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Necessary and sufficient conditions for the two parameter generalized Wilker-type inequalities

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Abstract
In the article, we provide the necessary and sufficient conditions for the parameters α and β such that the generalized Wilker-type inequality

$$\frac{2\beta}{\alpha+2\beta}\left(\frac{\sin x}{x}\right)^{\alpha}+\frac{\alpha}{\alpha+2\beta}\left(\frac{\tan x}{x}\right)^{\beta}-1>(<)0$$

holds for all $x \in (0, \pi/2)$.

MSC: 26D05; 33B10

Keywords: Wilker-type inequality; sine function; tangent function; necessary and sufficient condition

1 Introduction

The Wilker inequality [1, 2] for sine and tangent functions states that the inequality

$$\left(\frac{\sin x}{x}\right)^{2}+\frac{\tan x}{x}-2>0 \tag{1.1}$$

holds for all $x \in (0, \pi/2)$. The generalizations and improvements for the Wilker inequality (1.1) have been the subject of intensive research in the recent years. Wu and Srivastava [3] proved that the inequality

$$\frac{\lambda}{\lambda+\mu}\left(\frac{\sin x}{x}\right)^{p}+\frac{\mu}{\lambda+\mu}\left(\frac{\tan x}{x}\right)^{q}>1 \tag{1.2}$$

holds for all $x \in (0, \pi/2)$ if $\lambda > 0$, $\mu > 0$, $q > 0$ or $q \le \min\{-1, -\lambda/\mu\}$, and $p \le 2q\mu/\lambda$. Baricz and Sándor [4] generalized inequality (1.2) to the Bessel functions.

In [5], Zhu proved that the inequalities

$$\left(\frac{\sin x}{x}\right)^{2p}+\left(\frac{\tan x}{x}\right)^{p}>\left(\frac{x}{\sin x}\right)^{2p}+\left(\frac{x}{\tan x}\right)^{p}>2 \tag{1.3}$$

hold for $x \in (0, \pi/2)$ and $p \ge 1$. Matejíčka [6] presented the best possible parameter p such that the second inequality of (1.3) holds for $x \in (0, \pi/2)$.

Zhu [7] proved that the inequalities

$$(1-\lambda)\left(\frac{x}{\sin x}\right)^p+\lambda\left(\frac{x}{\tan x}\right)^p<1<(1-\eta)\left(\frac{x}{\sin x}\right)^p+\eta\left(\frac{x}{\tan x}\right)^p$$

are valid for all $x\in(0,\pi/2)$ if $(p,\lambda,\eta)\in\{(p,\lambda,\eta)|p\geq 1,\lambda\geq 1-(2/\pi)^p,\eta\leq 1/3\}\cup\{(p,\lambda,\eta)|0\leq p\leq 4/5,\lambda\geq 1/3,\eta\leq 1-(2/\pi)^p\}$.

In [8], Yang and Chu provided the necessary and sufficient condition for the parameter μ such that the generalized Wilker-type inequality

$$\frac{2}{\lambda+2}\left(\frac{\sin x}{x}\right)^{\lambda\mu}+\frac{\lambda}{\lambda+2}\left(\frac{\tan x}{x}\right)^{\mu}-1>(<)0$$

holds for any fixed $\lambda\geq 1$ and all $x\in(0,\pi/2)$.

Very recently, Chu *et al.* [9] proved that the two parameter generalized Wilker-type inequality

$$\frac{2\beta}{\alpha+2\beta}\left(\frac{\sin x}{x}\right)^{\alpha}+\frac{\alpha}{\alpha+2\beta}\left(\frac{\tan x}{x}\right)^{\beta}-1>0 \tag{1.4}$$

holds for all $x\in(0,\pi/2)$ if $(\alpha,\beta)\in E_0$, and inequality (1.4) is reversed if $(\alpha,\beta)\in E_1$, where

$$\begin{aligned}
E_0={}&\bigl\{(\alpha,\beta)|\alpha>0,\beta>0\bigr\}\cup\bigl\{(\alpha,\beta)|0<\alpha<-2\beta,\beta\geq -1\bigr\}\\
&\cup\left\{(\alpha,\beta)\middle|\beta>0,-\frac{12}{5}\leq\alpha+2\beta<0\right\}\\
&\cup\left\{(\alpha,\beta)\middle|\alpha\leq\frac{\pi^2}{4}-3,\beta\leq -1\right\}\\
&\cup\left\{(\alpha,\beta)\middle|\frac{\pi^2}{4}-3<\alpha<0,\beta\leq-\frac{37}{35},\alpha+2\beta+\frac{12}{5}\leq 0\right\},\\
E_1={}&\bigl\{(\alpha,\beta)|\alpha<0,\alpha+2\beta>0\bigr\}\cup\bigl\{(\alpha,\beta)|-1\leq\beta<0,\alpha+2\beta>0\bigr\}\\
&\cup\left\{(\alpha,\beta)\middle|-1\leq\beta<0,-2\beta-\frac{12}{5}\leq\alpha<0\right\}\cup\left\{(\alpha,\beta)\middle|0<\alpha\leq-2\beta-\frac{12}{5}\right\}.
\end{aligned}$$

The main purpose of this paper is to provide the necessary and sufficient conditions for the parameters α and β such that the generalized Wilker-type inequality (1.4) and its reversed inequality hold for all $x\in(0,\pi/2)$.

