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Generalized matrix summability of a conjugate derived Fourier series

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Abstract

The study of infinite matrices is important in the theory of summability and in approximation. In particular, Toeplitz matrices or regular matrices and almost regular matrices have been very useful in this context. In this paper, we propose to use a more general matrix method to obtain necessary and sufficient conditions to sum the conjugate derived Fourier series.

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1 Introduction

The idea of \mathfrak{B} -summability (or $F_{\mathfrak{B}}$ -convergence) was introduced by Bell [1] and Steiglitz [2]. It generalizes the notions of A -summability and almost convergence.

Let $\mathfrak{B} = (B_i)_{i=1}^{\infty}$ be a sequence of infinite matrices with $B_i = (b_{nk}(i))_{n,k=1}^{\infty}$. Then a bounded sequence $x = (x_k)_{k=1}^{\infty}$ is said to be \mathfrak{B} -summable (or $F_{\mathfrak{B}}$ -convergent) to the value L if $\lim_n (B_i x)_n = \lim_n \sum_k b_{nk}(i) x_k = L$, uniformly in $i \geq 0$. In this case, L is denoted $\mathfrak{B}\text{-}\lim x$.

Note that for $\mathfrak{B} = I$ (the unit matrix), $F_{\mathfrak{B}}$ -convergence is reduced to the ordinary convergence of x . For $\mathfrak{B} = \mathfrak{B}_1$, then $F_{\mathfrak{B}}$ -convergence is reduced to the almost convergence of x (see Lorentz [3]), where $\mathfrak{B}_1 = (b_{nk}^1(i))_{n,k=1}^{\infty}$ with

$$b_{nk}^1(i) = \begin{cases} \frac{1}{n+1}, & n \leq i \leq n+k, \\ 0 & \text{otherwise.} \end{cases}$$

In this paper, we use such type of matrices, which have many applications in various fields, to study the summability problem of the conjugate derived Fourier series.

Let $f(t)$ be a periodic function with period 2π and Lebesgue-integrable over $[-\pi, \pi]$. Then the Fourier series associated with f at any point x is defined by

$$f(x) \sim \frac{a_0}{2} + \sum_{j=1}^{\infty} (a_j \cos jx + b_j \sin jx), \quad (1.1)$$

and the conjugate series of the Fourier series (1.1) is

$$\sum_{j=1}^{\infty} (b_j \cos jx - a_j \sin jx). \quad (1.2)$$

The derived Fourier series is

$$\sum_{j=1}^{\infty} j(b_j \cos jx - a_j \sin jx), \quad (1.3)$$

and the conjugate derived Fourier series (omitting the minus sign) is

$$\sum_{j=1}^{\infty} j(a_j \cos jx + b_j \sin jx). \quad (1.4)$$

In this paper, we apply the notion of $F_{\mathfrak{B}}$ -convergence to study the summability problems of the conjugate derived Fourier series (1.4).

2 Preliminaries

For an infinite matrix $A = (a_{nk})_{n,k=1}^{\infty}$ of real or complex numbers, the A -transform of the sequence $x = (x_k)_{k=1}^{\infty}$ is defined by the sequence $Ax = (A_n(x))$, provided that $A_n(x) = \sum_k a_{nk}x_k$ converges for each $n \in \mathbb{N}$. If $x = (x_k) \in X$ implies that $Ax \in Y$, then we say that A defines a matrix transformation from a sequence space X into another sequence space Y , and we denote the class of such matrices by (X, Y) . Let c denote the space of all convergent sequences. We say that A is a conservative matrix if it transforms a convergent sequence into a convergent one. If, in addition, $\lim Ax = \lim x$, then A is called a regular matrix, and in this case, we write $A \in (c, c)_{\text{reg}}$. The well-known Silverman-Töeplitz conditions for the regularity of A are (see [4]):

- (i) $\|A\| = \sup_n \sum_k |a_{nk}| < \infty$,
- (ii) $\lim_n a_{nk} = 0$ for each k ,
- (iii) $\lim_n \sum_k b_{nk} = 1$.

