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Non linear analysis of a functionally graded beam with variable thickness

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The nonlinear analysis of a functionally graded beam under combined loads is investigated analytically in this paper. The point loads and moment is applied at the end of the beam. The area section is considered to vary continuously along the longitudinal axis of the beam. The most important idea for presentation of this paper is the application of the present method for a vast class of functionally graded and variable thickness beams. The gradation along the longitudinal axis of the beam is studied as a novel subject. Euler-Bernoulli theory is employed to derive the governing differential equation of a FG beam. The Adomian's Decomposition Method (ADM) is used for solution of the governing nonlinear differential equation. The analytical results are compared with those results obtained using the numerical simulation (FEM). This comparison indicates that the difference between them is not significant. The effect of non-homogenous coefficient on the rotation and deflection of the beam is studied for different values of non homogenous index. This study presents the general formulation to evaluate the deflection and rotation of the beams with variable properties.

Key words: Nonlinear, FG beam, ADM, rotation, deflection.

INTRODUCTION

There are two main theories to analyze the beam under specific loads. Euler Bernoulli and Timoshenko theories are two mentioned. Euler-Bernoulli theory considers the bending deformation and does not consider shear deformation. Timoshenko beam theory is the other theory that can be applied for wide beams and those classes of the beams that shear deformation is important in those analyses. Some difficulties arise in Timoshenko theory due to the complexity of boundary condition. Due to this problem, Euler-Bernoulli theory can be used for analysis of the problems. A beam with variable thickness and material properties along the longitudinal direction is considered for this paper.

Banerjee et al. (2008) studied the nonlinear deformation of beam made of isotropic material and constant area section. They employed the analytical and numerical methods for analysis of beam. Analytical method has been performed by Adomians Decomposition

Method and shooting method has been employed as a numerical scheme. The analytical results are compared with those results which are obtained using the numerical method. Kapuria et al. (2008) studied the vibration analysis of a functionally graded beam with consideration of the normal and shear stresses. They evaluated the normal and shear components of stress and strain. With the evaluation of the strain and kinetic energy, Hamilton principle has been employed to derive governing differential equation. They evaluated the natural frequency by taking the variation of Hamilton principle. Two-dimensional Static and dynamic behavior of a FG beam is studied by Li (2008). They used a two dimensional field for simulation of the displacement that are included deflection and rotation of the beam. Assumed field can be used to indicate presence of normal and shear stress in the beam. They used Timoshenko and Euler Bernoulli beam theories, simultaneously. It has been assumed that material properties are graded in the lateral direction.

Sina et al. (2009) studied the free vibration of a FG beam using an analytical method. They supposed a three

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components field of displacement that has been stated as a function of two variable of coordinate system in the plane of the beam. They employed Hamilton principle for derivation of the governing differential equation of the system.

Xiang and Yang (2008) studied the free and forced vibration of a laminated functionally graded beam with variable thickness under thermal load. Beam consists of a homogeneous substrate and two inhomogeneous functionally graded layers whose material composition follows a power law distribution along the thickness direction in terms of the volume fractions of the material constituents. Both the axial and rotary inertia of the beam are considered in that analysis. It is assumed that the beam may be clamped, hinged, or free at its ends and is subjected to one-dimensional steady state heat conduction in the thickness direction before undergoing the dynamic deformation. Exact solutions of the bending and free vibration of a functionally graded beam resting on a Winkler–Pasternak elastic foundation are presented based on the two-dimensional theory of elasticity by Ying et al. (2008). The beam is supposed to be orthotropic at any point, while material properties varying exponentially along the thickness direction. They employed the equation of motion in two dimensional space. The problem is solved by using the space-state method.

A functionally graded beam subjected to uniformly distributed transverse loading has been investigated by Sallai et al. (2009). Classical laminate beam, first-order shear deformation and high-order shear deformation theories have been considered for analysis of the problem.

Thermo-mechanical vibration analysis of the functionally graded (FG) beams and functionally graded sandwich (FGSW) beams were presented by Pradhan and Murmu (2009). The material properties of these beams are assumed to be varying in the thickness direction. The governing differential equations for beam vibration have been solved using the modified differential quadrature method (MDQM).

