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# Symmetric duality for a higher-order nondifferentiable multiobjective programming problem

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## Abstract

In this paper, a pair of Wolfe type higher-order nondifferentiable symmetric dual programs over arbitrary cones has been studied and then well-suited duality relations have been established considering  $K$ - $F$  convexity assumptions. An example which satisfies the weak duality relation has also been depicted.

**MSC:** 90C29; 90C30; 49N15

**Keywords:** symmetric duality; higher-order  $K$ - $F$  convexity; multiobjective programming; support function; efficient solutions

## 1 Introduction

Consider the following multiobjective programming problem:

$$(P) \quad K\text{-minimize } f(x) \\ \text{subject to } x \in X^0 = \{x \in S : -g(x) \in C\},$$

where  $S \subset R^n$  be open,  $f : S \rightarrow R^k$ ,  $g : S \rightarrow R^m$ ,  $K$ , and  $C$  are closed convex pointed cones with nonempty interiors in  $R^k$  and  $R^m$ , respectively.

Several researchers have studied the duality relations for different dual problems of (P) under various generalized convexity assumptions. Chen [1] considered a pair of symmetric higher-order Mond-Weir type nondifferentiable multiobjective programming problems and established duality relations under higher-order  $F$ -convexity assumptions. Later on, Agarwal *et al.* [2] have filled some of the gap in the work of Chen [1] and proved a strong duality theorem for a Mond-Weir type multiobjective higher-order nondifferentiable symmetric dual program. Khurana [3] considered a pair of Mond-Weir type symmetric dual multiobjective programs over arbitrary cones and established duality results under cone-pseudoinvex and strongly cone-pseudoinvex assumptions. Later on, Kim and Kim [4] extended the results in Khurana [3] to the nondifferentiable multiobjective symmetric dual problem. Gupta and Jayswal [5] studied the higher-order Mond-Weir type multiobjective symmetric duality over cones using higher-order cone-preinvex and cone-pseudoinvex functions, which further extends some of the results in [3, 6, 7].

Agarwal *et al.* [8] formulated a pair of Mond-Weir type nondifferentiable multiobjective higher-order symmetric dual programs over arbitrary cones and established duality

theorems under higher-order  $K$ - $F$  convexity assumptions. In the recent work of Suneja and Louhan [9], the authors have considered Wolfe and Mond-Weir type differentiable symmetric higher-order dual pairs. The Mond-Weir type model studied in [9] is similar to the problem considered in Gupta and Jayswal [5]. However, the strong duality result in [9] is for arbitrary cones in  $R^k$  instead of only those cones which contain the nonnegative orthant of  $R^k$  as considered in [5].

In the present paper, a pair of Wolfe type higher-order multiobjective nondifferentiable symmetric dual program have been formulated and we established weak, strong, and converse duality theorems under  $K$ - $F$  convexity assumptions. We also illustrate a nontrivial example of a function which satisfies the weak duality relation.

## 2 Definitions and preliminaries

Let  $C_1 \subseteq R^n$  and  $C_2 \subseteq R^m$  be closed convex cones with nonempty interiors and let  $S_1$  and  $S_2$  be nonempty open sets in  $R^n$  and  $R^m$ , respectively such that  $C_1 \times C_2 \subseteq S_1 \times S_2$ . For a real valued twice differentiable function  $f(x, y)$  defined on  $S_1 \times S_2$ ,  $\nabla_x f(x, y)$  denotes the gradient vector of  $f$  with respect to  $x$  at  $(x, y)$ ,  $\nabla_{xx} f(x, y)$  denotes the Hessian matrix with respect to  $x$  at  $(x, y)$ . Similarly,  $\nabla_y f(x, y)$ ,  $\nabla_{xy} f(x, y)$ , and  $\nabla_{yy} f(x, y)$  are also defined.

**Definition 2.1** [8] A point  $\bar{x} \in X^0$  is a weak efficient solution of (P) if there exists no  $x \in X^0$  such that

$$f(\bar{x}) - f(x) \in \text{int} K.$$

**Definition 2.2** [5] A point  $\bar{x} \in X^0$  is an efficient solution of (P) if there exists no  $x \in X^0$  such that

$$f(\bar{x}) - f(x) \in K \setminus \{0\}.$$

**Definition 2.3** The positive dual cone  $K^+$  of  $K$  is defined by

$$K^+ = \{y : x^T y \geq 0 \text{ for all } x \in K\}.$$

**Definition 2.4** For all  $(x, u) \in S_1 \times S_2$ , a functional  $F : S_1 \times S_2 \times R^n \rightarrow R$  is said to be sublinear with respect to the third variable, if

- (i)  $F(x, u; a_1 + a_2) \leq F(x, u; a_1) + F(x, u; a_2)$  for all  $a_1, a_2 \in R^n$ ,
- (ii)  $F(x, u; \beta a) = \beta F(x, u; a)$ , for all  $\beta \in R_+$  and for all  $a \in R^n$ .

For convenience, we write  $F(x, u; a) = F_{x,u}(a)$ .

