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# Some new integral inequalities on time scales containing integration on infinite intervals

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## Abstract

In this paper, some new Gronwall-Bellman type integral inequalities on time scales containing integration on infinite intervals are established. They provide new bounds for unknown functions concerned and can be used as a handy tool in the research of qualitative and quantitative properties of solutions of certain dynamic equations on time scales.

**MSC:** 26E70; 26D15; 26D10

**Keywords:** integral inequalities; time scales; dynamic equation; quantitative properties; qualitative properties

## 1 Introduction

The development of the theory of time scales was initiated by Hilger [1] as a theory capable to contain both difference and differential calculus in a consistent way. Since then many authors have expounded on various aspects of the theory of dynamic equations on time scales (for example, see [2–9], and the references therein). In these investigations, integral inequalities on time scales have been paid much attention by many authors, and a lot of integral inequalities on time scales have been established (for example, see [5–16] and the references therein), which are mainly designed to unify continuous and discrete analysis and play an important role in the research of boundedness, uniqueness, stability of solutions of dynamic equations on time scales. Among these inequalities, Gronwall-Bellman type integral inequalities are of particular importance as such inequalities provide explicit bounds for unknown functions. For related results, we refer to [10–16]. But to our knowledge, Gronwall-Bellman type integral inequalities on time scales containing integration on infinite intervals have been paid little attention so far in the literature.

In this paper, we establish some new Gronwall-Bellman type integral inequalities on time scales containing integration on infinite intervals. New explicit bounds for unknown functions concerned are obtained due to the presented inequalities. Also we present some applications for the established results.

## 2 Some preliminaries

Throughout this paper,  $\mathbb{R}$  denotes the set of real numbers and  $\mathbb{R}_+ = [0, \infty)$ , while  $\mathbb{Z}$  denotes the set of integers. For two given sets  $G, H$ , we denote the set of maps from  $G$  to  $H$  by  $(G, H)$ .

A time scale is an arbitrary nonempty closed subset of real numbers. In this paper,  $\mathbb{T}$  denotes an arbitrary time scale. On  $\mathbb{T}$  we define the forward and backward jump operators  $\sigma \in (\mathbb{T}, \mathbb{T})$  and  $\rho \in (\mathbb{T}, \mathbb{T})$  such that  $\sigma(t) = \inf\{s \in \mathbb{T}, s > t\}$ ,  $\rho(t) = \sup\{s \in \mathbb{T}, s < t\}$ .

**Definition 2.1** The graininess  $\mu \in (\mathbb{T}, \mathbb{R}_+)$  is defined by  $\mu(t) = \sigma(t) - t$ .

**Definition 2.2** The cylinder transformation  $\xi_h$  is defined by

$$\xi_h(z) = \begin{cases} \frac{\text{Log}(1+hz)}{h} & \text{if } h \neq 0 \text{ (for } z \neq -\frac{1}{h}\text{)}, \\ z & \text{if } h = 0, \end{cases}$$

where Log is the principal logarithm function.

**Definition 2.3** For  $p \in \mathfrak{R}$ , the exponential function is defined by

$$e_p(t, s) = \exp\left(\int_s^t \xi_{\mu(\tau)}(p(\tau)) \Delta \tau\right) \quad \text{for } s, t \in \mathbb{T}.$$

**Definition 2.4** If  $\sup_{t \in \mathbb{T}} t = \infty$ ,  $p \in \mathfrak{R}$ , we define

$$e_p(\infty, s) = \exp\left(\int_s^\infty \xi_{\mu(\tau)}(p(\tau)) \Delta \tau\right) \quad \text{for } t \in \mathbb{T}.$$

**Remark 2.1** If  $\mathbb{T} = \mathbb{R}$ , then we have

$$\begin{cases} e_p(t, s) = \exp\left(\int_s^t p(\tau) d\tau\right) & \text{for } s, t \in \mathbb{R}, \\ e_p(\infty, s) = \exp\left(\int_s^\infty p(\tau) d\tau\right) & \text{for } s \in \mathbb{R}. \end{cases}$$

If  $\mathbb{T} = \mathbb{Z}$ , then we have

$$\begin{cases} e_p(t, s) = \prod_{\tau=s}^{t-1} [1 + p(\tau)] & \text{for } s, t \in \mathbb{Z} \text{ and } s < t, \\ e_p(\infty, s) = \prod_{\tau=s}^\infty [1 + p(\tau)] & \text{for } s \in \mathbb{Z}. \end{cases}$$

The following two theorems include some known properties on the exponential function.

**Theorem 2.1** [17, Theorem 5.2] *If  $p \in \mathfrak{R}$ , then the following conclusions hold:*

- (i)  $e_p(t, t) \equiv 1$  and  $e_0(t, s) \equiv 1$ ,
  - (ii)  $e_p(\sigma(t), s) = (1 + \mu(t)p(t))e_p(t, s)$ ,
  - (iii) if  $p \in \mathfrak{R}^+$ , then  $e_p(t, s) > 0$ ,  $\forall s, t \in \mathbb{T}$ ,
  - (iv) if  $p \in \mathfrak{R}^+$ , then  $\ominus p \in \mathfrak{R}^+$ ,
  - (v)  $e_p(t, s) = \frac{1}{e_{\ominus p}(s, t)} = e_{\ominus p}(s, t)$ ,
- where  $\ominus p = -\frac{p}{1+\mu p}$ .