2 Lemmas

Lemma 2.1 (See [10], Lemma 2.3) *Let $-\infty<\alpha<\beta<\infty$, $f_1,f_2:[\alpha,\beta]\to\mathbb{R}$ be continuous on $[\alpha,\beta]$ and differentiable on (α,β), and $f_2'(x)\neq 0$ on (α,β). Then the inequality*

$$\frac{f_1(x)-f_1(\alpha)}{f_2(x)-f_2(\alpha)}>(<)\frac{f_1'(\alpha^+)}{f_2'(\alpha^+)}$$

holds for all $x\in(\alpha,\beta)$ if there exists $\eta\in(\alpha,\beta)$ such that $f_1'(x)/f_2'(x)$ is strictly increasing (*decreasing*) *on (α,η) and strictly decreasing* (*increasing*) *on (η,β), and*

$$\frac{f_1(\beta)-f_1(\alpha)}{f_2(\beta)-g_2(\alpha)}\geq(\leq)\frac{f_1'(\alpha^+)}{f_2'(\alpha^+)}\neq\infty.$$

Lemma 2.2 (See [9], Lemma 2.9) *Let* $\beta \in \mathbb{R}$, $x \in (0, \pi/2)$, *and* $\mathsf{F}(x)$, $\mathsf{G}(x)$, $\mathsf{H}(x)$ *and* $g(x)$ *be defined by*

$$\mathsf{F}(x) = \cos x(\sin x - x\cos x)^2(x - \sin x\cos x), \tag{2.1}$$

$$\mathsf{G}(x) = (x - \sin x\cos x)^2(\sin x - x\cos x), \tag{2.2}$$

$$\mathsf{H}(x) = x^3\left(\frac{\sin^2 x}{x^2} + \frac{\tan x}{x} - 2\right)\sin^2 x\cos x, \tag{2.3}$$

and

$$g(x) = \frac{\beta\mathsf{G}(x) + \mathsf{H}(x)}{\mathsf{F}(x)}, \tag{2.4}$$

respectively. Then the following statements are true:

(1) *The function* $g(x)$ *is strictly increasing from* $(0, \pi/2)$ *onto* $(2\beta + 12/5, 3 - \pi^2/4)$ *if* $\beta = -1$.

(2) *The function* $g(x)$ *is strictly increasing from* $(0, \pi/2)$ *onto* $(2\beta + 12/5, \infty)$ *if* $\beta > -1$.

(3) *The function* $g(x)$ *is strictly decreasing from* $(0, \pi/2)$ *onto* $(-\infty, 2\beta + 12/5)$ *if* $\beta \leq -37/35$.

Let $\alpha, \beta \in \mathbb{R}$, $x \in (0, \pi/2)$ and the functions $\mathsf{I}_\alpha(x)$, $\mathsf{J}_\beta(x)$ and $\mathsf{Q}_{\alpha,\beta}(x)$ be defined by

$$\mathsf{I}_\alpha(x) = \frac{1 - (\frac{\sin x}{x})^\alpha}{\alpha} \quad (\alpha \neq 0), \qquad \mathsf{I}_0(x) = \log x - \log(\sin x), \tag{2.5}$$

$$\mathsf{J}_\beta(x) = \frac{(\frac{\tan x}{x})^\beta - 1}{\beta} \quad (\beta \neq 0), \qquad \mathsf{J}_0(x) = \log(\tan x) - \log x, \tag{2.6}$$

and

$$\mathsf{Q}_{\alpha,\beta}(x) = \frac{\mathsf{I}_\alpha(x)}{\mathsf{J}_\beta(x)},$$

respectively.

Then it is not difficult to verify that

$$\mathsf{I}_\alpha(0^+) = \mathsf{J}_\beta(0^+) = 0,$$

$$\mathsf{Q}_{\alpha,\beta}(x) = \frac{\mathsf{I}_\alpha(x)}{\mathsf{J}_\beta(x)} = \frac{\mathsf{I}_\alpha(x) - \mathsf{I}_\alpha(0^+)}{\mathsf{J}_\beta(x) - \mathsf{J}_\beta(0^+)}, \tag{2.7}$$

$$\mathsf{Q}_{\alpha,\beta}(0^+) = \frac{1}{2}, \tag{2.8}$$

$$\mathsf{Q}_{\alpha,\beta}\left(\frac{\pi}{2}^-\right) = \frac{\beta}{\alpha}\left[\left(\frac{2}{\pi}\right)^\alpha - 1\right] \quad (\alpha \neq 0, \beta < 0), \tag{2.9}$$

$$\mathsf{Q}_{0,\beta}\left(\frac{\pi}{2}^-\right) = \lim_{\alpha \to 0} Q_{\alpha,\beta}\left(\frac{\pi}{2}^-\right) = \beta\log\frac{2}{\pi} \quad (\beta < 0). \tag{2.10}$$