**Definition 2.5** [8] Let  $F : S_1 \times S_2 \times R^n \rightarrow R$  be a sublinear functional with respect to the third variable. Also, let  $h_i : S_1 \times R^n \rightarrow R, i = 1, 2, \dots, k$  be a differentiable function. Then the function  $f : S_1 \times S_2 \rightarrow R^k$  is said to be higher-order  $K$ - $F$  convex in the first variable at  $u \in S_1$  for fixed  $v \in S_2$  with respect to  $h$ , such that for  $x \in S_1, p_i \in R^n, i = 1, 2, \dots, k$ ,

$$\begin{aligned} & (f_1(x, v) - f_1(u, v) - F_{x,u}(\nabla_x f_1(u, v) + \nabla_{p_1} h_1(u, p_1)) - h_1(u, p_1) + p_1^T [\nabla_{p_1} h_1(u, p_1)], \dots, \\ & f_k(x, v) - f_k(u, v) - F_{x,u}(\nabla_x f_k(u, v) + \nabla_{p_k} h_k(u, p_k)) - h_k(u, p_k) + p_k^T [\nabla_{p_k} h_k(u, p_k)]) \\ & \in K. \end{aligned}$$

**Definition 2.6** [10] Let  $\varphi$  be a compact convex set in  $R^n$ . The support function of  $\varphi$  is defined by

$$S(x|\varphi) = \max \{x^T y : y \in \varphi\}.$$

The subdifferentiable of  $S(x|\varphi)$  is given by

$$\partial S(x|\varphi) = \{z \in \varphi : z^T x = S(x|\varphi)\}.$$

For any set  $S \subset R^n$ , the normal cone to  $S$  at a point  $x \in S$  is defined by

$$N_S(x) = \{y \in R^n : y^T(z - x) \leq 0 \text{ for all } z \in S\}.$$

For each  $i = 1, 2, \dots, k$ , let  $f_i : S_1 \times S_2 \rightarrow R$ ,  $h_i : S_1 \times S_2 \times R^m \rightarrow R$  and  $g_i : S_1 \times S_2 \times R^n \rightarrow R$  be differentiable functions.  $p = (p_1, p_2, \dots, p_k)$  and  $r = (r_1, r_2, \dots, r_k)$ , for  $p_i \in R^m$  and  $r_i \in R^n$ ,  $i = 1, 2, \dots, k$ .  $C_1^+$  and  $C_2^+$  are the positive dual cones of  $C_1$  and  $C_2$ , respectively.  $D$  and  $E$  are the compact convex sets in  $R^n$  and  $R^m$ , respectively. Also, we use the following notations:

$$\begin{aligned} h(x, y, p) &= (h_1(x, y, p_1), h_2(x, y, p_2), \dots, h_k(x, y, p_k)), \\ g(u, v, r) &= (g_1(u, v, r_1), g_2(u, v, r_2), \dots, g_k(u, v, r_k)), \\ \nabla_p h(x, y, p) &= (\nabla_{p_1} h_1(x, y, p_1), \nabla_{p_2} h_2(x, y, p_2), \dots, \nabla_{p_k} h_k(x, y, p_k)), \\ \nabla_r g(u, v, r) &= (\nabla_{r_1} g_1(u, v, r_1), \nabla_{r_2} g_2(u, v, r_2), \dots, \nabla_{r_k} g_k(u, v, r_k)), \\ p^T \nabla_p h(x, y, p) &= (p_1^T \nabla_{p_1} h_1(x, y, p_1), p_2^T \nabla_{p_2} h_2(x, y, p_2), \dots, p_k^T \nabla_{p_k} h_k(x, y, p_k)) \quad \text{and} \\ r^T \nabla_r g(u, v, r) &= (r_1^T \nabla_{r_1} g_1(u, v, r_1), r_2^T \nabla_{r_2} g_2(u, v, r_2), \dots, r_k^T \nabla_{r_k} g_k(u, v, r_k)). \end{aligned}$$

### 3 Problem formulation

Consider the following pair of Wolfe type higher-order nondifferentiable multiobjective symmetric dual programs:

$$\begin{aligned} \text{(WHP)} \quad &K\text{-minimize } f(x, y) + h(x, y, p) + S(x|D)e - p^T \nabla_p h(x, y, p) \\ &- y^T \left( \sum_{i=1}^k \lambda_i \{ \nabla_y f_i(x, y) + \nabla_{p_i} h_i(x, y, p_i) \} e \right) \end{aligned}$$

subject to

$$- \left( \sum_{i=1}^k \lambda_i \{ \nabla_y f_i(x, y) + \nabla_{p_i} h_i(x, y, p_i) \} - z \right) \in C_2^+, \tag{1}$$

$$\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k) \in \text{int} K^+, \quad \lambda^T e = 1, \quad x \in C_1, \quad z \in E, \tag{2}$$

$$\text{(WHD)} \quad K\text{-maximize } f(u, v) + g(u, v, r) - S(v|E)e - r^T \nabla_r g(u, v, r)$$

$$- u^T \left( \sum_{i=1}^k \lambda_i \{ \nabla_u f_i(u, v) + \nabla_{r_i} g_i(u, v, r_i) \} e \right)$$

subject to

$$\left( \sum_{i=1}^k \lambda_i \{ \nabla_u f_i(u, v) + \nabla_{r_i} g_i(u, v, r_i) \} + w \right) \in C_1^+, \tag{3}$$

$$\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k) \in \text{int } K^+, \quad \lambda^T e = 1, \quad v \in C_2, \quad w \in D, \tag{4}$$

where  $e = (e_1, e_2, \dots, e_k) \in \text{int } K$  is fixed.

**Remark 3.1** If  $D = \{0\}$  and  $E = \{0\}$ , then our problems (WHP) and (WHD) become the problem studied in Suneja and Louhan [9].

Next, we will prove weak, strong, and converse duality results between (WHP) and (WHD).

