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On the Mazur-Ulam problem in non-Archimedean fuzzy 2-normed spaces

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Abstract
We study the notion of non-Archimedean fuzzy 2-normed space over a non-Archimedean field and prove that the Mazur-Ulam theorem holds under some conditions in the non-Archimedean fuzzy 2-normed space.
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1 Introduction

A mapping $f : X \longrightarrow Y$ is called an *isometry* if f satisfies

$$d_Y\big(f(x), f(y)\big) = d_X(x, y)$$

for all $x, y \in X$, where $d_X(\cdot,\cdot)$ and $d_Y(\cdot,\cdot)$ denote the metrics in the spaces X and Y, respectively.

The theory of isometric mappings originated in the classical paper [1] by Mazur and Ulam in 1932.

Mazur-Ulam theorem *Every isometry f of a normed real linear space X onto a normed real linear space is a linear mapping up to translation, that is, $x \mapsto f(x) - f(0)$ is linear, which amounts to the definition that f is affine.*

The Mazur-Ulam theorem is not true for a normed complex vector space. In addition, the onto assumption is also essential. Without this assumption, Baker [2] proved that an isometry from a normed real linear space into a strictly convex normed real linear space is affine.

Gähler [3, 4] introduced a new approach for a theory of 2-norm and n-norm on a linear space. Chu [5] studied the Mazur-Ulam theorem in linear 2-normed spaces. Recently, Moslehian and Sadeghi [6] introduced the Mazur-Ulam theorem in the non-Archimedean strictly convex normed spaces. Moreover, Mirmostafaee and Moslehian [7] introduced a non-Archimedean fuzzy norm on a linear space over a non-Archimedean field. In particular, Amyari and Sadeghi [8] proved Mazur-Ulam theorem under the condition of strict convexity in non-Archimedean 2-normed spaces.

In 1984, Katsaras [9] and Wu and Fang [10] introduced the notion of fuzzy norm, and also Wu and Fang gave the generalization of the Kolmogoroff normalized theorem for a fuzzy topological linear space. In addition, fuzzy n-normed linear spaces were studied by Narayanan and Vijayabalaji; see [11].

In this paper, we investigate the notion of non-Archimedean fuzzy 2-normed space over a linear ordered non-Archimedean field and prove that Mazur-Ulam theorem holds under some conditions in the non-Archimedean fuzzy 2-normed space.

Definition 1.1 A *non-Archimedean field* is a field $\mathcal{K}$ equipped with a (valuation) function from $\mathcal{K}$ into $[0, \infty)$ satisfying the following properties:

(1) $|a| \geq 0$ and equality holds if and only if $a = 0$,
(2) $|ab| = |a||b|$,
(3) $|a + b| \leq \max\{|a|, |b|\}$

for all $a, b \in \mathcal{K}$.

Clearly, $|1| = |-1| = 1$ and $|n| \leq 1$ for all $n \in \mathbb{N}$. An example of a non-Archimedean valuation is the function $|\cdot|$ taking everything except 0 into 1 and $|0| = 0$; see [12]. We call it a *non-Archimedean trivial valuation*. Also, the most important examples of non-Archimedean spaces are p-adic numbers; see [7].

Definition 1.2 Let X be a linear space over a field $\mathcal{K}$ with a non-Archimedean valuation $|\cdot|$. A function $\|\cdot\| : X \times X \longrightarrow [0, \infty)$ is said to be a *non-Archimedean 2-norm* if it satisfies the following properties:

(1) $\|x, y\| = 0$ if and only if x, y are linearly dependent,
(2) $\|x, y\| = \|y, x\|$,
(3) $\|cx, y\| = |c|\|x, y\|$,
(4) $\|x, y + z\| \leq \max\{\|x, y\|, \|x, z\|\}$

for all $x, y, z \in X$ and $c \in \mathcal{K}$. Then $(X, \|\cdot\|)$ is called a *non-Archimedean 2-normed space*.