Lemma 2.3 (See [9], Lemma 2.10) *Let* $x \in (0, \pi/2)$ *and* $\mathsf{Q}_{\alpha,\beta}(x)$ *be defined by* (2.7). *Then the following statements are true*:

(1) *If* $\alpha + 2\beta + 12/5 \geq 0$ *and* $\beta \geq -1$, *then* $\mathsf{Q}_{\alpha,\beta}(x)$ *is strictly decreasing on* $(0, \pi/2)$.
(2) *If* $\alpha \leq \pi^2/4 - 3$ *and* $-37/35 < \beta \leq -1$, *then* $\mathsf{Q}_{\alpha,\beta}(x)$ *is strictly increasing on* $(0, \pi/2)$.
(3) *If* $\alpha + 2\beta + 12/5 \leq 0$ *and* $\beta \leq -37/35$, *then* $\mathsf{Q}_{\alpha,\beta}(x)$ *is strictly increasing on* $(0, \pi/2)$.

Lemma 2.4 *Let* $x \in (0, \pi/2)$, $\mathsf{Q}_{\alpha,\beta}(x)$ *be defined by* (2.7) *and the function* $x \to \mathsf{D}(\alpha, \beta; x)$ *be defined by*

$$\mathsf{D}(\alpha, \beta; x) = \mathsf{Q}_{\alpha,\beta}(x) - \frac{1}{2}. \tag{2.11}$$

Then the following statements are true:
(1) *If* $\alpha \in \mathbb{R}$ *is fixed and* $\beta < 0$, *then there exists a unique solution* $\beta = \beta(\alpha)$ *given by*

$$\beta(\alpha) = \frac{\alpha}{2[(\frac{2}{\pi})^\alpha - 1]} \quad (\alpha \neq 0), \qquad \beta(0) = \frac{1}{2 \log \frac{2}{\pi}} \tag{2.12}$$

satisfies the equation $\mathsf{D}(\alpha, \beta; \frac{\pi}{2}^-) = 0$ *such that* $\mathsf{D}(\alpha, \beta; \frac{\pi}{2}^-) > 0$ *for* $\beta < \beta(\alpha)$ *and* $\mathsf{D}(\alpha, \beta; \frac{\pi}{2}^-) < 0$ *for* $\beta > \beta(\alpha)$.
(2) *If* $\beta < 0$ *is fixed, then there exists a unique solution* $\alpha = \alpha(\beta)$ *satisfies the equation* $\mathsf{D}(\alpha, \beta; \frac{\pi}{2}^-) = 0$ *such that* $\mathsf{D}(\alpha, \beta; \frac{\pi}{2}^-) > 0$ *for* $\alpha < \alpha(\beta)$ *and* $\mathsf{D}(\alpha, \beta; \frac{\pi}{2}^-) < 0$ *for* $\alpha > \alpha(\beta)$. *In particular, one has*

$$\alpha_0 = \alpha(-1) = -0.44367302\cdots, \qquad \alpha_0^* = \alpha\left(-\frac{37}{35}\right) = -0.20340978\cdots. \tag{2.13}$$

(3) *The two functions* $\alpha \to \beta(\alpha)$ *and* $\beta \to \alpha(\beta)$ *are strictly decreasing.*

Proof Part (1) follows easily from (2.9)-(2.11) and the fact that $[(2/\pi)^\alpha - 1]/\alpha < 0$.
(2) It follows from (2.9) and (2.11) that

$$\lim_{\alpha \to -\infty} \mathsf{D}\left(\alpha, \beta; \frac{\pi}{2}^-\right) = \infty, \qquad \lim_{\alpha \to \infty} \mathsf{D}\left(\alpha, \beta; \frac{\pi}{2}^-\right) = -\frac{1}{2}. \tag{2.14}$$

Note that

$$\frac{d}{d\alpha}\left[\frac{(\frac{2}{\pi})^\alpha - 1}{\alpha}\right] = \frac{(\frac{2}{\pi})^\alpha}{\alpha^2}\left[\log\left(\frac{2}{\pi}\right)^\alpha + \left(\frac{2}{\pi}\right)^{-\alpha} - 1\right] > 0 \tag{2.15}$$

for $\alpha \neq 0$.

From (2.9), (2.11), and (2.15) we clearly see that the function $\alpha \to \mathsf{D}(\alpha, \beta; \frac{\pi}{2}^-)$ is strictly decreasing. Therefore, there exists a unique solution $\alpha = \alpha(\beta)$ that satisfies the equation $\mathsf{D}(\alpha, \beta; \frac{\pi}{2}^-) = 0$ such that $\mathsf{D}(\alpha, \beta; \frac{\pi}{2}^-) > 0$ for $\alpha < \alpha(\beta)$ and $\mathsf{D}(\alpha, \beta; \frac{\pi}{2}^-) < 0$ for $\alpha > \alpha(\beta)$ follows from (2.14) and the monotonicity of the function $\alpha \to \mathsf{D}(\alpha, \beta; \frac{\pi}{2}^-)$. Numerical computations show that

$$\alpha(-1) = -0.44367302\cdots, \qquad \alpha\left(-\frac{37}{35}\right) = -0.20340978\cdots.$$