**Theorem 3.1** (Weak duality) *Let  $(x, y, \lambda, z, p)$  and  $(u, v, \lambda, w, r)$  be feasible solutions for (WHP) and (WHD), respectively. Assume the following conditions hold:*

- (I)  $f(\cdot, v) + (\cdot)^T w$  is higher-order  $K$ - $F$  convex at  $u$  with respect to  $g(u, v, r)$  for fixed  $v$ ,
- (II)  $-f(x, \cdot) + (\cdot)^T z$  is higher-order  $K$ - $G$  convex at  $y$  with respect to  $-h(x, y, p)$  for fixed  $x$ ,
- (III)  $R_+^k \subseteq K$ ,

where  $F : S_1 \times S_2 \times R^n \rightarrow R$  and  $G : S_1 \times S_2 \times R^m \rightarrow R$  are the sublinear functionals with respect to the third variable and satisfy the following conditions:

$$F_{x,u}(a) + u^T a \geq 0 \quad \text{for all } a \in C_1^+, \tag{A}$$

$$G_{v,y}(b) + b^T y \geq 0 \quad \text{for all } b \in C_2^+. \tag{B}$$

Then

$$\begin{aligned} & \left[ f(u, v) + g(u, v, r) - S(v|E)e - r^T \nabla_r g(u, v, r) - u^T \sum_{i=1}^k \lambda_i \{ \nabla_u f_i(u, v) + \nabla_{r_i} g_i(u, v, r_i) \} e \right] \\ & - \left[ f(x, y) + h(x, y, p) + S(x|D)e - p^T \nabla_p h(x, y, p) \right. \\ & \left. - y^T \sum_{i=1}^k \lambda_i \{ \nabla_y f_i(x, y) + \nabla_{p_i} h_i(x, y, p_i) \} e \right] \\ & \notin K \setminus \{0\}. \end{aligned} \tag{5}$$

*Proof* We shall obtain the proof by contradiction. Let (5) not hold. Then

$$\begin{aligned} & \left[ f(u, v) + g(u, v, r) - S(v|E)e - r^T \nabla_r g(u, v, r) - u^T \sum_{i=1}^k \lambda_i \{ \nabla_u f_i(u, v) + \nabla_{r_i} g_i(u, v, r_i) \} e \right] \\ & - \left[ f(x, y) + h(x, y, p) + S(x|D)e - p^T \nabla_p h(x, y, p) \right. \\ & \left. - y^T \sum_{i=1}^k \lambda_i \{ \nabla_y f_i(x, y) + \nabla_{p_i} h_i(x, y, p_i) \} e \right] \\ & \in K \setminus \{0\}. \end{aligned}$$

It follows from  $\lambda \in \text{int } K^+$  and  $\lambda^T e = 1$  that

$$\begin{aligned} & \left[ \sum_{i=1}^k \lambda_i \{f_i(u, v) + g_i(u, v, r_i) - r_i^T \nabla_{r_i} g_i(u, v, r_i)\} - S(v|E) \right. \\ & \quad \left. - u^T \sum_{i=1}^k \lambda_i \{ \nabla_u f_i(u, v) + \nabla_{r_i} g_i(u, v, r_i) \} \right] \\ & \quad - \left[ \sum_{i=1}^k \lambda_i \{f_i(x, y) + h_i(x, y, p_i) - p_i^T \nabla_{p_i} h_i(x, y, p_i)\} \right. \\ & \quad \left. + S(x|D) - y^T \sum_{i=1}^k \lambda_i \{ \nabla_y f_i(x, y) + \nabla_{p_i} h_i(x, y, p_i) \} \right] > 0. \end{aligned} \tag{6}$$

Now, since  $f(\cdot, v) + (\cdot)^T w$  is higher-order  $K$ - $F$  convex at  $u$  with respect to  $g(u, v, r)$  for fixed  $v$ , we get

$$\begin{aligned} & (f_1(x, v) + x^T w e_1 - f_1(u, v) - u^T w e_1 - F_{x,u} [ \nabla_u f_1(u, v) + w e_1 + \nabla_{r_1} g_1(u, v, r_1) ] \\ & \quad - g_1(u, v, r_1) + r_1^T \nabla_{r_1} g_1(u, v, r_1), \dots, f_k(x, v) + x^T w e_k - f_k(u, v) - u^T w e_k \\ & \quad - F_{x,u} [ \nabla_u f_k(u, v) + w e_k + \nabla_{r_k} g_k(u, v, r_k) ] - g_k(u, v, r_k) + r_k^T \nabla_{r_k} g_k(u, v, r_k)) \in K. \end{aligned}$$

Using  $\lambda \in \text{int } K^+$  and  $\lambda^T e = 1$ , it follows that

$$\begin{aligned} & \sum_{i=1}^k \lambda_i \{f_i(x, v) - f_i(u, v) - g_i(u, v, r_i) + r_i^T \nabla_{r_i} g_i(u, v, r_i)\} + x^T w - u^T w \\ & \geq \sum_{i=1}^k \lambda_i F_{x,u} [ (\nabla_u f_i(u, v) + \nabla_{r_i} g_i(u, v, r_i)) + w e_i ]. \end{aligned}$$

Since  $\lambda \in \text{int } K^+ \subseteq \text{int } R_+^k$  (by hypothesis (III)), hence  $\lambda > 0$ . Therefore, using (2) and sub-linearity of  $F$  in the above expression, we obtain

$$\begin{aligned} & \sum_{i=1}^k \lambda_i \{f_i(x, v) - f_i(u, v) - g_i(u, v, r_i) + r_i^T \nabla_{r_i} g_i(u, v, r_i)\} + x^T w - u^T w \\ & \geq \sum_{i=1}^k F_{x,u} [ \lambda_i (\nabla_u f_i(u, v) + \nabla_{r_i} g_i(u, v, r_i)) + w ]. \end{aligned}$$

It follows from (A) and the dual constraint (3) that

$$\begin{aligned} & \sum_{i=1}^k \lambda_i \{f_i(x, v) - f_i(u, v) - g_i(u, v, r_i) + r_i^T \nabla_{r_i} g_i(u, v, r_i)\} + x^T w - u^T w \\ & \geq -u^T \left( \sum_{i=1}^k \lambda_i [ \nabla_u f_i(u, v) + \nabla_{r_i} g_i(u, v, r_i) ] + w \right) \end{aligned} \tag{7}$$

for  $a = (\sum_{i=1}^k \lambda_i [ \nabla_u f_i(u, v) + \nabla_{r_i} g_i(u, v, r_i) ] + w) \in C_1^+$ .

