that, the computational time and the accuracy of approximation solution are more advantageous. In addition we can cope with ill conditioning in some large scale problems with this technique because we can work with the smaller system of equations. Therefore the domain decomposition method can be an appropriate substitution for the classical ones, that is, without domain decomposition. It should be taken into account that collocation on the Chebyshev-Gauss-Lobatto is more accurate than that of the uniform grid. The proposed method can be extended to solve the nonlinear PDEs.

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