(3) The function $\alpha \to \beta(\alpha)$ is strictly decreasing follows easily from (2.12) and (2.15). The function $\beta \to \alpha(\beta)$ is strictly decreasing due to it is the inverse function of $\alpha \to \beta(\alpha)$. $\square$

Lemma 2.5 *Let $\beta(\alpha)$ be defined by* (2.12). *Then*

$$\alpha_1 = -0.36131140\cdots \tag{2.16}$$

is the unique solution of the equation $\beta(\alpha) = -\alpha/2 - 6/5$ such that $\beta(\alpha) < -\alpha/2 - 6/5$ for $\alpha < \alpha_1$ and $\beta(\alpha) > -\alpha/2 - 6/5$ for $\alpha > \alpha_1$.

Proof Let $P(\alpha) = \beta(\alpha) + \alpha/2 + 6/5$. Then from (2.12) we clearly see that

$$P(\alpha) = \frac{(\frac{2}{\pi})^\alpha}{2}\frac{\alpha}{(\frac{2}{\pi})^\alpha - 1} + \frac{6}{5},$$

$$\lim_{\alpha\to-\infty} P(\alpha) = -\infty, \qquad \lim_{\alpha\to\infty} P(\alpha) = \frac{6}{5}, \tag{2.17}$$

$$\frac{dP(\alpha)}{d\alpha} = -\frac{(\frac{2}{\pi})^\alpha}{2}\frac{\log(\frac{2}{\pi})^\alpha - (\frac{2}{\pi})^\alpha + 1}{[(\frac{2}{\pi})^\alpha - 1]^2} > 0 \tag{2.18}$$

for $\alpha \neq 0$, where the last of (2.18) due to $\log x - x + 1 < 0$ for all $x > 0$ with $x \neq 1$.

Inequality (2.18) implies that the function $\alpha \to P(\alpha)$ is strictly increasing on $(0,\infty)$. Therefore, there exists a unique $\alpha = \alpha_1$ that satisfies the equation $\beta(\alpha) = -\alpha/2 - 6/5$ such that $\beta(\alpha) < -\alpha/2 - 6/5$ for $\alpha < \alpha_1$ and $\beta(\alpha) > -\alpha/2 - 6/5$ for $\alpha > \alpha_1$ follows from (2.17) and the monotonicity of the function $\alpha \to P(\alpha)$. Numerical computations show that $\alpha_1 = -0.36131140\cdots$. □

Lemma 2.6 *Let $\mathsf{Q}_{\alpha,\beta}(x)$, $\beta(\alpha)$, α_0 and α_0^* be defined by* (2.7), (2.12), *and* (2.13), *respectively. Then the following statements are true*:

(1) *If $\alpha \geq -2/7 = -0.28571428\cdots$, then the inequality $\mathsf{Q}_{\alpha,\beta}(x) > 1/2$ holds for all $x \in (0,\pi/2)$ if and only if $\beta \leq -\alpha/2 - 6/5$.*

(2) *If $\alpha \geq \alpha_0^*$, then the inequality $\mathsf{Q}_{\alpha,\beta}(x) < 1/2$ holds for all $x \in (0,\pi/2)$ if and only if $\beta \geq \beta(\alpha)$.*

(3) *If $\alpha \leq -2/5$, then the inequality $\mathsf{Q}_{\alpha,\beta}(x) < 1/2$ holds for all $x \in (0,\pi/2)$ if and only if $\beta \geq -\alpha/2 - 6/5$.*

(4) *If $\alpha \leq \alpha_0$, then the inequality $\mathsf{Q}_{\alpha,\beta}(x) > 1/2$ holds for all $x \in (0,\pi/2)$ if and only if $\beta \leq \beta(\alpha)$.*

Proof (1) If $\alpha \geq -2/7$ and $\mathsf{Q}_{\alpha,\beta}(x) > 1/2$ for all $x \in (0,\pi/2)$, then from (2.5)-(2.7) one has

$$\lim_{x\to0^+} x^{-2}\left[\mathsf{Q}_{\alpha,\beta}(x) - \frac{1}{2}\right] = \lim_{x\to0^+} x^{-2}\left[-\frac{5\alpha+10\beta+12}{120}x^2 + o\left(x^2\right)\right] = -\frac{5\alpha+10\beta+12}{120} \geq 0,$$

which implies that $\beta \leq -\alpha/2 - 6/5$.

If $\alpha \geq -2/7$ and $\beta \leq -\alpha/2 - 6/5$, then we clearly see

$$\alpha + 2\beta + \frac{12}{5} \leq 0, \quad \beta \leq -\frac{37}{35}. \tag{2.19}$$

Therefore, $\mathsf{Q}_{\alpha,\beta}(x) > 1/2$ for all $x \in (0,\pi/2)$ follows from Lemma 2.3(3) and (2.8) together with (2.19).