Similarly, using hypothesis (II), (B),  $\lambda > 0$ , (1), (2), and sublinearity of  $G$ , we obtain

$$\begin{aligned} & \sum_{i=1}^k \lambda_i \{f_i(x, y) - f_i(x, v) + h_i(x, y, p_i) - p_i^T \nabla_{p_i} h_i(x, y, p_i)\} + v^T z - y^T z \\ & \geq y^T \left( \sum_{i=1}^k \lambda_i [\nabla_y f_i(x, y) + \nabla_{p_i} h_i(x, y, p_i)] - z \right) \end{aligned} \tag{8}$$

for  $b = -(\sum_{i=1}^k \lambda_i [\nabla_y f_i(x, y) + \nabla_{p_i} h_i(x, y, p_i)] - z) \in C_2^+$ .

Now, adding (7) and (8), we have

$$\begin{aligned} & \sum_{i=1}^k \lambda_i \{f_i(x, y) - f_i(u, v) - g_i(u, v, r_i) + r_i^T \nabla_{r_i} g_i(u, v, r_i) \\ & \quad + h_i(x, y, p_i) - p_i^T \nabla_{p_i} h_i(x, y, p_i)\} + x^T w + v^T z \\ & \geq -u^T \sum_{i=1}^k \lambda_i [\nabla_u f_i(u, v) + \nabla_{r_i} g_i(u, v, r_i)] \\ & \quad + y^T \sum_{i=1}^k \lambda_i [\nabla_y f_i(x, y) + \nabla_{p_i} h_i(x, y, p_i)]. \end{aligned}$$

Finally, it follows from  $x^T w \leq S(x|D)$  and  $v^T z \leq S(v|E)$  that

$$\begin{aligned} & \sum_{i=1}^k \lambda_i \{f_i(x, y) + h_i(x, y, p_i) - p_i^T \nabla_{p_i} h_i(x, y, p_i)\} \\ & \quad + S(x|D) - y^T \sum_{i=1}^k \lambda_i [\nabla_y f_i(x, y) + \nabla_{p_i} h_i(x, y, p_i)] \\ & \geq \sum_{i=1}^k \lambda_i \{f_i(u, v) + g_i(u, v, r_i) - r_i^T \nabla_{r_i} g_i(u, v, r_i)\} \\ & \quad - S(v|E) - u^T \sum_{i=1}^k \lambda_i [\nabla_u f_i(u, v) + \nabla_{r_i} g_i(u, v, r_i)], \end{aligned}$$

which contradicts (6). Hence the result. □

**Example 3.1** Let  $k = 2$ ,  $n = m = 1$ . Let  $S_1 = S_2 = R_+ = \{x \in R : x \geq 0\}$ ,  $C_1 = C_2 = R_+$ , and  $K = \{(x, y) \in R^2 : x \geq 0, y \geq -x\}$ .

Then  $C_1^+ = C_2^+ = R_+$  and  $K^+ = \{(x, y) \in R^2 : x \geq 0, x \geq y\}$ . Obviously,  $R_+^2 \subseteq K$ .

Let  $f : S_1 \times S_2 \rightarrow R$ ,  $g : S_1 \times S_2 \times R^n \rightarrow R$  and  $h : S_1 \times S_2 \times R^m \rightarrow R$  be defined as

$$f(x, y) = (x^2 - y^2, x^2), \quad g(u, v, r) = (-r_1 u, -r_2 u) \quad \text{and} \quad h(x, y, p) = (p_1 y, p_2 y).$$

Let  $D = [0, 1]$  and  $E = \{0\}$ . Then  $S(x|D) = \frac{x+|x|}{2}$  and  $S(v|E) = 0$ . Suppose  $(e_1, e_2) = (1, 1) \in \text{int} K$ . Also, suppose the sublinear functionals  $F$  and  $G$  are defined as

$$F_{x,u}(a) = x^T a \quad \text{and} \quad G_{v,y}(b) = b^T v.$$

Now, substituting the above defined expressions in the problems (WHP) and (WHD), we get

$$(EP) \quad K\text{-minimize} \left( x^2 - y^2 + \frac{x + |x|}{2} + (\lambda_1 - \lambda_2)y^2, x^2 + \frac{x + |x|}{2} + (\lambda_1 - \lambda_2)y^2 \right)$$

subject to

$$(\lambda_1 - \lambda_2)y \geq 0,$$

$$\lambda_1 > 0, \quad \lambda_1 - \lambda_2 > 0, \quad \lambda_1 + \lambda_2 = 1, \quad x \geq 0,$$

$$(ED) \quad K\text{-maximize} (u^2 - v^2 - (\lambda_1 + \lambda_2)u^2, u^2 - (\lambda_1 + \lambda_2)u^2)$$

subject to

$$(\lambda_1 + \lambda_2)u + w \geq 0,$$

$$\lambda_1 > 0, \quad \lambda_1 - \lambda_2 > 0, \quad \lambda_1 + \lambda_2 = 1, \quad v \geq 0, \quad w \in [0, 1].$$

Now, we shall show that for the primal-dual pair (EP) and (ED), the hypotheses of Theorem 3.1 hold.