Definition 1.3 Let X be a linear space over a field $\mathcal{K}$ with a non-Archimedean valuation $|\cdot|$. A function $N : X^2 \times \mathbb{R} \longrightarrow [0, 1]$ is said to be a *non-Archimedean fuzzy 2-norm* on X if for all $x, y \in X$ and all $s, t \in \mathbb{R}$,

(N1) $N(x, y, t) = 0$ for $t \leq 0$,
(N2) for $t > 0$, $N(x, y, t) = 1$ if and only if x and y are linearly dependent,
(N3) $N(x, y, t) = N(y, x, t)$,
(N4) $N(x, cy, t) = N(y, x, \frac{t}{|c|})$ for $c \neq 0$,
(N5) $N(x, y + z, \max\{s, t\}) \geq \min\{N(x, y, s), N(x, z, t)\}$,
(N6) $N(x, y, *)$ is a nondecreasing function of $\mathbb{R}$ and $\lim_{t\to\infty} N(x, y, t) = 1$.

The pair (X, N) is called a *non-Archimedean fuzzy 2-normed space.*

The property (N4) implies that $N(-x, y, t) = N(x, y, t)$ for all $x, y \in X$ and $t > 0$. It is easy to show that (N5) is equivalent to the following condition:

$$N(x, y + z, t) \geq \min\bigl\{N(x, y, t), N(x, z, t)\bigr\} \quad \text{for all } x, y, z \in X \text{ and } t \in \mathbb{R}.$$

Example 1.4 Let $(X, \|\cdot, \cdot\|)$ be a non-Archimedean 2-normed space. Define

$$N(x, y, t) = \begin{cases} \frac{t}{t + \|x, y\|} & \text{when } t > 0, t \in \mathbb{R}, \\ 0 & \text{when } t \leq 0, \end{cases}$$

where $x, y \in X$. Then (X, N) is a non-Archimedean fuzzy 2-normed space.

Definition 1.5 A non-Archimedean fuzzy 2-normed space is said to be *strictly convex* if $N(x, y + z, \max\{s, t\}) = \min\{N(x, y, s), N(x, z, t)\}$ and $N(x, y, s) = N(x, z, t)$ imply $y = z$ and $s = t$.

Definition 1.6 Let (X, N) and (Y, N) be two non-Archimedean fuzzy 2-normed spaces. We call $f : (X, N) \longrightarrow (Y, N)$ a *fuzzy 2-isometry* if $N(a - c, b - c, t) = N(f(a) - f(c), f(b) - f(c), t)$ for all $a, b, c \in X$ and $t > 0$.

Definition 1.7 Let X be a non-Archimedean fuzzy 2-normed space, and let a, b, c be mutually disjoint elements of X. Then a, b and c are said to be *collinear* if $b - c = r(a - c)$ for some real number r.

We denote the set of all elements of $\mathcal{K}$ whose norms are 1 by $\mathcal{C}$, that is,

$$\mathcal{C} = \big\{r \in \mathcal{K} \,|\, |r| = 1\big\}.$$

2 Main results

Lemma 2.1 *Let (X, N) be a non-Archimedean fuzzy 2-normed space over a linear ordered non-Archimedean field $\mathcal{K}$. Then*

$$N(x, y, t) = N(x, y + rx, t) \quad \text{for all } r \in \mathcal{K}.$$

Proof Let $x, y \in X$ and let $r \in \mathcal{K}$. Without loss of generality, we may assume $t > 0$. Then

$$N(x, y + rx, t) \geq \min\big\{N(x, y, t), N(x, rx, t)\big\} = N(x, y, t).$$

Conversely,

$$\begin{aligned} N(x, y, t) &= N(x, y + rx - rx, t) \geq \min\big\{N(x, y + rx, t), N(x, rx, t)\big\} \\ &= N(x, y + rx, t). \end{aligned}$$

Thus $N(x, y, t) = N(x, y + rx, t)$ for all $r \in \mathcal{K}$. □

Lemma 2.2 *Let (X, N) be a non-Archimedean fuzzy 2-normed space over a linear ordered non-Archimedean field $\mathcal{K}$ with $\mathcal{C} = \{2^n | n \in \mathbb{Z}\}$, and let $a, b, c \in X$ and $t > 0$. Suppose that X is strictly convex. Then $\alpha = \frac{a+b}{2}$ is the unique element of X such that*

$$N(a - c, a - \alpha, t) = N(b - \alpha, b - c, t) = N(a - c, b - c, t),$$

where a, b and α are collinear.