Rahimi and Davoodinik (2010) presented analytical solution of a beam that has been graded along the thickness direction by using the Adomian's Decomposition Method. The geometric nonlinearity has been considered in the problem.

In this paper, the effect of geometric nonlinearity is analytically studied on the behavior of a FG beams. In this study, nonlinear analysis of beam that is graded in the longitudinal direction and with variable thickness is considered. Non homogenous modulus of elasticity and area section can be represented as a function of longitudinal coordinate. In the previous study, nonlinear analysis of isotropic beam with constant area section has been studied (Banerjee et al., 2008). Therefore, this advancement (non homogenous material property along the longitudinal direction and variable thickness) in analysis of the beam can be considered as novelty of this

paper.

FORMULATION

In this study, it is supposed that the beam is subjected to vertical and horizontal forces and point moment that is applied at the end of the beam. Three types of area sections and material properties can be classified:

1. In the first type, it is supposed that the area section of the beam in the longitudinal direction to be constant and modulus of elasticity to be as a function of longitudinal coordinate. In this type, due to identity of the area section and center of mass, moment of inertia must be constant.

$$E = E(s), I = \text{const}$$

2. In the second type, it is supposed that the area section of the beam varies arbitrary and modulus of elasticity to be constant, in the longitudinal direction. In this type, due to varying of the area section, moment of inertia must be variable. $E = \text{const}, I = I(s)$.

3. In the third type, it is supposed that both the area section of the beam and modulus of elasticity in the longitudinal direction vary arbitrary. $E = E(s), I = I(s)$.

The comprehensive differential equation and solution of a FG beam under above conditions is presented in this study. As mentioned above, parameter $EI(s)$ is variable in three types. General nonlinear differential equation of a FG beam is developed by Euler- Bernoulli theory.

Governing nonlinear differential equation

Used model is shown in Figure 1. Point loads nP, P act on the beam in horizontal and vertical direction, respectively. Point moment M acts about the outward axis.

To start the formulation of the problem, an arbitrary area section of the beam can be free from the body in coordinate (s). Considering the resultant of moments in this section, from the strength of material, we know that, the resultant of moments in every section is equal to multiplication of gradient of rotation of the beam in rotational stiffness $EI(x)$. Also, considering the end of the beam with coordinate (a, b), resultant of moment can be obtained as follows (Banerjee et al., 2008):

$$M(x, y) = P(a - x) + nP(b - y) \quad (1)$$

$$M(x, y) = EI(s) \times \frac{\partial \theta(s)}{\partial s} \quad (2)$$

$$EI(s) \times \frac{\partial \theta(s)}{\partial s} = P(a - x) + nP(b - y) \quad (3)$$