(A.1)  $f(\cdot, v) + (\cdot)^T we$  is higher-order  $K$ - $F$  convex at  $u = 0 \in S_1$  with respect to  $g(u, v, r)$  for fixed  $v$  and for all  $x \in S_1, r_1, r_2 \in R$ , and we have

$$\begin{aligned} & (f_1(x, v) + x^T we_1 - f_1(u, v) - u^T we_1 - F_{x,u}[\nabla_u f_1(u, v) + we_1 + \nabla_{r_1} g_1(u, v, r_1)] \\ & \quad - g_1(u, v, r_1) + r_1^T \nabla_{r_1} g_1(u, v, r_1), f_2(x, v) + x^T we_2 - f_2(u, v) - u^T we_2 \\ & \quad - F_{x,u}[\nabla_u f_2(u, v) + we_2 + \nabla_{r_2} g_2(u, v, r_2)] \\ & \quad - g_2(u, v, r_2) + r_2^T \nabla_{r_2} g_2(u, v, r_2)) \\ & = (x^2, x^2) \in K. \end{aligned}$$

(A.2)  $-f(x, \cdot) + (\cdot)^T ze$  is higher-order  $K$ - $G$  convex at  $y = 0 \in S_2$  with respect to  $-h(x, y, p)$  for fixed  $x$  and for all  $v \in S_2, p_1, p_2 \in R$ , and we have

$$\begin{aligned} & (-f_1(x, v) + v^T ze_1 + f_1(x, y) - y^T ze_1 - G_{v,y}[-\nabla_y f_1(x, y) + ze_1 - \nabla_{p_1} h_1(x, y, p_1)] \\ & \quad + h_1(x, y, p_1) - p_1^T \nabla_{p_1} h_1(x, y, p_1), -f_2(x, v) + v^T ze_2 + f_2(x, y) - y^T ze_2 \\ & \quad - G_{v,y}[-\nabla_y f_2(x, y) + ze_2 - \nabla_{p_2} h_2(x, y, p_2)] \\ & \quad + h_2(x, y, p_2) - p_2^T \nabla_{p_2} h_2(x, y, p_2)) \\ & = (v^2, 0) \in K. \end{aligned}$$

(A.3)

$$F_{x,u}(a) + u^T a = (x + u)^T a \geq 0, \quad \forall a \in C_1^+ \text{ and } \forall x, u \in S_1,$$

$$G_{v,y}(b) + b^T y = (v + y)^T b \geq 0, \quad \forall b \in C_2^+ \text{ and } \forall v, y \in S_2.$$

The points  $(x, y, \lambda_1, \lambda_2, z, p_1, p_2) = (1, 0, \frac{3}{4}, \frac{1}{4}, 0, 1, 1)$  and  $(u, v, \lambda_1, \lambda_2, w, r_1, r_2) = (0, 1, \frac{3}{4}, \frac{1}{4}, 1, 1, 1)$  are feasible for the problems (EP) and (ED), respectively. These feasible points do

satisfy the result of the weak duality theorem since

$$\begin{aligned} & \left[ f(u, v) + g(u, v, r) - S(v|E)e - r^T \nabla_r g(u, v, r) - u^T \sum_{i=1}^k \lambda_i \{ \nabla_u f(u, v) + \nabla_{r_i} g_i(u, v, r_i) \} e \right] \\ & - \left[ f(x, y) + h(x, y, p) + S(x|D)e - p^T \nabla_p h(x, y, p) \right. \\ & \left. - y^T \sum_{i=1}^k \lambda_i \{ \nabla_y f(x, y) + \nabla_{p_i} h_i(x, y, p_i) \} e \right] \\ & = (-3, -2) \notin K \setminus \{0\}. \end{aligned}$$

**Theorem 3.2** (Strong duality) *Let  $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{z}, \bar{p})$  be a weak efficient solution of (WHD). Let*

- (I) *the Hessian matrix  $\nabla_{p_i p_i} h_i(\bar{x}, \bar{y}, \bar{p}_i)$  for all  $i = 1, 2, \dots, k$  be positive or negative definite;*
- (II)  *$\bar{p}_i \neq 0$ , for some  $i \in \{1, 2, \dots, k\}$  imply that  $\sum_{i=1}^k \xi_i \nabla_{yy} f_i(\bar{x}, \bar{y}) \bar{p}_i \neq 0$  for all  $\xi \in K^+$ ;*
- (III)  *$\sum_{i=1}^k \xi_i \nabla_{yy} f_i(\bar{x}, \bar{y}) \bar{p}_i \notin \text{span}\{ \nabla_y f_i(\bar{x}, \bar{y}) + \nabla_y h_i(\bar{x}, \bar{y}, \bar{p}_i), \nabla_y f_i(\bar{x}, \bar{y}) + \nabla_{p_i} h_i(\bar{x}, \bar{y}, \bar{p}_i) \}$ ,  $i = 1, 2, \dots, k\} \setminus \{0\}$ , for all  $\xi \in K^+$ ;*
- (IV) *the set of vectors  $\{ \nabla_y f_i(\bar{x}, \bar{y}) : i = 1, 2, \dots, k \}$  be linearly independent;*
- (V)  *$\nabla_y h_i(\bar{x}, \bar{y}, 0) = 0 = \nabla_{p_i} h_i(\bar{x}, \bar{y}, 0)$ ,  $h_i(\bar{x}, \bar{y}, 0) = g_i(\bar{x}, \bar{y}, 0)$ ,  $\nabla_x h_i(\bar{x}, \bar{y}, 0) = \nabla_{r_i} g_i(\bar{x}, \bar{y}, 0)$ , for all  $i = \{1, 2, \dots, k\}$ .*

Then

- (I) *there exists  $\bar{w} \in D$  such that  $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{w}, \bar{r} = 0)$  is feasible for (WHD) and*
- (II) *the objective values of (WHP) and (WHD) are equal.*

Also, if the hypotheses of Theorem 3.1 are satisfied for all feasible solutions of (WHP) and (WHD), then  $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{w}, \bar{r} = 0)$  is an efficient solution for (WHD).