**Remark 2.2** If  $s = \infty$ , then Theorem 2.1(iii), (v) still holds.

**Theorem 2.2** [17, Theorem 5.1] *If  $p \in \mathfrak{R}$ , and fix  $t_0 \in \mathbb{T}$ , then the exponential function  $e_p(t, t_0)$  is the unique solution of the following initial value problem:*

$$\begin{cases} y^\Delta(t) = p(t)y(t), \\ y(t_0) = 1. \end{cases}$$

For more details about the calculus of time scales, we refer to [18, 19].

The following lemma is important for proving our results.

**Lemma 2.1** *Suppose that  $\sup_{t \in \mathbb{T}^\kappa} t = \infty$ ,  $u, q \in C_{rd}$ ,  $p \in \mathfrak{R}^+$ , and  $u$  is delta differential at  $t \in \mathbb{T}^\kappa$ , then*

$$u^\Delta(t) \geq p(t)u(t) - q(t), \quad t \in \mathbb{T}^\kappa \tag{2.1}$$

implies

$$u(t) \leq u(\infty)e_{\ominus p}(\infty, t) + \int_t^\infty q(s)e_p(t, \sigma(s)) \Delta s, \quad t \in \mathbb{T}^\kappa. \tag{2.2}$$

*Proof* Since  $p \in \mathfrak{R}^+$ , then from Theorem 2.1(iv) we have  $\ominus p \in \mathfrak{R}^+$ , and furthermore, from Theorem 2.1(iii) we obtain  $e_{\ominus p}(t, \alpha) > 0, \forall \alpha \in \mathbb{T}^\kappa$ .

So,

$$[u(t)e_{\ominus p}(t, \alpha)]^\Delta = [e_{\ominus p}(t, \alpha)]^\Delta u(t) + e_{\ominus p}(\sigma(t), \alpha)u^\Delta(t). \tag{2.3}$$

On the other hand, from Theorem 2.2 we have

$$[e_{\ominus p}(t, \alpha)]^\Delta = (\ominus p)(t)e_{\ominus p}(t, \alpha). \tag{2.4}$$

So, combining (2.3), (2.4) and Theorem 2.1, it follows that

$$\begin{aligned} [u(t)e_{\ominus p}(t, \alpha)]^\Delta &= (\ominus p)(t)e_{\ominus p}(t, \alpha)u(t) + e_{\ominus p}(\sigma(t), \alpha)u^\Delta(t) \\ &= e_{\ominus p}(\sigma(t), \alpha) \left[ \frac{(\ominus p)(t)}{1 + \mu(t)(\ominus p)(t)} u(t) + u^\Delta(t) \right] \\ &= e_{\ominus p}(\sigma(t), \alpha) [u^\Delta(t) - p(t)u(t)]. \end{aligned} \tag{2.5}$$

Substituting  $t$  with  $s$  and an integration for (2.5) with respect to  $s$  from  $\alpha$  to  $\infty$  yield

$$u(\infty)e_{\ominus p}(\infty, \alpha) - u(\alpha)e_{\ominus p}(\alpha, \alpha) = \int_\alpha^\infty e_{\ominus p}(\sigma(s), \alpha) [u^\Delta(s) - p(s)u(s)] \Delta s. \tag{2.6}$$

Since  $e_{\ominus p}(\alpha, \alpha) = 1$ , from (2.1) and (2.6) we have

$$u(\infty)e_{\ominus p}(\infty, \alpha) - u(\alpha) \geq - \int_\alpha^\infty e_{\ominus p}(\sigma(s), \alpha) q(s) \Delta s = - \int_\alpha^\infty e_p(\alpha, \sigma(s)) q(s) \Delta s,$$

which is followed by

$$u(\alpha) \leq u(\infty)e_{\ominus p}(\infty, \alpha) + \int_{\alpha}^{\infty} e_p(\alpha, \sigma(s))q(s)\Delta s. \tag{2.7}$$

Since  $\alpha \in \mathbb{T}^{\kappa}$  is arbitrary, after substituting  $\alpha$  with  $t$ , we obtain the desired inequality.  $\square$

**Lemma 2.2** [20] *Assume that  $a \geq 0, p \geq q \geq 0$ , and  $p \neq 0$ , then for any  $K > 0$*

$$a^{\frac{q}{p}} \leq \frac{q}{p}K^{\frac{q-p}{p}}a + \frac{p-q}{p}K^{\frac{q}{p}}.$$

### 3 Main results

**Theorem 3.1** *Suppose that  $\sup_{t \in \mathbb{T}^{\kappa}} t = \infty, u, a, f, g, h \in C_{rd}(\mathbb{T}, \mathbb{R}_+)$ , and  $a$  is decreasing. If  $u(t)$  satisfies the following inequality:*

$$u(t) \leq a(t) + \int_t^{\infty} \left[ f(s)u(s) + g(s) \int_s^{\infty} h(\xi)u(\xi)\Delta\xi \right] \Delta s, \quad t \in \mathbb{T}^{\kappa}, \tag{3.1}$$

then

$$u(t) \leq a(t)e_{\ominus \tilde{f}}(\infty, t), \quad t \in \mathbb{T}^{\kappa}, \tag{3.2}$$

provided that  $1 + \mu(t)\tilde{f}(t) > 0$ , where

$$\tilde{f}(t) = - \left[ f(t) + g(t) \int_t^{\infty} h(\xi)\Delta\xi \right]. \tag{3.3}$$