These regular matrices play a very important role in summing nonconvergent or convergent sequences and series. Our methods are based on the previous techniques, that is, we put conditions on the entries of a matrix through which we try to sum certain sequences or series. There are special well-known regular summability matrices, Cesàro, Riesz, Nörlund, Euler, and Hankel matrices, which have various applications in different fields of mathematics and other sciences.

Theorem A *The method \mathfrak{B} is regular (see [1, 2]) if and only if*

- (i) $\|\mathfrak{B}\| < \infty$,
- (ii) $\lim_n b_{nk}(i) = 0$ for all $k \geq 1$ uniformly in i , and
- (iii) $\lim_n \sum_k b_{nk}(i) = 1$ uniformly in i ,

where

$$\|\mathfrak{B}\| = \sup_{n,i} \sum_k |b_{nk}(i)| < \infty,$$

which means that there exists a constant M such that

$$\sum_k |b_{nk}(i)| \leq M$$

for all n, i and the series $\sum_k |b_{nk}(i)|$ converges uniformly in i for each n .

Let $\sum_n a_n$ be a given infinite series with the sequence of partial sums (s_n) . Let (p_n) be a sequence of positive real numbers such that

$$P_n = \sum_{v=0}^n p_v \rightarrow \infty, \quad n \rightarrow \infty \quad (P_{-j} = p_{-j} = 0, \quad j \geq 0).$$

We write

$$t_n = \frac{1}{P_n} \sum_{v=0}^n p_v s_v,$$

which denotes the (\bar{N}, p_n) -mean of the sequence (s_n) generated by the sequence of coefficients (p_n) . The series $\sum_n a_n$ is said to be (\bar{N}, p_n) -summable to l if $t_n \rightarrow l$ as $n \rightarrow \infty$ (see [5]). The (\bar{N}, p_n) -summability method is regular if (i) $P_n \rightarrow \infty$ as $n \rightarrow \infty$ and (ii) $\sum_{v=0}^n p_v \leq C |P_n|$ as $n \rightarrow \infty$.

We write

$$T_n = \frac{1}{(1+q)^n} \sum_{v=0}^n \binom{n}{v} q^{n-v} s_v,$$

which denotes the (E, q) -mean of the sequence (s_n) . The series $\sum_n a_n$ is said to be (E, q) -summable to l if $T_n \rightarrow l$ as $n \rightarrow \infty$ (see [5]). The (E, q) -summability method is regular.

Further, we write

$$\tau_n = \frac{1}{(1+q)^n} \sum_{k=0}^n \binom{n}{k} q^{n-k} \frac{1}{P_k} \sum_{v=0}^k p_v s_v,$$

which denotes the (E, q) -transform of the (\bar{N}, p_n) -mean of the sequence (s_n) . The series $\sum_n a_n$ is said to be $(E, q)(\bar{N}, p_n)$ -summable to l if $\tau_n \rightarrow l$ as $n \rightarrow \infty$ (see [6]).

We also need the following result on the weak convergence of sequences in the Banach space of all continuous functions defined on a finite closed interval (see [7, 8]).

Lemma B $\lim_{n \rightarrow \infty} \int_0^\pi g_n dh_x = 0$ for all $h_x \in BV[0, \pi]$ if and only if $\|g_n\| < \infty$ for all n and $\lim_{n \rightarrow \infty} g_n = 0$.

We need the following well-known Dirichlet-Jordan criterion for Fourier series (see [8, 9]).

Lemma C The trigonometric Fourier series of a 2π -periodic function f of bounded variation converges to $[f(x+0) - f(x-0)]/2$ for every x , and this convergence is uniform on every closed interval on which f is continuous.