*Proof* Since  $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{z}, \bar{p})$  is a weak efficient solution for (WHP), by the Fritz John necessary optimality conditions [11], there exist  $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k) \in K^+$ ,  $\beta \in C_2$ , and  $\eta \in R$  such that

$$\begin{aligned} & \left[ \sum_{i=1}^k \alpha_i (\nabla_x f_i(\bar{x}, \bar{y}) + \nabla_x h_i(\bar{x}, \bar{y}, \bar{p}_i)) + (\alpha^T e) \gamma + \sum_{i=1}^k \bar{\lambda}_i \{ \beta - (\alpha^T e) \bar{y} \} \nabla_{xy} f_i(\bar{x}, \bar{y}) \right. \\ & \left. + \sum_{i=1}^k \nabla_{p_i x} h_i(\bar{x}, \bar{y}, \bar{p}_i) (\bar{\lambda}_i \beta - \bar{\lambda}_i (\alpha^T e) \bar{y} - \alpha_i \bar{p}_i) \right]^T (x - \bar{x}) \geq 0 \quad \text{for all } x \in C_1, \end{aligned} \tag{9}$$

$$\begin{aligned} & \sum_{i=1}^k \alpha_i [\nabla_y f_i(\bar{x}, \bar{y}) + \nabla_y h_i(\bar{x}, \bar{y}, \bar{p}_i)] - \sum_{i=1}^k (\alpha^T e) \bar{\lambda}_i \{ \nabla_y f_i(\bar{x}, \bar{y}) + \nabla_{p_i} h_i(\bar{x}, \bar{y}, \bar{p}_i) \} \\ & + \sum_{i=1}^k \bar{\lambda}_i \nabla_{yy} f_i(\bar{x}, \bar{y}) (\beta - (\alpha^T e) \bar{y}) \\ & + \sum_{i=1}^k \nabla_{p_i y} h_i(\bar{x}, \bar{y}, \bar{p}_i) (\bar{\lambda}_i \beta - \bar{\lambda}_i (\alpha^T e) \bar{y} - \alpha_i \bar{p}_i) = 0, \end{aligned} \tag{10}$$

$$\{ \beta - (\alpha^T e) \bar{y} \}^T [\nabla_y f_i(\bar{x}, \bar{y}) + \nabla_{p_i} h_i(\bar{x}, \bar{y}, \bar{p}_i)] + \eta e_i = 0, \quad i = 1, 2, \dots, k, \tag{11}$$

$$\nabla_{p_i p_i} h_i(\bar{x}, \bar{y}, \bar{p}_i) (\bar{\lambda}_i \beta - \bar{\lambda}_i (\alpha^T e) \bar{y} - \alpha_i \bar{p}_i) = 0, \quad i = 1, 2, \dots, k, \tag{12}$$

$$\beta^T \left( \sum_{i=1}^k \bar{\lambda}_i \{ \nabla_{\mathcal{Y}} f_i(\bar{x}, \bar{y}) + \nabla_{p_i} h_i(\bar{x}, \bar{y}, \bar{p}_i) \} - \bar{z} \right) = 0, \quad i = 1, 2, \dots, k, \tag{13}$$

$$\eta^T (\bar{\lambda}^T e - 1) = 0, \tag{14}$$

$$\beta \in N_E(\bar{z}), \tag{15}$$

$$\gamma \in D, \quad \gamma^T \bar{x} = S(\bar{x}|D), \tag{16}$$

$$(\alpha, \beta, \eta) \neq 0. \tag{17}$$

Now, hypothesis (I) and (12) imply that

$$\bar{\lambda}_i \beta - \bar{\lambda}_i (\alpha^T e) \bar{y} - \alpha_i \bar{p}_i = 0, \quad i = 1, 2, \dots, k. \tag{18}$$

Using (18) in (10), we have

$$\begin{aligned} \sum_{i=1}^k \alpha_i \nabla_{\mathcal{Y}\mathcal{Y}} f_i(\bar{x}, \bar{y}) \bar{p}_i &= (\alpha^T e) \sum_{i=1}^k \bar{\lambda}_i (\nabla_{\mathcal{Y}} f_i(\bar{x}, \bar{y}) + \nabla_{p_i} h_i(\bar{x}, \bar{y}, \bar{p}_i)) \\ &\quad - \sum_{i=1}^k \alpha_i (\nabla_{\mathcal{Y}} f_i(\bar{x}, \bar{y}) + \nabla_{p_i} h_i(\bar{x}, \bar{y}, \bar{p}_i)), \end{aligned} \tag{19}$$

which yields

$$\begin{aligned} \sum_{i=1}^k \alpha_i \nabla_{\mathcal{Y}\mathcal{Y}} f_i(\bar{x}, \bar{y}) \bar{p}_i &\in \text{span} \{ \nabla_{\mathcal{Y}} f_i(\bar{x}, \bar{y}) + \nabla_{p_i} h_i(\bar{x}, \bar{y}, \bar{p}_i), \nabla_{\mathcal{Y}} f_i(\bar{x}, \bar{y}) \\ &\quad + \nabla_{p_i} h_i(\bar{x}, \bar{y}, \bar{p}_i), i = 1, 2, \dots, k \}. \end{aligned} \tag{20}$$

Now, we claim that  $\bar{p}_i = 0$  for all  $i = 1, 2, \dots, k$ . On the contrary, suppose that for some  $i \in \{1, 2, \dots, k\}$ ,  $\bar{p}_i \neq 0$ , then using hypothesis (II), we have

$$\sum_{i=1}^k \alpha_i \nabla_{\mathcal{Y}\mathcal{Y}} f_i(\bar{x}, \bar{y}) \bar{p}_i \neq 0. \tag{21}$$

This contradicts hypothesis (III) (by (20) and (21)). Hence,

$$\bar{p}_i = 0 \quad \text{for all } i = 1, 2, \dots, k. \tag{22}$$

Using (22) in (18), we have  $\bar{\lambda}_i \beta = \bar{\lambda}_i (\alpha^T e) \bar{y}$ ,  $i = 1, 2, \dots, k$ .