*Proof* Fix  $T \in \mathbb{T}^{\kappa}$ , and let  $t \in [T, \infty) \cap \mathbb{T}^{\kappa}$ . Since  $a(t)$  is decreasing on  $\mathbb{T}^{\kappa}$ , from (3.1) we have

$$u(t) \leq a(T) + \int_t^{\infty} \left[ f(s)u(s) + g(s) \int_s^{\infty} h(\xi)u(\xi)\Delta\xi \right] \Delta s, \quad t \in [T, \infty) \cap \mathbb{T}^{\kappa}. \tag{3.4}$$

Denote the right-hand side of (3.4) by  $v(t)$ . Then it follows that

$$u(t) \leq v(t), \quad t \in [T, \infty) \cap \mathbb{T}^{\kappa}; \tag{3.5}$$

and furthermore,

$$\begin{aligned} v^{\Delta}(t) &= - \left[ f(t)u(t) + g(t) \int_t^{\infty} h(\xi)u(\xi)\Delta\xi \right] \\ &\geq - \left[ f(t) + g(t) \int_t^{\infty} h(\xi)\Delta\xi \right] v(t) = \tilde{f}(t)v(t), \end{aligned} \tag{3.6}$$

where  $\tilde{f}$  is defined in (3.3).

Under the condition  $1 + \mu(t)\tilde{f}(t) > 0$ , in fact we have  $\tilde{f} \in \mathfrak{R}^+$ . So, a suitable application of Lemma 2.1 yields

$$v(t) \leq v(\infty)e_{\ominus \tilde{f}}(\infty, t), \quad t \in [T, \infty) \cap \mathbb{T}^{\kappa}. \tag{3.7}$$

Since  $v(\infty) = a(T)$ , then (3.7) can be rewritten as

$$v(t) \leq a(T)e_{\ominus \tilde{f}}(\infty, t), \quad t \in [T, \infty) \cap \mathbb{T}^{\kappa}. \tag{3.8}$$

Combining (3.8) with (3.5), we have

$$u(t) \leq a(T)e_{\ominus \tilde{f}}(\infty, t), \quad t \in [T, \infty) \cap \mathbb{T}^{\kappa}. \tag{3.9}$$

Setting  $t = T$  in (3.9), since  $T$  is selected from  $\mathbb{T}^{\kappa}$  arbitrarily, after substituting  $T$  with  $t$ , we obtain the desired inequality (3.2).  $\square$

Since  $\mathbb{T}$  is an arbitrary time scale, if we take  $\mathbb{T}$  for some peculiar cases, then we have the following corollaries.

**Corollary 3.1** *Suppose that  $\mathbb{T} = \mathbb{R}$ .  $u, a, f, g, h \in C(\mathbb{R}, \mathbb{R}_+)$ , and  $a$  is decreasing on  $\mathbb{R}$ . If  $u(t)$  satisfies the following inequality:*

$$u(t) \leq a(t) + \int_t^\infty \left[ f(s)u(s) + g(s) \int_s^\infty h(\xi)u(\xi) d\xi \right] ds, \quad t \in \mathbb{R}, \tag{3.10}$$

then

$$u(t) \leq a(t) \exp\left(-\int_t^\infty \tilde{f}(\tau) d\tau\right), \quad t \in \mathbb{R}, \tag{3.11}$$

where  $\tilde{f}(t) = -[f(t) + g(t) \int_t^\infty h(\xi) d\xi]$ .

*Proof* When  $\mathbb{T} = \mathbb{R}$ , we have  $\mu(t) = \sigma(t) - t \equiv 0$ , and then  $\ominus \tilde{f} = -\frac{\tilde{f}}{1+\mu \tilde{f}} = -\tilde{f}$ . Obviously,  $1 + \mu(t)\tilde{f}(t) = 1 > 0$ . Furthermore, from Remark 2.1 we have

$$e_{\ominus \tilde{f}}(\infty, t) = \exp\left(\int_t^\infty (\ominus \tilde{f})(\tau) d\tau\right) = \exp\left(-\int_t^\infty \tilde{f}(\tau) d\tau\right). \tag{3.12}$$

Combining (3.12) and (3.2), we obtain the desired result.  $\square$

**Corollary 3.2** *Suppose that  $\mathbb{T} = \mathbb{Z}$ .  $u, a, f, g, h \in (\mathbb{Z}, \mathbb{R}_+)$ , and  $a$  is decreasing on  $\mathbb{Z}$ . If  $u(n)$  satisfies the following inequality:*

$$u(n) \leq a(n) + \sum_{s=n}^\infty \left[ f(s)u(s) + g(s) \sum_{\xi=s}^\infty h(\xi)u(\xi) \right], \quad n \in \mathbb{Z}, \tag{3.13}$$

then

$$u(n) \leq a(n) \prod_{\tau=n}^\infty \frac{1}{1+\tilde{f}(\tau)}, \quad n \in \mathbb{Z}, \tag{3.14}$$

provided that  $1 + \tilde{f}(n) > 0$ , where  $\tilde{f}(n) = -[f(n) + g(n) \sum_{\xi=n}^\infty h(\xi)]$ .