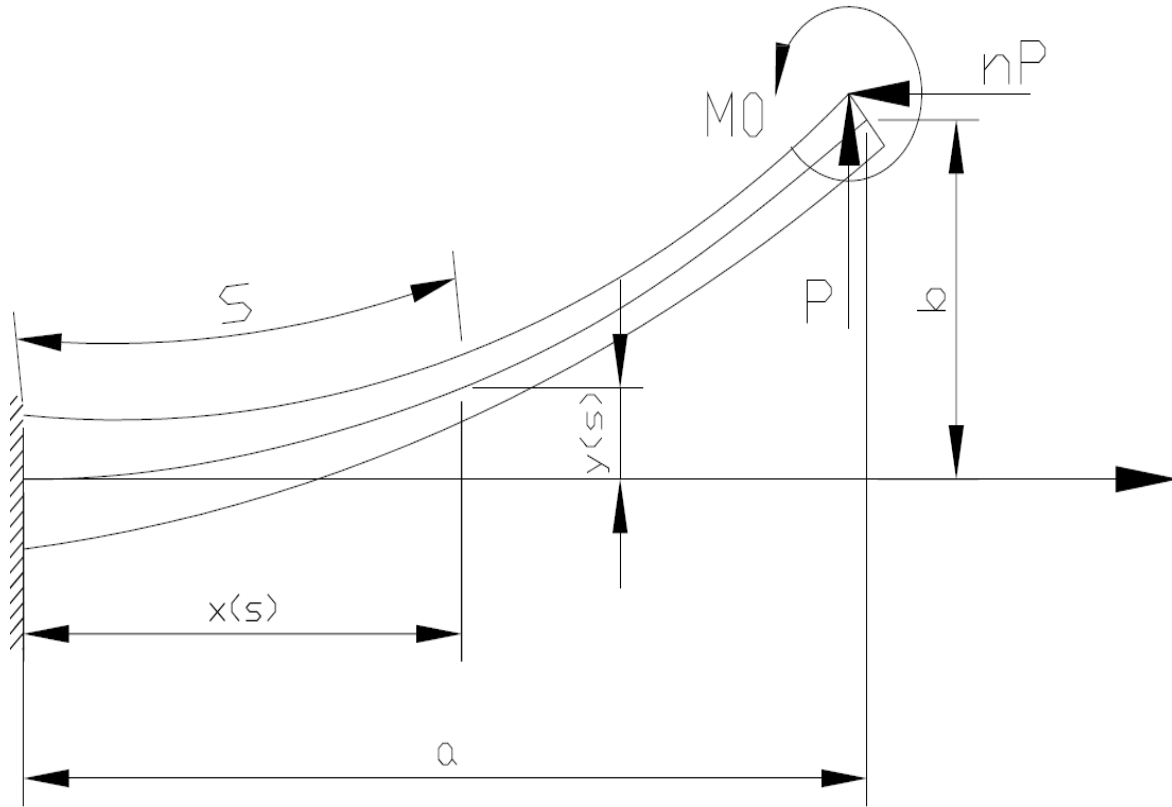


Figure 1. The schematic figure of a FG beam with variable thickness under point loads

Due to simultaneously existence of two components of coordinate system x, y in Equation (3), it is inevitable to eliminate them. Differentiating of Equation (1) with respect to (s) and regarding the Equation (4), the nonlinear differential equation can be derived as follows (Equation 5):

$$\begin{aligned} \frac{\partial}{\partial s} \left(EI(s) \times \frac{\partial \theta(s)}{\partial s} \right) &= -P \times \left(\frac{\partial x}{\partial s} + n \times \frac{\partial y}{\partial s} \right) \\ \frac{\partial x}{\partial s} &= \cos \theta, \quad \frac{\partial y}{\partial s} = \sin \theta \\ \frac{\partial}{\partial s} \left(EI(s) \times \frac{\partial \theta(s)}{\partial s} \right) &= -P \times (\cos \theta + n \times \sin \theta) \end{aligned} \quad (4)$$

$$\begin{aligned} \frac{\partial}{\partial s} \left(EI(s) \times \frac{\partial \theta(s)}{\partial s} \right) + EI(s) \times \frac{\partial^2 \theta(s)}{\partial s^2} &= -P \times (\cos \theta + n \times \sin \theta) \\ \rightarrow \frac{\partial^2 \theta(s)}{\partial s^2} + \frac{1}{EI(s)} \frac{\partial}{\partial s} \left(EI(s) \times \frac{\partial \theta(s)}{\partial s} \right) &= -\frac{P}{EI(s)} \times (\cos \theta + n \times \sin \theta) \end{aligned} \quad (5)$$

$$EI(s) = EI(0) \times e^{-bs} = a \times e^{-bs} \quad (5a)$$

$$B.C : \theta(0) = 0, \quad EI(L) \times \frac{\partial \theta(L)}{\partial s} = M_0 \quad (5b)$$

Equation (5) is the general nonlinear differential equation of a FG beam that is graded along the longitudinal direction. Equation (5a) is the assumed function for simulation of the variable rotational stiffness $EI(s)$ and Equation (5b) is the boundary condition of the nonlinear differential equation (Equation 5).

If modulus of elasticity and moment of inertia at the total length of beam is assumed to be constant, then second term in the left hand of Equation (5) must be vanished. This result is accordance with the literature (Banerjee et al., 2008). The obtained differential equation can be applied for isotropic and constant area section.

For analytical solution of a FG beam (Equation 5), Adomian's decomposition method (ADM) is employed. This method can be used to solve the nonlinear differential equation. The procedure of ADM is next described.