(2) If $\alpha \geq \alpha_0^*$ and $\mathsf{Q}_{\alpha,\beta}(x) < 1/2$ for all $x \in (0, \pi/2)$, then from (2.11) and Lemma 2.4(1) we clearly see that $\mathsf{D}(\alpha, \beta; \frac{\pi}{2}^-) \leq 0$ and $\beta \geq \beta(\alpha)$.

Next, we prove that $\mathsf{Q}_{\alpha,\beta}(x) < 1/2$ for all $x \in (0, \pi/2)$ if $\alpha \geq \alpha_0^*$ and $\beta \geq \beta(\alpha)$. It follows from (2.6) and (2.7) together with the fact that

$$\frac{\partial \mathsf{J}_\beta(x)}{\partial \beta} = \frac{(\frac{\tan x}{x})^\beta}{\beta^2}\left[\log\left(\frac{\tan x}{x}\right)^\beta + \left(\frac{x}{\tan x}\right)^\beta - 1\right] > 0$$

for $x \in (0, \pi/2)$ and $\beta \neq 0$ that the function $\beta \to \mathsf{Q}_{\alpha,\beta}(x)$ is strictly decreasing. Therefore, it suffices to prove that $\mathsf{Q}_{\alpha,\beta}(x) < 1/2$ for all $x \in (0, \pi/2)$ if $\alpha \geq \alpha_0^*$ and $\beta = \beta(\alpha)$.

From (2.13) and Lemma 2.4(3) we get

$$\beta = \beta(\alpha) \leq \beta\left(\alpha_0^*\right) = -\frac{37}{35}. \tag{2.20}$$

Let $\alpha\beta \neq 0$, $\mathsf{F}(x)$, $\mathsf{G}(x)$, $\mathsf{H}(x)$, $g(x)$, $\mathsf{I}_\alpha(x)$ and $\mathsf{J}_\beta(x)$ be defined by (2.1)-(2.6), respectively. Then simple computations lead to

$$\left[\frac{\mathsf{I}'_\alpha(x)}{\mathsf{J}'_\beta(x)}\right]' = -\frac{\cos^\beta x \sin^{\alpha-\beta-1} x}{(x - \sin x \cos x)^2}\left[g(x) + \alpha\right]\mathsf{F}(x)x^{\beta-\alpha-1}, \tag{2.21}$$

$$\mathsf{J}'_\beta(x) = \frac{2x - \sin(2x)}{2x^2\cos^2 x}\left(\frac{\tan x}{x}\right)^{\beta-1} > 0 \tag{2.22}$$

for $x \in (0, \pi/2)$.

Let $\alpha_1 = -0.36131140\cdots$ be defined by (2.16). Then it follows from Lemma 2.2(3), Lemma 2.5, and (2.20) together with $\alpha \geq \alpha_0^* = -0.20340978\cdots > \alpha_1$ that the function $x \to g(x) + \alpha$ is strictly decreasing on $(0, \pi/2)$ and

$$\lim_{x\to 0^+}\left[g(x) + \alpha\right] = \alpha + 2\beta + \frac{12}{5} > 0, \qquad \lim_{x\to \frac{\pi}{2}^-}\left[g(x) + \alpha\right] = -\infty. \tag{2.23}$$

From (2.21) and (2.23) together with the monotonicity of the function $x \to g(x) + \alpha$ on the interval $(0, \pi/2)$ we clearly see that there exists $x_0 \in (0, \pi/2)$ such that the function $x \to \mathsf{I}'_\alpha(x)/\mathsf{J}'_\beta(x)$ is strictly decreasing on $(0, x_0)$ and strictly increasing on $(x_0, \pi/2)$.

Note that

$$\frac{\mathsf{I}_\alpha(\frac{\pi}{2}^-) - \mathsf{I}_\alpha(0^+)}{\mathsf{J}_{\beta(\alpha)}(\frac{\pi}{2}^-) - \mathsf{J}_{\beta(\alpha)}(0^+)} = \mathsf{Q}_{\alpha,\beta(\alpha)}\left(\frac{\pi}{2}^-\right) = \mathsf{D}\left(\alpha, \beta(\alpha); \frac{\pi}{2}^-\right) + \frac{1}{2} = \frac{1}{2}. \tag{2.24}$$

Therefore, $\mathsf{Q}_{\alpha,\beta}(x) < 1/2$ for all $x \in (0, \pi/2)$ follows from Lemma 2.1, (2.7), (2.22), (2.24), and the piecewise monotonicity of the function $x \to \mathsf{I}'_\alpha(x)/\mathsf{J}'_\beta(x)$ on the interval $(0, \pi/2)$.

(3) If $\alpha \leq -2/5$ and $\mathsf{Q}_{\alpha,\beta}(x) < 1/2$ for all $x \in (0, \pi/2)$, then from (2.5)-(2.7) we have

$$\lim_{x\to 0^+} x^{-2}\left[\mathsf{Q}_{\alpha,\beta}(x) - \frac{1}{2}\right] = \lim_{x\to 0^+} x^{-2}\left[-\frac{5\alpha + 10\beta + 12}{120}x^2 + o\left(x^2\right)\right] = -\frac{5\alpha + 10\beta + 12}{120} \leq 0,$$

which implies that $\beta \geq -\alpha/2 - 6/5$.