*Proof* When  $\mathbb{T} = \mathbb{Z}$ , we have  $\mu(n) = \sigma(n) - n \equiv 1, \forall n \in \mathbb{Z}$ , and then  $\Theta \tilde{f} = -\frac{\tilde{f}}{1+\mu f} = -\frac{\tilde{f}}{1+f}$ . Furthermore, from Remark 2.1 we have

$$e_{\Theta \tilde{f}}(\infty, n) = \prod_{\tau=n}^{\infty} [1 + (\Theta \tilde{f})(\tau)] = \prod_{\tau=n}^{\infty} \frac{1}{1 + \tilde{f}(\tau)}. \tag{3.15}$$

Combining (3.15) and (3.2), we obtain the desired result. □

**Theorem 3.2** *Suppose that  $\sup_{t \in \mathbb{T}^{\kappa}} t = \infty$ ,  $u, a, f, g$  are defined as in Theorem 3.1.  $p$  is a positive number with  $p \geq 1$ . If  $u(t)$  satisfies the following inequality:*

$$u^p(t) \leq a(t) + \int_t^{\infty} [f(s)u(s) + g(s)] \Delta s, \quad t \in \mathbb{T}^{\kappa}, \tag{3.16}$$

then

$$u(t) \leq \left\{ a(t)e_{\Theta \hat{f}}(\infty, t) + \int_t^{\infty} \hat{g}(s)e_{\hat{f}}(t, \sigma(s)) \Delta s \right\}^{\frac{1}{p}}, \quad t \in \mathbb{T}^{\kappa}, \tag{3.17}$$

provided  $1 + \mu(t)\hat{f}(t) > 0$ , where

$$\begin{cases} \hat{f}(t) = -f(t)\frac{1}{p}K^{\frac{1-p}{p}}, \\ \hat{g}(t) = f(t)\frac{p-1}{p}K^{\frac{1}{p}} + g(t), \quad \forall K > 0. \end{cases} \tag{3.18}$$

*Proof* Fix  $T \in \mathbb{T}^{\kappa}$ . Let  $t \in [T, \infty) \cap \mathbb{T}^{\kappa}$ , and

$$v(t) = a(T) + \int_t^{\infty} [f(s)u(s) + g(s)] \Delta s, \quad t \in \mathbb{T}^{\kappa}. \tag{3.19}$$

Then

$$u(t) \leq v^{\frac{1}{p}}(t), \quad t \in [T, \infty) \cap \mathbb{T}^{\kappa}, \tag{3.20}$$

and furthermore,

$$v^{\Delta}(t) = -[f(t)u(t) + g(t)] \geq -[f(t)v^{\frac{1}{p}}(t) + g(t)], \quad t \in [T, \infty) \cap \mathbb{T}^{\kappa}. \tag{3.21}$$

On the other hand, from Lemma 2.2 we have

$$v^{\frac{1}{p}}(t) \leq \frac{1}{p}K^{\frac{1-p}{p}}v(t) + \frac{p-1}{p}K^{\frac{1}{p}}, \quad \forall K > 0. \tag{3.22}$$

So, combining (3.21) and (3.22), we obtain

$$\begin{aligned} v^{\Delta}(t) &\geq -\left[ f(t) \left( \frac{1}{p}K^{\frac{1-p}{p}}v(t) + \frac{p-1}{p}K^{\frac{1}{p}} \right) + g(t) \right] \\ &= \hat{f}(t)v(t) - \hat{g}(t), \quad t \in [T, \infty) \cap \mathbb{T}^{\kappa}, \end{aligned} \tag{3.23}$$

where  $\hat{f}$  and  $\hat{g}$  are defined in (3.18).

Since  $v(\infty) = a(T)$ , and  $1 + \mu(t)\widehat{f}(t) > 0$ , we have  $\widehat{f}(t) \in \mathfrak{N}^+$ , and a suitable application of Lemma 2.1 yields

$$v(t) \leq a(T)e_{\ominus\widehat{f}}(\infty, t) + \int_t^\infty \widehat{g}(s)e_{\widehat{f}}(t, \sigma(s))\Delta s, \quad t \in [T, \infty) \cap \mathbb{T}^\kappa. \quad (3.24)$$

Combining (3.24) and (3.20), we have

$$u(t) \leq \left\{ a(T)e_{\ominus\widehat{f}}(\infty, t) + \int_t^\infty \widehat{g}(s)e_{\widehat{f}}(t, \sigma(s))\Delta s \right\}^{\frac{1}{p}}, \quad t \in [T, \infty) \cap \mathbb{T}^\kappa. \quad (3.25)$$

Setting  $t = T$  in (3.25), and considering  $T$  is selected from  $\mathbb{T}^\kappa$  arbitrarily, after substituting  $T$  with  $t$ , we obtain the desired inequality (3.17).  $\square$

**Corollary 3.3** *Suppose that  $\mathbb{T} = \mathbb{R}$ .  $u, a, f, g \in C(\mathbb{R}, \mathbb{R}_+)$ , and  $a$  is decreasing on  $\mathbb{R}$ .  $p$  is defined as in Theorem 3.2. If  $u(t)$  satisfies the following inequality:*

$$u^p(t) \leq a(t) + \int_t^\infty [f(s)u(s) + g(s)] ds, \quad t \in \mathbb{R}, \quad (3.26)$$

then

$$u(t) \leq \left\{ a(t) \exp\left(-\int_t^\infty \widehat{f}(\tau) d\tau\right) + \int_t^\infty \widehat{g}(s) \exp\left(\int_s^t \widehat{f}(\tau) d\tau\right) ds \right\}^{\frac{1}{p}}, \quad t \in \mathbb{R}, \quad (3.27)$$

where  $\widehat{f}, \widehat{g}$  are the same as in Theorem 3.2.