3 Main results

We write

$$\phi_x(t) = f(x+t) + f(x-t)$$

and

$$h_x(t) = \frac{\phi_x(t)}{4 \sin \frac{t}{2}}.$$

Theorem 3.1 *Let $f(x)$ be a periodic function with period 2π and Lebesgue-integrable over $[-\pi, \pi]$, and let $\tilde{S}'_k(x)$ denote the partial sums of series (1.4). Let the method \mathfrak{B} be regular. Then, for every $x \in [-\pi, \pi]$ for which $h_x(t)$ is continuous and of bounded variation on $[0, \pi]$,*

$$\lim_n \sum_k b_{nk}(i) \tilde{S}'_k(x) = \frac{1}{4\pi} \int_0^\pi \operatorname{cosec}^2 \frac{t}{2} \phi_x(t) dt - h_x(0+) \quad (3.1)$$

uniformly in i if and only if

$$\lim_n \sum_k b_{nk}(i) \cos\left(k + \frac{1}{2}\right)t = 0 \quad \text{for every } t \in [0, \pi]. \quad (3.2)$$

Proof We have

$$\begin{aligned} \tilde{S}'_k(x) &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(u) \frac{\partial}{\partial x} \left(\sum_{p=1}^k \sin p(u-x) du \right) \\ &= -\frac{1}{\pi} \int_0^\pi \frac{d}{dt} \left(\frac{\cos \frac{1}{2}t - \cos(k + \frac{1}{2})t}{2 \sin \frac{1}{2}t} \right) (f(x+t) + f(x-t)) dt \\ &= -\frac{1}{\pi} \int_0^\pi \frac{d}{dt} \left(\frac{1}{2} \cos \frac{1}{2}t \right) \phi_x(t) dt + \frac{1}{\pi} \int_0^\pi \frac{d}{dt} \left(\frac{\cos(k + \frac{1}{2})t}{2 \sin \frac{1}{2}t} \right) \phi_x(t) dt \\ &= \frac{1}{4\pi} \int_0^\pi \operatorname{cosec}^2 \frac{t}{2} \phi_x(t) dt - \frac{2}{\pi} \int_0^\pi \left(k + \frac{1}{2} \right) \sin \left(k + \frac{1}{2} \right) t h_x(t) dt \\ &\quad - \frac{1}{\pi} \int_0^\pi \frac{\cos(k + \frac{1}{2})t}{\tan \frac{1}{2}t} h_x(t) dt \\ &= \frac{1}{4\pi} \int_0^\pi \operatorname{cosec}^2 \frac{t}{2} \phi_x(t) dt + \frac{2}{\pi} \int_0^\pi \cos \left(k + \frac{1}{2} \right) t dh_x(t) \\ &\quad - \frac{1}{\pi} \int_0^\pi \frac{\cos(k + \frac{1}{2})t}{\tan \frac{1}{2}t} h_x(t) dt \\ &= \frac{1}{4\pi} \int_0^\pi \operatorname{cosec}^2 \frac{t}{2} \phi_x(t) dt + \frac{2}{\pi} \int_0^\pi \cos \left(k + \frac{1}{2} \right) t dh_x(t) - M_k, \\ \tilde{S}'_k(x) &= \frac{1}{4\pi} \int_0^\pi \operatorname{cosec}^2 \frac{t}{2} \phi_x(t) dt + \frac{2}{\pi} \int_0^\pi \cos \left(k + \frac{1}{2} \right) t dh_x(t) - M_k, \end{aligned} \quad (3.3)$$

where

$$M_k = \frac{1}{\pi} \int_0^\pi \frac{\cos(k + \frac{1}{2})t}{\tan \frac{1}{2}t} h_x(t) dt,$$

and

$$\begin{aligned}\sum_k b_{nk}(i) \tilde{S}'_k(x) &= \sum_k b_{nk}(i) \frac{1}{4\pi} \int_0^\pi \operatorname{cosec}^2 \frac{t}{2} \phi_x(t) dt \\ &\quad + \frac{2}{\pi} \int_0^\pi D_n(t) dh_x(t) - \sum_k b_{nk}(i) M_k\end{aligned}\quad (3.4)$$

with

$$D_n(t) = \sum_k b_{nk}(i) \cos\left(k + \frac{1}{2}\right)t. \quad (3.5)$$

Since $h_x(t)$ is of bounded variation on $[0, \pi]$ and tends to $h_x(0+)$, $h_x(t) \cos \frac{t}{2}$ has the same properties. Therefore by Lemma C we have

$$M_k(t) \rightarrow h_x(0+) \quad \text{as } n \rightarrow \infty. \quad (3.6)$$