Proof Let $\alpha = \frac{a+b}{2} \in X$ and $t > 0$. By Lemma 2.1, we have

$$\begin{aligned} N(a - c, a - \alpha, t) &= N\left(a - c, a - \frac{a + b}{2}, t\right) \\ &= N\left(a - c, \frac{a - b}{2}, t\right) \end{aligned}$$

$$
\begin{aligned}
&= N\bigl(a-c, a-b, |2|t\bigr) \\
&= N(a-c, a-b, t) \\
&= N(a-c, b-c, t).
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
N(b-\alpha, b-c, t) &= N\left(b - \frac{a+b}{2}, b-c, t\right) = N(b-a, b-c, t) \\
&= N(a-c, b-c, t).
\end{aligned}
$$

Hence we have $N(a-c, a-\alpha, t) = N(a-c, b-c, t) = N(b-\alpha, b-c, t)$, that is, the existence part holds. To show the uniqueness part, assume that β is an element of X such that

$$
N(a-c, a-\beta, t) = N(b-\beta, b-c, t) = N(a-c, b-c, t),
$$

where a, b and β are collinear. Since a, b and β are collinear, there exists a real number s such that

$$
\beta = sa + (1-s)b.
$$

We may assume $s \neq 0$ and $s \neq 1$.

$$
\begin{aligned}
N(a-c, b-c, t) = N(a-c, a-\beta, t) &= N\bigl(a-c, a-\bigl(sa+(1-s)b\bigr), t\bigr) \\
&= N\left(a-c, a-b, \frac{t}{|1-s|}\right) \\
&= N\left(a-c, b-c, \frac{t}{|1-s|}\right).
\end{aligned}
$$

Similarly, we have

$$
N(a-c, b-c, t) = N\left(a-c, b-c, \frac{t}{|s|}\right),
$$

that is,

$$
N(a-c, b-c, t) = N\left(a-c, b-c, \frac{t}{|1-s|}\right) = N\left(a-c, b-c, \frac{t}{|s|}\right).
$$

We note that

$$
\begin{aligned}
&N\left(a-c+a-c, b-c, \max\left\{\frac{t}{|s|}, \frac{t}{|1-s|}\right\}\right) \\
&\quad \geq \min\left\{N\left(a-c, b-c, \frac{t}{|s|}\right), N\left(a-c, b-c, \frac{t}{|1-s|}\right)\right\} \\
&\quad = N\left(a-c, b-c, \frac{t}{|s|}\right) = N\left(a-c, b-c, \frac{t}{|1-s|}\right),
\end{aligned}
$$

and

$$\begin{aligned}
&N\left(a-c+a-c, b-c, \max\left\{\frac{t}{|s|}, \frac{t}{|1-s|}\right\}\right)\\
&\quad = N\left(2(a-c), b-c, \max\left\{\frac{t}{|s|}, \frac{t}{|1-s|}\right\}\right)\\
&\quad = N\left(a-c, b-c, \max\left\{\frac{t}{|s|}, \frac{t}{|1-s|}\right\}\right).
\end{aligned}$$

The previous note implies that

$$N(a-c, b-c, t) = N\left(a-c, b-c, \frac{t}{|s|}\right) = N\left(a-c, b-c, \frac{t}{|1-s|}\right).$$

The strict convexity of X implies that $|s| = |1-s| = 1$. Then there exist elements t_1 and t_2 in $\mathbb{Z}$ such that $1-s = 2^{t_1}$ and $s = 2^{t_2}$. Since $2^{t_1} + 2^{t_2} = 1$, we know that $t_1, t_2 < 0$. Without loss of generality, we let $1-s = 2^{-n_1}$ and $s = 2^{-n_2}$ with $n_1 \geq n_2$. If $n_1 \gneqq n_2$, then

$$1 = 2^{-n_1} + 2^{-n_2} = 2^{-n_1}\left(1 + 2^{n_1 - n_2}\right).$$

Hence $2^{n_1} = 1 + 2^{n_1 - n_2}$. This is a contradiction. Thus $n_1 = n_2$, that is, $s = \frac{1}{2}$. This implies that $\beta = \frac{a+b}{2} = \alpha$. Therefore the proof is completed. □