**Corollary 3.4** *Suppose that  $\mathbb{T} = \mathbb{Z}$ .  $u, a, f, g \in (\mathbb{Z}, \mathbb{R}_+)$ , and  $a$  is decreasing on  $\mathbb{Z}$ .  $p$  is defined as in Theorem 3.2. If  $u(n)$  satisfies the following inequality:*

$$u^p(n) \leq a(n) + \sum_{s=n}^\infty [f(s)u(s) + g(s)], \quad n \in \mathbb{Z}, \quad (3.28)$$

then

$$u(n) \leq \left\{ a(n) \prod_{\tau=n}^\infty \frac{1}{1+\widehat{f}(\tau)} + \sum_{s=n}^\infty \widehat{g}(s) \left( \prod_{\tau=n}^s \frac{1}{1+\widehat{f}(\tau)} \right) \right\}^{\frac{1}{p}}, \quad n \in \mathbb{Z}, \quad (3.29)$$

where  $\widehat{f}, \widehat{g}$  are the same as in Theorem 3.2.

The proofs for Corollaries 3.3 and 3.4 are similar to those for Corollaries 3.1 and 3.2, and we omit them here.

**Theorem 3.3** *Suppose that  $\sup_{t \in \mathbb{T}^\kappa} t = \infty$ ,  $u, a, f, g, h$  are defined as in Theorem 3.1, and  $p \geq 1$ . If  $u(t)$  satisfies the following inequality:*

$$u^p(t) \leq a(t) + \int_t^\infty [f(s)u^p(s) + g(s)u(s) + h(s)]\Delta s, \quad t \in \mathbb{T}^\kappa, \quad (3.30)$$

then

$$u(t) \leq \left\{ \left[ a(t)e_{\ominus H_1}(\infty, t) + \int_t^\infty H_2(s)e_{H_1}(t, \sigma(s)) \Delta s \right] e_{\ominus(-f)}(\infty, t) \right\}^{\frac{1}{p}}, \quad (3.31)$$

provided that  $1 - \mu(t)f(t) > 0$ ,  $1 + \mu(t)H_1(t) > 0$ , where

$$\begin{cases} H_1(t) = -g(t)\frac{1}{p}K^{\frac{1-p}{p}}e_{\ominus(-f)}(\infty, t), \\ H_2(t) = g(t)\frac{p-1}{p}K^{\frac{1}{p}} + h(t), \quad \forall K > 0. \end{cases} \quad (3.32)$$

*Proof* Let

$$v(t) = a(t) + \int_t^\infty [g(s)u(s) + h(s)] \Delta s, \quad t \in \mathbb{T}^\kappa. \quad (3.33)$$

Then

$$u^p(t) \leq v(t) + \int_t^\infty f(s)u^p(s) \Delta s, \quad t \in \mathbb{T}^\kappa. \quad (3.34)$$

Since  $1 - \mu(t)f(t) > 0$ , we have  $-f \in \mathfrak{R}^+$ , and furthermore  $\ominus(-f) \in \mathfrak{R}^+$ . Treating  $u^p(t)$  as a variable, by Corollary 3.5 we obtain

$$u^p(t) \leq v(t)e_{\ominus(-f)}(\infty, t), \quad t \in \mathbb{T}^\kappa. \quad (3.35)$$

Now fix  $T \in \mathbb{T}^\kappa$ . Let  $t \in [T, \infty) \cap \mathbb{T}^\kappa$ , and

$$y(t) = a(T) + \int_t^\infty [g(s)u(s) + h(s)] \Delta s, \quad t \in \mathbb{T}^\kappa. \quad (3.36)$$

Then

$$v(t) \leq y(t), \quad t \in [T, \infty) \cap \mathbb{T}^\kappa. \quad (3.37)$$

Since  $\ominus(-f) \in \mathfrak{R}^+$ , we have  $e_{\ominus(-f)}(\infty, t) > 0$ . From (3.35) and Lemma 2.2, we obtain

$$\begin{aligned} y^\Delta(t) &= -[g(t)u(t) + h(t)] \geq -[g(t)(v(t)e_{\ominus(-f)}(\infty, t))^{\frac{1}{p}} + h(t)] \\ &\geq -\left[ g(t) \left( \frac{1}{p}K^{\frac{1-p}{p}}v(t)e_{\ominus(-f)}(\infty, t) + \frac{p-1}{p}K^{\frac{1}{p}} \right) + h(t) \right] \\ &\geq -\left[ g(t) \left( \frac{1}{p}K^{\frac{1-p}{p}}y(t)e_{\ominus(-f)}(\infty, t) + \frac{p-1}{p}K^{\frac{1}{p}} \right) + h(t) \right] \\ &= H_1(t)y(t) - H_2(t), \end{aligned} \quad (3.38)$$

where  $H_1, H_2$  are defined in (3.32).