Since \mathfrak{B} is regular, by Theorem A we have

$$\lim_n \sum_k b_{nk}(i) M_k(t) = h_x(0+) \quad (3.7)$$

and

$$\lim_n \sum_k b_{nk}(i) \frac{1}{4\pi} \int_0^\pi \operatorname{cosec}^2 \frac{t}{2} \phi_x(t) dt = \frac{1}{4\pi} \int_0^\pi \operatorname{cosec}^2 \frac{t}{2} \phi_x(t) dt.$$

Next, we have to show that (3.2) holds if and only if

$$\lim_n \int_0^\pi D_n(t) dh_x(t) = 0. \quad (3.8)$$

By Lemma B we have that (3.8) holds if and only if

$$|D_n(t)| \leq K \quad \text{for all } n \text{ and } t \in [0, \pi] \quad (3.9)$$

and (3.2) holds, where K is a positive constant.

Since condition (3.9) is satisfied by condition (i) of Theorem A, it follows that (3.8) holds if and only if (3.2) holds.

This completes the proof of the theorem. \square

Theorem 3.2 *Let $f(x)$ be a periodic function with period 2π and Lebesgue-integrable over $[-\pi, \pi]$. Then, for every $x \in [-\pi, \pi]$ for which $h_x(t)$ is continuous and of bounded variation on $[0, \pi]$, $(\tilde{S}'_k(x))$ is (E, q) -summable to $\frac{1}{4\pi} \int_0^\pi \operatorname{cosec}^2 \frac{t}{2} \phi_x(t) dt - h_x(0+)$ if and only if $(\cos(k + \frac{1}{2})t)_k$ is (E, q) -summable to 0 for every $t \in [0, \pi]$.*

Proof It follows easily by choosing $i = 0$ and $b_{nk}(i) = b_{nk}$ with

$$b_{nk} = \begin{cases} \frac{1}{(1+q)^n} \binom{n}{k} q^{n-k}, & 0 \leq k \leq n, \quad q > 0, \\ 0 & \text{otherwise.} \end{cases} \quad \square$$

Theorem 3.3 *Let $f(x)$ be a periodic function with period 2π and Lebesgue-integrable over $[-\pi, \pi]$. Then, for every $x \in [-\pi, \pi]$ for which $h_x(t)$ is continuous and of bounded variation on $[0, \pi]$, $(\tilde{S}_k(x))$ is $(E, q)(\bar{N}, p_n)$ -summable to $\frac{1}{4\pi} \int_0^\pi \operatorname{cosec}^2 \frac{t}{2} \phi_x(t) dt - h_x(0+)$ if and only if $(\cos(k + \frac{1}{2})t)_k$ is $(E, q)(\bar{N}, p_n)$ -summable to 0 for every $t \in [0, \pi]$.*

Proof It follows easily by choosing $i = 0$ and $b_{nk}(i) = c_{nk}$ with

$$c_{nk} = \begin{cases} \binom{n}{k} q^{n-k} \frac{1}{p_k} \sum_{v=0}^k p_v, & 0 \leq k \leq n, q > 0, \\ 0 & \text{otherwise.} \end{cases}$$

□

4 Conclusion

In this paper, we have characterized the $F_{\mathfrak{B}}$ -convergence of conjugate derived Fourier series. We deduced special cases to obtain necessary and sufficient conditions for (E, q) -summability and $(E, q)(\bar{N}, p_n)$ -summability of the conjugate derived Fourier series.

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Competing interests

The authors declare that they have no competing interests.

Authors' contributions

Both authors have read the manuscript and agreed to its content and are responsible for all aspects of the accuracy and integrity of the manuscript.

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