Adomians decomposition method (ADM)

Here, detail of ADM is demonstrated. This method can be used for solution of linear and nonlinear, homogenous and non-homogenous one dimensional differential equations. Arefi and Rahimi (2011, 2012) suggested the other method for solution of nonlinear problems. In the first step, components of nonlinear differential equation are divided to four elements. The 1st component is the highest order of the linear derivatives in the differential equation.

The second component is the lower order of variable of problem. The third component is the nonlinear component and fourth component is the non-homogenous term in the differential equation. In the general form and based on above expressions, we can show the nonlinear differential equation as follows:

$$L\theta + R\theta + N\theta = g$$

L : Highest order operator of differential equation

R : Lower order operator

N : Nonlinear operator of differential equation

g : Nonhomogenous term of differential equation

$$\frac{\partial^2 \theta(s)}{\partial s^2} + \frac{1}{EI(s)} \frac{\partial}{\partial s} (EI(s)) \times \frac{\partial \theta(s)}{\partial s} = -\frac{P}{EI(s)} \times (\cos \theta + n \times \sin \theta) \quad (6a)$$

$$L := \frac{\partial^2}{\partial s^2} \quad (6b)$$

$$R := \frac{1}{EI(s)} \frac{\partial}{\partial s} (EI(s)) \times \frac{\partial}{\partial s} \quad (6c)$$

$$N := \frac{P}{EI(s)} \times (\cos(\dots) + n \times \sin(\dots)) \quad (6d)$$

Components of the general nonlinear differential equation are shown above. Nonlinear differential equation can be multiplied by inverse of higher order of differential equation, L^{-1} as follows:

$$\begin{aligned} L^{-1}[L\theta + R\theta + N\theta] &= 0 \rightarrow \\ \theta + L^{-1}[R\theta] + L^{-1}[N\theta] &= 0 \end{aligned} \quad (7)$$

In the following, last two terms of Equation (7) can be transferred to right hand of Equation (7) as follows:

$$\theta = -L^{-1}[R\theta] - L^{-1}[N\theta] \quad (8a)$$

Applying successive approximation method on Equation (8a) can be performed as follows (Hojjati and Jafari, 2008):

$$\theta_n(s) = -L^{-1}(R\theta_{n-1}) - L^{-1}(N\theta_{n-1}) \quad n=1,2,3,\dots \quad (8b)$$

The main problems that we may encounter in this procedure are classified as follows:

1. What is the zero th order solution of θ defined as θ_0 ? θ_0 can be obtained by disregarding the nonlinear term and solving the linearized differential equation. In this step, boundary condition must be imposed on the derived homogenous solution. The other step of solution can be determined by Equation (8b).
2. In the other steps, two constants must be derived in every step of solution. (Due to NDE is second order) as mentioned above, the

actual boundary condition must be imposed on the solution of zero th order. Therefore, the homogenous boundary conditions must be imposed on the constants of integration.

The final solution of the governing non-linear differential equation must be evaluated by summation of the obtained solutions in every step as follows:

$$\theta(s) = \sum_{n=0}^l \theta_n(s) \quad (9)$$

Where, l is the number of assumed steps.

Obtaining the zero th order solution

As mentioned previously, θ_0 can be obtained by disregarding the nonlinear term and solving the modified differential equation as follows:

$$\frac{\partial^2 \theta_0(s)}{\partial s^2} + \frac{1}{EI(s)} \frac{\partial}{\partial s} (EI(s)) \times \frac{\partial \theta_0(s)}{\partial s} = 0 \quad (10)$$

$$EI(s) = ae^{-bs} \rightarrow \frac{1}{EI(s)} \frac{\partial}{\partial s} (EI(s)) = \frac{-abe^{bs}}{ae^{bs}} = -b \quad (11)$$

$$\frac{\partial^2 \theta_0(s)}{\partial s^2} - b \times \frac{\partial \theta_0(s)}{\partial s} = 0 \quad (11a)$$

$$\frac{\partial \theta_0(s)}{\partial s} = f \quad (11b)$$

$$\frac{\partial f}{\partial s} - b \times f = 0 \rightarrow f = \frac{\partial \theta_0(s)}{\partial s} = c' \times e^{bs}$$

$$\theta_0(s) = c_2 e^{bs} + c_1 \quad (12)$$

Imposing the boundary condition θ_0 can be obtained.