Since  $1 + \mu(t)H_1(t) > 0$ , we have  $H_1 \in \mathfrak{R}^+$ , and a suitable application of Lemma 2.1 yields

$$y(t) \leq y(\infty)e_{\ominus H_1}(\infty, t) + \int_t^\infty H_2(s)e_{H_1}(t, \sigma(s)) \Delta s. \quad (3.39)$$

Since  $y(\infty) = a(T)$ , we obtain

$$y(t) \leq a(T)e_{\ominus H_1}(\infty, t) + \int_t^\infty H_2(s)e_{H_1}(t, \sigma(s)) \Delta s. \tag{3.40}$$

Combining (3.35), (3.37) and (3.40), we obtain

$$u(t) \leq \left\{ \left[ a(T)e_{\ominus H_1}(\infty, t) + \int_t^\infty H_2(s)e_{H_1}(t, \sigma(s)) \Delta s \right] e_{\ominus(-f)}(\infty, t) \right\}^{\frac{1}{p}},$$

$$t \in [T, \infty) \cap \mathbb{T}^\kappa. \tag{3.41}$$

Setting  $t = T$  in (3.41) yields

$$u(T) \leq \left\{ \left[ a(T)e_{\ominus H_1}(\infty, T) + \int_T^\infty H_2(s)e_{H_1}(T, \sigma(s)) \Delta s \right] e_{\ominus(-f)}(\infty, T) \right\}^{\frac{1}{p}}. \tag{3.42}$$

Since  $T$  is selected from  $\mathbb{T}^\kappa$  arbitrarily, substituting  $T$  with  $t$  in (3.42), we obtain the desired inequality (3.31). □

**Corollary 3.5** *Suppose that  $\mathbb{T} = \mathbb{R}$ .  $u, a, f, g, h \in C(\mathbb{R}, \mathbb{R}_+)$ , and  $a$  is decreasing on  $\mathbb{R}$ .  $p$  is defined as in Theorem 3.3. If  $u(t)$  satisfies the following inequality:*

$$u^p(t) \leq a(t) + \int_t^\infty [f(s)u^p(s) + g(s)u(s) + h(s)] ds, \quad t \in \mathbb{R}, \tag{3.43}$$

then

$$u(t) \leq \left\{ \left[ a(t) \exp\left(-\int_t^\infty H_1(\tau) d\tau\right) + \int_t^\infty H_2(s) \exp\left(\int_s^t H_1(\tau) d\tau\right) ds \right] \times \exp\left(\int_t^\infty f(\tau) d\tau\right) \right\}^{\frac{1}{p}}, \quad t \in \mathbb{R}, \tag{3.44}$$

where  $H_1(t) = -g(t)\frac{1}{p}K^{\frac{1-p}{p}} \exp(\int_t^\infty f(\tau) d\tau)$ , and  $H_2$  is defined as in Theorem 3.3.

**Corollary 3.6** *Suppose that  $\mathbb{T} = \mathbb{Z}$ .  $u, a, f, g \in (\mathbb{Z}, \mathbb{R}_+)$ , and  $a$  is decreasing on  $\mathbb{Z}$ .  $p$  is the same as in Theorem 3.3. If  $u(n)$  satisfies the following inequality:*

$$u^p(n) \leq a(n) + \sum_{s=n}^\infty [f(s)u^p(s) + g(s)u(s) + h(s)], \quad n \in \mathbb{Z}, \tag{3.45}$$

then

$$u(n) \leq \left\{ \left[ a(n) \prod_{\tau=n}^\infty \frac{1}{1+H_1(\tau)} + \sum_{s=n}^\infty \left[ H_2(s) \left( \prod_{\tau=n}^s \frac{1}{1+H_1(\tau)} \right) \right] \right] \prod_{\tau=n}^\infty \frac{1}{1-f(\tau)} \right\}^{\frac{1}{p}},$$

$$n \in \mathbb{Z}, \tag{3.46}$$

where  $H_1(\tau) = -g(\tau)\frac{1}{p}K^{\frac{1-p}{p}} \prod_{\xi=\tau}^\infty \frac{1}{1-f(\xi)}$ , and  $H_2$  is defined as in Theorem 3.3.

**Theorem 3.4** *Suppose that  $\sup_{t \in \mathbb{T}^\kappa} t = \infty$ ,  $u, a, f$  are defined as in Theorem 3.1, and  $p \geq 1$ .  $L \in C(\mathbb{T}^\kappa \times \mathbb{R}_+, \mathbb{R}_+)$ , and  $0 \leq L(s, x) - L(s, y) \leq M(s, y)(x - y)$  for  $x \geq y \geq 0$ , where  $M \in C(\mathbb{T}^\kappa \times \mathbb{R}_+, \mathbb{R}_+)$ . If  $u(t)$  satisfies the following inequality:*

$$u^p(t) \leq a(t) + \int_t^\infty [f(s)u^p(s) + L(s, u(s))] \Delta s, \quad t \in \mathbb{T}^\kappa, \tag{3.47}$$

then

$$u(t) \leq \left\{ \left[ a(t)e_{\ominus \widehat{H}_1}(\infty, t) + \int_t^\infty \widehat{H}_2(s)e_{\widehat{H}_1}(t, \sigma(s)) \Delta s \right] e_{\ominus(-f)}(\infty, t) \right\}^{\frac{1}{p}}, \tag{3.48}$$

provided that  $1 - \mu(t)f(t) > 0$ ,  $1 + \mu(t)\widehat{H}_1(t) > 0$ , where

$$\begin{cases} \widehat{H}_1(t) = -M(t, (e_{\ominus(-f)}(\infty, t))^{\frac{1}{p}} \frac{p-1}{p} K^{\frac{1}{p}}) (e_{\ominus(-f)}(\infty, t))^{\frac{1}{p}} \frac{1}{p} K^{\frac{1-p}{p}}, \\ \widehat{H}_2(t) = L(t, (e_{\ominus(-f)}(\infty, t))^{\frac{1}{p}} \frac{p-1}{p} K^{\frac{1}{p}}), \quad \forall K > 0. \end{cases} \tag{3.49}$$