If $\alpha \leq -2/5$ and $\beta \geq -\alpha/2 - 6/5$, then we clearly see that

$$\alpha + 2\beta + \frac{12}{5} \geq 0, \quad \beta \geq -1. \tag{2.25}$$

Therefore, $\mathsf{Q}_{\alpha,\beta}(x) < 1/2$ for $x \in (0,\pi/2)$ follows easily from Lemma 2.3(1), (2.8), and (2.25).

(4) If $\alpha \le \alpha_0$ and $\mathsf{Q}_{\alpha,\beta}(x) > 1/2$ for all $x \in (0,\pi/2)$, then (2.11) and Lemma 2.4(1) lead to the conclusion that $\mathsf{D}(\alpha,\beta;\frac{\pi}{2}^-) \ge 0$ and $\beta \le \beta(\alpha)$.

Next, we prove that $\mathsf{Q}_{\alpha,\beta}(x) > 1/2$ for all $x \in (0,\pi/2)$ if $\alpha \le \alpha_0$ and $\beta \le \beta(\alpha)$. Since the function $\beta \to \mathsf{Q}_{\alpha,\beta}(x)$ is strictly decreasing which was proved in part (2), we only need to prove that $\mathsf{Q}_{\alpha,\beta}(x) > 1/2$ for all $x \in (0,\pi/2)$ if $\alpha \le \alpha_0$ and $\beta = \beta(\alpha)$. It follows from Lemma 2.2(1) and (2), Lemma 2.4(3), Lemma 2.5, and $\alpha \le \alpha_0 < \alpha_1$ that $\beta \ge \beta(\alpha_0) = -1$ and the function $g(x) + \alpha$ is strictly increasing on $(0,\pi/2)$ such that

$$\lim_{x\to 0^+}\bigl[g(x)+\alpha\bigr] = \alpha + 2\beta + \frac{12}{5} < 0, \tag{2.26}$$

$$\begin{aligned}\lim_{x\to \frac{\pi}{2}^-}\bigl[g(x)+\alpha\bigr] &= \begin{cases}\alpha + \infty, & \beta(\alpha) > -1,\\ \alpha + 3 - \frac{\pi^2}{4}, & \beta(\alpha) = -1,\end{cases}\\ &= \begin{cases}\infty, & \beta > -1,\\ \alpha_0 + 3 - \frac{\pi^2}{4} > 0, & \beta = -1.\end{cases}\end{aligned} \tag{2.27}$$

From (2.21), (2.26), and (2.27) we clearly see that there exists $x^* \in (0,\pi/2)$ such that the function $x \to \mathsf{I}'_\alpha(x)/\mathsf{J}'_\beta(x)$ is strictly increasing on $(0,x^*)$ and strictly decreasing on $(x^*,\pi/2)$. Therefore, $\mathsf{Q}_{\alpha,\beta}(x) > 1/2$ for all $x \in (0,\pi/2)$ follows from Lemma 2.1, (2.7), (2.22), (2.24), and the piecewise monotonicity of the function $x \to \mathsf{I}'_\alpha(x)/\mathsf{J}'_\beta(x)$ on the interval $(0,\pi/2)$. □

Lemma 2.7 *Let* $\mathsf{Q}_{\alpha,\beta}(x)$, α_0, α_0^* *and* $\alpha(\beta)$ *be defined by* (2.7) *and Lemma* 2.4, *respectively. Then the following statements are true*:

(1) *If* $\beta \ge -1$, *then the inequality* $\mathsf{Q}_{\alpha,\beta}(x) < 1/2$ *holds for all* $x \in (0,\pi/2)$ *if and only if* $\alpha \ge -2\beta - 12/5$.

(2) *If* $-1 \le \beta < 0$, *then the inequality* $\mathsf{Q}_{\alpha,\beta}(x) > 1/2$ *holds for all* $x \in (0,\pi/2)$ *if and only if* $\alpha \le \alpha(\beta)$.

(3) *If* $\beta \le -37/35$, *then the inequality* $\mathsf{Q}_{\alpha,\beta}(x) > 1/2$ *holds for all* $x \in (0,\pi/2)$ *if and only if* $\alpha \le -2\beta - 12/5$.

(4) *If* $\beta \le -37/35$, *then the inequality* $\mathsf{Q}_{\alpha,\beta}(x) < 1/2$ *holds for all* $x \in (0,\pi/2)$ *if and only if* $\alpha \ge \alpha(\beta)$.

Proof (1) If $\beta \ge -1$ and $\mathsf{Q}_{\alpha,\beta}(x) < 1/2$ for all $x \in (0,\pi/2)$, then from (2.5)-(2.7) we get

$$\lim_{x\to 0^+} x^{-2}\left[\mathsf{Q}_{\alpha,\beta}(x) - \frac{1}{2}\right] = \lim_{x\to 0^+} x^{-2}\left[-\frac{5\alpha+10\beta+12}{120}x^2 + o\bigl(x^2\bigr)\right] = -\frac{5\alpha+10\beta+12}{120} \le 0,$$

which implies that $\alpha \ge -2\beta - 12/5$.