Since  $\bar{\lambda} \in \text{int} K^+$ ,  $\bar{\lambda}_i \neq 0$  for at least one  $i$ ,

$$\beta = (\alpha^T e) \bar{y}. \tag{23}$$

It follows from (11) and (23) that  $\eta e_i = 0$ ,  $i = 1, 2, \dots, k$ , which from  $e \in \text{int} K$  implies  $\eta = 0$ .

From (19), (22), and hypothesis (V), we get

$$\sum_{i=1}^k \{ \alpha_i - (\alpha^T e) \bar{\lambda}_i \} \nabla_{\mathcal{Y}} f_i(\bar{x}, \bar{y}) = 0,$$

which from hypothesis (IV) yields

$$\alpha = (\alpha^T e)\bar{\lambda}. \tag{24}$$

Now, if  $\alpha = 0$ , then  $\alpha^T e = 0$ . Therefore, from (23), we get  $\beta = 0$  and hence,  $(\alpha, \beta, \eta) = 0$ . This contradicts (17). Thus  $\alpha \neq 0$ . Since  $\alpha \in K^+$  and  $e \in \text{int}K$ , we have

$$\alpha^T e > 0. \tag{25}$$

From (23) and (25), we obtain

$$\bar{y} = \frac{\beta}{(\alpha^T e)} \in C_2.$$

Further, using inequalities (18), (23)-(25) in (9), we obtain

$$\left[ \sum_{i=1}^k \bar{\lambda}_i \{ \nabla_x f_i(\bar{x}, \bar{y}) + \nabla_x h_i(\bar{x}, \bar{y}, \bar{p}_i) \} + \gamma \right]^T (x - \bar{x}) \geq 0 \quad \text{for all } x \in C_1.$$

For  $\bar{r} = 0$ , it follows from (22) and hypothesis (V) that

$$\left[ \sum_{i=1}^k \bar{\lambda}_i \{ \nabla_x f_i(\bar{x}, \bar{y}) + \nabla_{r_i} g_i(\bar{x}, \bar{y}, \bar{r}_i) \} + \gamma \right]^T (x - \bar{x}) \geq 0. \tag{26}$$

Let  $x \in C_1$ . Then  $\bar{x} + x \in C_1$  and hence from (26), we have

$$\left[ \sum_{i=1}^k \bar{\lambda}_i \{ \nabla_x f_i(\bar{x}, \bar{y}) + \nabla_{r_i} g_i(\bar{x}, \bar{y}, \bar{r}_i) \} + \gamma \right]^T x \geq 0 \quad \text{for all } x \in C_1.$$

Therefore,  $[\sum_{i=1}^k \bar{\lambda}_i \{ \nabla_x f_i(\bar{x}, \bar{y}) + \nabla_{r_i} g_i(\bar{x}, \bar{y}, \bar{r}_i) \} + \gamma] \in C_1^+$ .

Thus,  $(\bar{x}, \bar{y}, \bar{\lambda}, \gamma = \bar{w}, \bar{r} = 0)$  is a feasible solution for the dual problem.

Consider  $x = 0$  and  $x = 2\bar{x}$  in (26), we get

$$\left[ \sum_{i=1}^k \bar{\lambda}_i \{ \nabla_x f_i(\bar{x}, \bar{y}) + \nabla_{r_i} g_i(\bar{x}, \bar{y}, \bar{r}_i) \} + \gamma \right]^T \bar{x} = 0,$$

which implies that

$$\bar{x}^T \sum_{i=1}^k \bar{\lambda}_i \{ \nabla_x f_i(\bar{x}, \bar{y}) + \nabla_{r_i} g_i(\bar{x}, \bar{y}, \bar{r}_i) \} = -\bar{x}^T \gamma = -S(\bar{x}|D). \tag{27}$$

Now, (15) and (23) yield  $(\alpha^T e)\bar{y} \in N_E(\bar{z})$ . Since  $\alpha^T e > 0$ ,  $\bar{y} \in N_E(\bar{z})$ .

Again as  $E$  is a compact convex set in  $R^m$ ,  $\bar{y}^T \bar{z} = S(\bar{y}|E)$ .

Further, (13), (23), and (25) yield

$$\bar{y}^T \sum_{i=1}^k \bar{\lambda}_i \{ \nabla_y f_i(\bar{x}, \bar{y}) + \nabla_{p_i} h_i(\bar{x}, \bar{y}, \bar{p}_i) \} = \bar{y}^T \bar{z} = S(\bar{y}|E). \tag{28}$$

By hypothesis (V) for  $\bar{r} = 0$ , (22), (27)-(28), we obtain

$$\begin{aligned} & f(\bar{x}, \bar{y}) + h(\bar{x}, \bar{y}, \bar{p}) + S(\bar{x}|D)e - \bar{p}^T \nabla_p h(\bar{x}, \bar{y}, \bar{p}) - \bar{y}^T \sum_{i=1}^k \bar{\lambda}_i \{ \nabla_y f_i(\bar{x}, \bar{y}) + \nabla_{p_i} h_i(\bar{x}, \bar{y}, \bar{p}_i) \} e \\ & = f(\bar{x}, \bar{y}) + g(\bar{x}, \bar{y}, \bar{r}) - S(\bar{y}|E)e - \bar{r}^T \nabla_r g(\bar{x}, \bar{y}, \bar{r}) - \bar{x}^T \sum_{i=1}^k \bar{\lambda}_i \{ \nabla_x f_i(\bar{x}, \bar{y}) + \nabla_{r_i} g_i(\bar{x}, \bar{y}, \bar{r}_i) \} e. \end{aligned}$$

Hence, the two objective values are equal.