*Proof* Let

$$v(t) = a(t) + \int_t^\infty L(s, u(s)) \Delta s, \quad t \in \mathbb{T}^\kappa. \tag{3.50}$$

Then

$$u^p(t) \leq v(t) + \int_t^\infty f(s)u^p(s) \Delta s, \quad t \in \mathbb{T}^\kappa. \tag{3.51}$$

Similar to the process of (3.34) to (3.35), we obtain

$$u^p(t) \leq v(t)e_{\ominus(-f)}(\infty, t), \quad t \in \mathbb{T}^\kappa. \tag{3.52}$$

Now fix  $T \in \mathbb{T}^\kappa$ . Let  $t \in [T, \infty) \cap \mathbb{T}^\kappa$ , and

$$y(t) = a(T) + \int_t^\infty L(s, u(s)) \Delta s, \quad t \in \mathbb{T}^\kappa. \tag{3.53}$$

Then

$$v(t) \leq y(t), \quad t \in [T, \infty) \cap \mathbb{T}^\kappa. \tag{3.54}$$

Since  $1 - \mu(t)f(t) > 0$ , we have  $\ominus(-f) \in \mathfrak{R}^+$ , and furthermore  $e_{\ominus(-f)}(\infty, t) > 0$ . From (3.52), (3.54) and Lemma 2.2, we obtain

$$\begin{aligned} y^\Delta(t) &= -L(t, u(t)) \geq -L(t, (y(t)(e_{\ominus(-f)}(\infty, t))^{\frac{1}{p}})) \\ &\geq -L\left(t, (e_{\ominus(-f)}(\infty, t))^{\frac{1}{p}} \frac{1}{p} K^{\frac{1-p}{p}} y(t) + (e_{\ominus(-f)}(\infty, t))^{\frac{1}{p}} \frac{p-1}{p} K^{\frac{1}{p}}\right) \\ &= -\left[ L\left(t, (e_{\ominus(-f)}(\infty, t))^{\frac{1}{p}} \frac{1}{p} K^{\frac{1-p}{p}} y(t) + (e_{\ominus(-f)}(\infty, t))^{\frac{1}{p}} \frac{p-1}{p} K^{\frac{1}{p}}\right) \right. \\ &\quad \left. - L\left(t, (e_{\ominus(-f)}(\infty, t))^{\frac{1}{p}} \frac{p-1}{p} K^{\frac{1}{p}}\right) + L\left(t, (e_{\ominus(-f)}(\infty, t))^{\frac{1}{p}} \frac{p-1}{p} K^{\frac{1}{p}}\right) \right] \end{aligned}$$

$$\begin{aligned} &\geq -\left[ M\left(t, (e_{\ominus(-f)}(\infty, t))^{\frac{1}{p}} \frac{p-1}{p} K^{\frac{1}{p}}\right) (e_{\ominus(-f)}(\infty, t))^{\frac{1}{p}} \frac{1}{p} K^{\frac{1-p}{p}} y(t) \right. \\ &\quad \left. + L\left(t, (e_{\ominus(-f)}(\infty, t))^{\frac{1}{p}} \frac{p-1}{p} K^{\frac{1}{p}}\right) \right] \\ &= \widehat{H}_1(t)y(t) - \widehat{H}_2(t), \end{aligned} \tag{3.55}$$

where  $\widehat{H}_1, \widehat{H}_2$  are defined in (3.49).

We note that the structure of (3.55) is just similar to (3.38). So, following in a similar manner as the process of (3.38)-(3.40), we deduce

$$y(t) \leq a(T)e_{\ominus\widehat{H}_1}(\infty, t) + \int_t^\infty \widehat{H}_2(s)e_{\widehat{H}_1}(t, \sigma(s)) \Delta s. \tag{3.56}$$

Combining (3.52), (3.54) and (3.56), we obtain

$$\begin{aligned} u(t) &\leq \left\{ \left[ a(T)e_{\ominus\widehat{H}_1}(\infty, t) + \int_t^\infty \widehat{H}_2(s)e_{\widehat{H}_1}(t, \sigma(s)) \Delta s \right] e_{\ominus(-f)}(\infty, t) \right\}^{\frac{1}{p}}, \\ t &\in [T, \infty) \cap \mathbb{T}^\kappa. \end{aligned} \tag{3.57}$$

Setting  $t = T$  in (3.57), since  $T$  is selected from  $\mathbb{T}^\kappa$  arbitrarily, after substituting  $T$  with  $t$ , we obtain the desired inequality (3.48).  $\square$

**Remark 3.1** If we take  $\mathbb{T} = \mathbb{R}$  or  $\mathbb{T} = \mathbb{Z}$  in Theorem 3.4, then we obtain another two corollaries, which are similar to Corollaries 3.1-3.6.