Theorem 2.3 *Let X and Y be non-Archimedean fuzzy 2-normed spaces over a linear ordered non-Archimedean field $\mathcal{K}$ with $\mathcal{C} = \{2^n | n \in \mathbb{Z}\}$. Let X and Y be strict convexities. Suppose that $f: X \longrightarrow Y$ is a fuzzy 2-isometry satisfying that $f(a), f(b)$ and $f(c)$ are collinear when a, b and c are collinear. Then $f(x) - f(0)$ is additive.*

Proof Let $g(x) = f(x) - f(0)$. Since f is a fuzzy 2-isometry, so is g. It is easy to show that if a, b and c are collinear, then $g(a)$, $g(b)$ and $g(c)$ are collinear. Since $g: X \longrightarrow Y$ is a fuzzy 2-isometry, we have

$$\begin{aligned}
N\left(g(a)-g(c), g(a)-g\left(\frac{a+b}{2}\right), t\right) &= N\left(a-c, a-\frac{a+b}{2}, t\right)\\
&= N(a-c, a-b, t) = N(a-c, b-c, t)\\
&= N\big(g(a)-g(c), g(b)-g(c), t\big).
\end{aligned}$$

Similarly, we get $N(g(b) - g(\frac{a+b}{2}), g(b) - g(c), t) = N(g(a) - g(c), g(b) - g(c), t)$. Hence

$$\begin{aligned}
N\left(g(a)-g(c), g(a)-g\left(\frac{a+b}{2}\right), t\right) &= N\left(g(b)-g\left(\frac{a+b}{2}\right), g(b)-g(c), t\right)\\
&= N\big(g(a)-g(c), g(b)-g(c), t\big).
\end{aligned}$$

By the uniqueness of Lemma 2.2, we have $g(\frac{a+b}{2}) = \frac{g(a)+g(b)}{2}$ for all $a, b \in X$. Thus $f(x) - f(0)$ is additive, as desired. □

Example 2.4 Let $\mathcal{K} = \mathbb{Z}_3$, where $\mathbb{Z}_3 = \{0, 1, 2\}$. Suppose that the field $\mathcal{K}$ has a non-Archimedean trivial valuation $|\cdot|$. Then $|2| = 1$, that is, $\mathcal{C} = \{2^n | n \in \mathbb{Z}\}$.

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References

1. Mazur, S, Ulam, S: Sur les transformation isométriques d'espaces vectoriels normés. C. R. Math. Acad. Sci. Paris **194**, 946-948 (1932)
2. Baker, JA: Isometries in normed spaces. Am. Math. Mon. **78**, 655-658 (1971)
3. Gähler, S: Linear 2-normierte Raume. Math. Nachr. **28**, 1-43 (1964)
4. Gähler, S: Untersuchungen über verallgemeinerte *m*-metrische Räume, I, II, III. Math. Nachr. **40**, 165-189 (1969)
5. Chu, H-Y: On the Mazur-Ulam problem in linear 2-normed spaces. J. Math. Anal. Appl. **327**, 1041-1045 (2007)
6. Moslehian, MS, Sadeghi, G: A Mazur-Ulam theorem in non-Archimedean normed spaces. Nonlinear Anal. **69**, 3405-3408 (2008)
7. Mirmostafaee, AK, Moslehian, MS: Stability of additive mappings in non-Archimedean fuzzy normed spaces. Fuzzy Sets Syst. **160**, 1643-1652 (2009)
8. Amyari, M, Sadeghi, G: Isometries in non-Archimedean 2-normed spaces. Preprint
9. Katsaras, AK: Fuzzy topological vector spaces II. Fuzzy Sets Syst. **12**, 143-154 (1984)
10. Wu, C, Fang, J: Fuzzy generalization of Kolmogoroffs theorem. J. Harbin Inst. Technol. **1**, 1-7 (1984)
11. Narayanan, A, Vijayabalaji, S: Fuzzy *n*-normed linear spaces. Int. J. Math. Math. Sci. **2005**(24), 3963-3977 (2005)
12. Narici, L, Beckenstein, E: Strange terrain-non-Archimedean spaces. Am. Math. Mon. **88**, 667-676 (1981)

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