Now, let  $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{w}, \bar{r} = 0)$  be not an efficient solution of (WHD), then there exists a point  $(\hat{u}, \hat{v}, \hat{\lambda}, \hat{w}, \hat{r})$  feasible for (WHD) such that

$$\begin{aligned} & \left[ f(\hat{u}, \hat{v}) + g(\hat{u}, \hat{v}, \hat{r}) - S(\hat{v}|E)e - \hat{r}^T \nabla_r g(\hat{u}, \hat{v}, \hat{r}) - \hat{u}^T \sum_{i=1}^k \bar{\lambda}_i \{ \nabla_u f_i(\hat{u}, \hat{v}) + \nabla_{r_i} g_i(\hat{u}, \hat{v}, \hat{r}_i) \} e \right] \\ & - \left[ f(\bar{x}, \bar{y}) + g(\bar{x}, \bar{y}, \bar{r}) - S(\bar{y}|E)e - \bar{r}^T \nabla_r g(\bar{x}, \bar{y}, \bar{r}) \right. \\ & \left. - \bar{x}^T \sum_{i=1}^k \bar{\lambda}_i \{ \nabla_x f_i(\bar{x}, \bar{y}) + \nabla_{r_i} g_i(\bar{x}, \bar{y}, \bar{r}_i) \} e \right] \\ & \in K \setminus \{0\}. \end{aligned}$$

From (27), (28), and hypothesis (V) for  $\bar{r} = 0$  and  $\bar{p} = 0$ , we obtain

$$\begin{aligned} & \left[ f(\hat{u}, \hat{v}) + g(\hat{u}, \hat{v}, \hat{r}) - S(\hat{v}|E)e - \hat{r}^T \nabla_r g(\hat{u}, \hat{v}, \hat{r}) - \hat{u}^T \sum_{i=1}^k \bar{\lambda}_i \{ \nabla_u f_i(\hat{u}, \hat{v}) + \nabla_{r_i} g_i(\hat{u}, \hat{v}, \hat{r}_i) \} e \right] \\ & - \left[ f(\bar{x}, \bar{y}) + h(\bar{x}, \bar{y}, \bar{p}) + S(\bar{x}|D)e - \bar{p}^T \nabla_p h(\bar{x}, \bar{y}, \bar{p}) \right. \\ & \left. - \bar{y}^T \sum_{i=1}^k \bar{\lambda}_i \{ \nabla_y f_i(\bar{x}, \bar{y}) + \nabla_{p_i} h_i(\bar{x}, \bar{y}, \bar{p}_i) \} e \right] \\ & \in K \setminus \{0\}, \end{aligned}$$

which contradicts Theorem 3.1. Hence,  $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{w}, \bar{r} = 0)$  is the efficient solution of (WHD).  $\square$

**Theorem 3.3** (Converse duality) *Let  $(\bar{u}, \bar{v}, \bar{\lambda}, \bar{w}, \bar{r})$  be a weak efficient solution of (WHP). Let*

- (I) *the Hessian matrix  $\nabla_{r_i r_i} g_i(\bar{u}, \bar{v}, \bar{r}_i)$  for all  $i = 1, 2, \dots, k$  be positive or negative definite;*
- (II)  *$\bar{r}_i \neq 0$ , for some  $i \in \{1, 2, \dots, k\}$  implies that  $\sum_{i=1}^k \xi_i \nabla_{uu} f_i(\bar{u}, \bar{v}) \bar{r}_i \neq 0$  for all  $\xi \in K^+$ ;*
- (III)  *$\sum_{i=1}^k \xi_i \nabla_{uu} f_i(\bar{u}, \bar{v}) \bar{r}_i \notin \text{span}\{ \nabla_u f_i(\bar{u}, \bar{v}) + \nabla_u g_i(\bar{u}, \bar{v}, \bar{r}_i), \nabla_u f_i(\bar{u}, \bar{v}) + \nabla_{r_i} g_i(\bar{u}, \bar{v}, \bar{r}_i), i = 1, 2, \dots, k \} \setminus \{0\}$ , for all  $\xi \in K^+$ ;*
- (IV) *the set of vectors  $\{ \nabla_u f_i(\bar{u}, \bar{v}) : i = 1, 2, \dots, k \}$  be linearly independent;*
- (V)  *$\nabla_u g_i(\bar{u}, \bar{v}, 0) = 0 = \nabla_{r_i} g_i(\bar{u}, \bar{v}, 0)$ ,  $g_i(\bar{u}, \bar{v}, 0) = h_i(\bar{u}, \bar{v}, 0)$ ,  $\nabla_v g_i(\bar{u}, \bar{v}, 0) = \nabla_{p_i} h_i(\bar{u}, \bar{v}, 0)$ , for all  $i = \{1, 2, \dots, k\}$ .*

*Then*

- (I) there exists  $\bar{z} \in E$  such that  $(\bar{u}, \bar{v}, \bar{\lambda}, \bar{z}, \bar{p} = 0)$  is feasible for (WHP) and  
(II) the objective values of (WHP) and (WHD) are equal.

Also, if the hypotheses of Theorem 3.1 are satisfied for all feasible solutions of (WHP) and (WHD), then  $(\bar{u}, \bar{v}, \bar{\lambda}, \bar{z}, \bar{p} = 0)$  is an efficient solution for (WHP).

*Proof* The proof follows along the lines of Theorem 3.2. □

#### Competing interests

The authors declare that they have no competing interests.

#### Authors' contributions

All the authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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