### 4 Applications

In this section, we give some applications for the results presented above. In Examples 1 and 3, new explicit bounds for solutions of certain dynamic equations on time scales are derived by the results established, while in Example 2, we deal with the quantitative properties of solutions of a kind of a dynamical equation on time scales.

**Example 1** Consider the following dynamic integral equation on time scales:

$$u^p(t) = C + \int_t^\infty F(s, u(s)) \Delta s, \quad t \in \mathbb{T}^\kappa, \tag{4.1}$$

with the condition  $u(\infty) = C$ , where  $u \in C_{rd}(\mathbb{T}, \mathbb{R})$ ,  $p$  is a positive number with  $p \geq 1$ .

Assume that  $|F(t, u)| \leq f(t)|u| + g(t)$ , where  $f, g, h \in C_{rd}(\mathbb{T}, \mathbb{R}_+)$ , then

$$|u(t)| \leq \left\{ |C|e_{\ominus\widehat{f}}(\infty, t) + \int_t^\infty \widehat{g}(s)e_{\widehat{f}}(t, \sigma(s)) \Delta s \right\}^{\frac{1}{p}}, \quad t \in \mathbb{T}^\kappa, \tag{4.2}$$

provided  $1 + \mu(t)\widehat{f}(t) > 0$ , where  $\widehat{f}(t), \widehat{g}(t)$  are defined the same as in Theorem 3.2.

In fact, from (4.1) we have

$$|u(t)|^p \leq |C| + \int_t^\infty |F(s, u(s))| \Delta s \leq |C| + \int_t^\infty [f(s)|u(s)| + g(s)] \Delta s.$$

Then a suitable application of Theorem 3.2 yields the desired result.

**Example 2** Consider the following dynamic integral equation on time scales:

$$u(t) = C + \int_t^\infty F(s, u(s)) \Delta s, \quad t \in \mathbb{T}^\kappa, \tag{4.3}$$

where  $u$  is the same as in Example 1.

Assume that  $|F(t, u_1) - F(t, u_2)| \leq f(t)|u_1(t) - u_2(t)|$ , where  $f \in C_{\text{rd}}(\mathbb{T}, \mathbb{R}_+)$ , then Eq. (4.3) has at most one solution.

In fact, if  $u_1(t)$  and  $u_2(t)$  are two solutions to Eq. (4.3), then

$$\begin{aligned} |u_1(t) - u_2(t)| &= \left| \int_t^\infty F(s, u_1(s)) - F(s, u_2(s)) \Delta s \right| \leq \int_t^\infty |F(s, u_1(s)) - F(s, u_2(s))| \Delta s \\ &\leq \int_t^\infty f(s) |u_1(s) - u_2(s)| \Delta s. \end{aligned} \tag{4.4}$$

Then a suitable application of Theorem 3.1 yields  $|u_1(t) - u_2(t)| \leq 0$ , which implies  $u_1(t) \equiv u_2(t)$ .

**Example 3** Consider the following dynamic differential equation on time scales:

$$(u^p(t))^\Delta = -G(t, u(t)), \quad t \in \mathbb{T}^\kappa, \tag{4.5}$$

with the condition  $u(\infty) = C$ , where  $u \in C_{\text{rd}}(\mathbb{T}, \mathbb{R})$ , and  $p$  is a constant with  $p \geq 1$ .

Assume that  $|G(t, u)| \leq f(t)|u|^p + L(t, |u(t)|)$ , where  $f, L$  are defined the same as in Theorem 3.4, then

$$|u(t)| \leq \left\{ \left[ |C| e_{\ominus \widehat{H}_1}(\infty, t) + \int_t^\infty \widehat{H}_2(s) e_{\widehat{H}_1}(t, \sigma(s)) \Delta s \right] e_{\ominus(-f)}(\infty, t) \right\}^{\frac{1}{p}}, \tag{4.6}$$

provided that  $1 - \mu(t)f(t) > 0$ ,  $1 + \mu(t)\widehat{H}_1(t) > 0$ , where  $\widehat{H}_1(t), \widehat{H}_2(t)$  are the same as in Theorem 3.4.

In fact, considering  $u(\infty) = C$ , then the equivalent form of (4.5) is denoted by

$$u^p(t) = C + \int_t^\infty G(s, u(s)) \Delta s, \quad t \in \mathbb{T}^\kappa. \tag{4.7}$$

Furthermore,

$$\begin{aligned} |u(t)|^p &\leq |C| + \int_t^\infty |G(s, u(s))| \Delta s \\ &\leq |C| + \int_t^\infty [f(s)|u(s)|^p + L(s, |u(s)|)] \Delta s, \quad t \in \mathbb{T}^\kappa. \end{aligned} \tag{4.8}$$

Then a suitable application of Theorem 3.4 yields the desired inequality (4.6).

### 5 Conclusions

We have established some new Gronwall-Bellman type integral inequalities on time scales containing integration on infinite intervals, which can be used in the research of boundedness and other qualitative properties as well as quantitative properties of solutions of

certain dynamic equations on time scales. From the corollaries in Section 3, one can see the established inequalities unify continuous and discrete analysis. Finally, we note that the process of Theorems 3.1-3.4 can be applied to establish delay inequalities with two independent variables on time scales.

#### Competing interests

The authors declare that they have no competing interests.

#### Authors' contributions

ZM and CW carried out the main part of this article. All authors read and approved the final manuscript.

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