If $\beta \ge -1$ and $\alpha \ge -2\beta - 12/5$, then $\mathsf{Q}_{\alpha,\beta}(x) < 1/2$ for all $x \in (0,\pi/2)$ follows from (2.8) and Lemma 2.3(1).

(2) If $-1 \le \beta < 0$ and $\mathsf{Q}_{\alpha,\beta}(x) > 1/2$ for all $x \in (0,\pi/2)$, then (2.11) and Lemma 2.4(2) lead to the conclusion that $\mathsf{D}(\alpha,\beta;\frac{\pi}{2}^-) \ge 0$ and $\alpha \le \alpha(\beta)$.

Next, we prove that $\mathrm{Q}_{\alpha,\beta}(x) > 1/2$ for all $x \in (0, \pi/2)$ if $-1 \leq \beta < 0$ and $\alpha \leq \alpha(\beta)$. It follows from $-1 \leq \beta < 0$ and $\alpha \leq \alpha(\beta)$ together with Lemma 2.4(3) that

$$\alpha \leq \alpha(-1) = \alpha_0, \quad \beta \leq \beta(\alpha). \tag{2.28}$$

Therefore, $\mathrm{Q}_{\alpha,\beta}(x) > 1/2$ for all $x \in (0, \pi/2)$ follows from Lemma 2.6(4) and (2.28).

(3) If $\beta \leq -37/35$ and $\mathrm{Q}_{\alpha,\beta}(x) > 1/2$ for all $x \in (0, \pi/2)$, then from (2.5)-(2.7) we have

$$\lim_{x \to 0^+} x^{-2}\left[\mathrm{Q}_{\alpha,\beta}(x) - \frac{1}{2}\right] = \lim_{x \to 0^+} x^{-2}\left[-\frac{5\alpha + 10\beta + 12}{120}x^2 + o\left(x^2\right)\right] = -\frac{5\alpha + 10\beta + 12}{120} \geq 0,$$

which implies that $\alpha \leq -2\beta - 12/5$.

If $\beta \leq -37/35$ and $\alpha \leq -2\beta - 12/5$, then $\mathrm{Q}_{\alpha,\beta}(x) > 1/2$ for all $x \in (0, \pi/2)$ follows from (2.8) and Lemma 2.3(3).

(4) If $\beta \leq -37/35$ and $\mathrm{Q}_{\alpha,\beta}(x) < 1/2$ for all $x \in (0, \pi/2)$, then (2.11) and Lemma 2.4(2) lead to the conclusion that $\mathrm{D}(\alpha, \beta; \frac{\pi}{2}^-) \leq 0$ and $\alpha \geq \alpha(\beta)$.

Next, we prove that $\mathrm{Q}_{\alpha,\beta}(x) < 1/2$ for all $x \in (0, \pi/2)$ if $\beta \leq -37/35$ and $\alpha \geq \alpha(\beta)$. It follows from $\beta \leq -37/35$ and $\alpha \geq \alpha(\beta)$ together with Lemma 2.4(3) that

$$\alpha \geq \alpha\left(-\frac{37}{35}\right) = \alpha_0^*, \quad \beta \geq \beta(\alpha). \tag{2.29}$$

Therefore, the desired result follows from Lemma 2.6(2) and (2.29). □

3 Main results

Let $\alpha, \beta \in \mathbb{R}$ with $\alpha\beta(\alpha + 2\beta) \neq 0$ and $\mathrm{Q}_{\alpha,\beta}(x)$ be defined by (2.7), then we clearly see that the generalized Wilker-type inequality

$$\frac{2\beta}{\alpha + 2\beta}\left(\frac{\sin x}{x}\right)^\alpha + \frac{\alpha}{\alpha + 2\beta}\left(\frac{\tan x}{x}\right)^\beta - 1 > 0 \tag{3.1}$$

holds for all $x \in (0, \pi/2)$ if and only if $\mathrm{Q}_{\alpha,\beta}(x) < 1/2$ and $\alpha\beta(\alpha + 2\beta) > 0$ or $\mathrm{Q}_{\alpha,\beta}(x) > 1/2$ and $\alpha\beta(\alpha + 2\beta) < 0$, while the generalized Wilker-type inequality

$$\frac{2\beta}{\alpha + 2\beta}\left(\frac{\sin x}{x}\right)^\alpha + \frac{\alpha}{\alpha + 2\beta}\left(\frac{\tan x}{x}\right)^\beta - 1 < 0 \tag{3.2}$$

holds for all $x \in (0, \pi/2)$ if and only if $\mathrm{Q}_{\alpha,\beta}(x) < 1/2$ and $\alpha\beta(\alpha + 2\beta) < 0$ or $\mathrm{Q}_{\alpha,\beta}(x) > 1/2$ and $\alpha\beta(\alpha + 2\beta) > 0$.

From Lemmas 2.6 and 2.7 together with inequalities (3.1) and (3.2) we get Theorems 3.1 and 3.2 immediately.

Theorem 3.1 *Let $\alpha, \beta \in \mathbb{R}$ with $\alpha\beta(\alpha + 2\beta) \neq 0$, $\beta(\alpha)$, α_0 and α_0^* be defined by* (2.12) *and* (2.13), *respectively. Then the following statements are true*:

(1) *If $\alpha \geq -2/7$, then inequality* (3.1) *holds for all $x \in (0, \pi/2)$ if and only if $(\alpha, \beta) \in \{(\alpha, \beta) | \beta \leq -\alpha/2 - 6/5, \alpha\beta(\alpha + 2\beta) < 0\}$ and inequality* (3.2) *holds for all $x \in (0, \pi/2)$ if and only if $(\alpha, \beta) \in \{(\alpha, \beta) | \beta \leq -\alpha/2 - 6/5, \alpha\beta(\alpha + 2\beta) > 0\}$.*

(2) *If* $\alpha \geq \alpha_0^*$, *then inequality* (3.1) *holds for all* $x \in (0, \pi/2)$ *if and only if* $(\alpha, \beta) \in \{(\alpha, \beta) | \beta \geq \beta(\alpha), \alpha\beta(\alpha + 2\beta) > 0\}$ *and inequality* (3.2) *holds for all* $x \in (0, \pi/2)$ *if and only if* $(\alpha, \beta) \in \{(\alpha, \beta) | \beta \geq \beta(\alpha), \alpha\beta(\alpha + 2\beta) < 0\}$.

(3) *If* $\alpha \leq -2/5$, *then inequality* (3.1) *holds for all* $x \in (0, \pi/2)$ *if and only if* $(\alpha, \beta) \in \{(\alpha, \beta) | \beta \geq -\alpha/2 - 6/5, \alpha\beta(\alpha + 2\beta) > 0\}$ *and inequality* (3.2) *holds for all* $x \in (0, \pi/2)$ *if and only if* $(\alpha, \beta) \in \{(\alpha, \beta) | \beta \geq -\alpha/2 - 6/5, \alpha\beta(\alpha + 2\beta) < 0\}$.

(4) *If* $\alpha \leq \alpha_0$, *then inequality* (3.1) *holds for all* $x \in (0, \pi/2)$ *if and only if* $(\alpha, \beta) \in \{(\alpha, \beta) | \beta \leq \beta(\alpha), \alpha\beta(\alpha + 2\beta) < 0\}$ *and inequality* (3.2) *holds for all* $x \in (0, \pi/2)$ *if and only if* $(\alpha, \beta) \in \{(\alpha, \beta) | \beta \leq \beta(\alpha), \alpha\beta(\alpha + 2\beta) > 0\}$.

Theorem 3.2 *Let* $\alpha, \beta \in \mathbb{R}$ *with* $\alpha\beta(\alpha + 2\beta) \neq 0$, α_0, α_0^*, *and* $\alpha(\beta)$ *be defined by Lemma* 2.4. *Then the following statements are true*:

(1) *If* $\beta \geq -1$, *then inequality* (3.1) *holds for all* $x \in (0, \pi/2)$ *if and only if* $(\alpha, \beta) \in \{(\alpha, \beta) | \alpha \geq -2\beta - 12/5, \alpha\beta(\alpha + 2\beta) > 0\}$ *and inequality* (3.2) *holds for all* $x \in (0, \pi/2)$ *if and only if* $(\alpha, \beta) \in \{(\alpha, \beta) | \alpha \geq -2\beta - 12/5, \alpha\beta(\alpha + 2\beta) < 0\}$.

(2) *If* $-1 \leq \beta < 0$, *then inequality* (3.1) *holds for all* $x \in (0, \pi/2)$ *if and only if* $(\alpha, \beta) \in \{(\alpha, \beta) | \alpha \leq \alpha(\beta), \alpha\beta(\alpha + 2\beta) < 0\}$ *and inequality* (3.2) *holds for all* $x \in (0, \pi/2)$ *if and only if* $(\alpha, \beta) \in \{(\alpha, \beta) | \alpha \leq \alpha(\beta), \alpha\beta(\alpha + 2\beta) > 0\}$.

(3) *If* $\beta \leq -37/35$, *then inequality* (3.1) *holds for all* $x \in (0, \pi/2)$ *if and only if* $(\alpha, \beta) \in \{(\alpha, \beta) | \alpha \leq -2\beta - 12/5, \alpha\beta(\alpha + 2\beta) < 0\} \cup \{(\alpha, \beta) | \alpha \geq \alpha(\beta), \alpha\beta(\alpha + 2\beta) > 0\}$ *and inequality* (3.2) *holds for all* $x \in (0, \pi/2)$ *if and only if* $(\alpha, \beta) \in \{(\alpha, \beta) | \alpha \leq -2\beta - 12/5, \alpha\beta(\alpha + 2\beta) > 0\} \cup \{(\alpha, \beta) | \alpha \geq \alpha(\beta), \alpha\beta(\alpha + 2\beta) < 0\}$.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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