$$B.C : \theta(0) = 0, \quad EI(L) \times \frac{\partial \theta(L)}{\partial s} = M_0 \quad (13)$$

$$\theta_0(s) = c_2 e^{bs} + c_1 \quad (13a)$$

$$\begin{cases} c_2 + c_1 = 0 \\ ae^{-bL} \times c_2 be^{bL} = M_0 \end{cases} \quad (13b)$$

$$c_2 = -c_1 = \frac{M_0}{ab} \quad (13c)$$

$$\theta_0(s) = c_2 e^{bs} + c_1 = \frac{M_0}{ab} [-1 + e^{bs}] \quad (14)$$

As mentioned above, θ_0 is a function of M_0 only and P is not appeared in Equation (14). For this reason, this solution is incorrect and must be modified by successive approximation method. In this step, we can obtain other step of solution using θ_0 .

Obtaining the first order solution

As shown in Equation 8, the general solution can be divided to linear and nonlinear solutions. Linear and nonlinear solutions of n th step are presented as θ_{n1}, θ_{n2} , respectively.

$$\theta_{11} = -\int \int \left[\frac{1}{EI(s)} \frac{\partial}{\partial s} (EI(s)) \right] \frac{\partial \theta_0}{\partial s} ds ds$$

$$\frac{1}{EI(s)} \frac{\partial}{\partial s} (EI(s)) = \frac{-abe^{bs}}{ae^{bs}} = -b \rightarrow \theta_{11} = -b \int \int \frac{\partial \theta_0}{\partial s} ds ds = \frac{M_0 e^{bs}}{ab} \quad (15)$$

We must apply most regardless in obtaining the nonlinear part of the solution. In the nonlinear part of solution, exists $\sin \theta(s)$ and $\cos \theta(s)$ terms. Calculating the integral of these terms is very difficult and using the explicit analytical method is not accessible. For this reason, Taylor expansion can be applied as follows:

$$\sin \theta = \theta - \frac{\theta^3}{6} + \frac{\theta^5}{120} - \dots \quad (16a)$$

$$\cos \theta = 1 - \frac{\theta^2}{2} + \frac{\theta^4}{24} - \dots \quad (16b)$$

This replacement simplifies the solution procedure. Result of the nonlinear integration is presented only for two terms as follows:

$$\theta_{12} = -P \times \int \int \left[\frac{\cos \theta_0 + n \times \sin \theta_0}{EI(s)} \right] ds ds = -\frac{P}{a} \times \int \int \left[\frac{1 - \frac{\theta_0^2}{2} + n \times \left[\theta_0 - \frac{\theta_0^3}{6} \right]}{EI(s)} \right] ds ds =$$

$$-P \times \left[\frac{e^{bs}}{b^2} - \frac{11 M_0^2 e^{bs}}{24 a^2 b^4} + \frac{11 M_0^2 (e^{bs})^2}{48 a^2 b^4} - \frac{11 M_0^2 (e^{bs})^3}{216 a^2 b^4} - \frac{n M_0 e^{bs}}{ab^3} + \frac{1 n M_0 (e^{bs})^2}{4 ab^3} \right.$$

$$\left. + \frac{1 n M_0^3 (e^{bs})}{3 a^3 b^5} - \frac{1 n M_0^3 (e^{bs})^2}{4 a^3 b^5} + \frac{1 n M_0^3 (e^{bs})^3}{9 a^3 b^5} - \frac{1 n M_0^3 (e^{bs})^4}{48 a^3 b^5} \right] \quad (17)$$

Solution of the 1st step can be defined by summation of θ_{11}, θ_{12} and one linear term $C_1 s + C_2$ as follows: (last two terms arises from two times integration with respect to s)

$$\theta_1 = \theta_{11} + \theta_{12} + c_1 s + c_2 \quad (18)$$

If it is supposed that $N\theta = f(\theta)$ then for $n \geq 1$, successive approximation can be presented as follows:

$$\theta_n(s) = -L^{-1}(Ru_{n-1}) - L^{-1}(A_{n-1})$$

$$n=1,2,3,\dots$$

$$\begin{cases} A_0 = f(\theta_0) \\ A_1 = \theta_1 \frac{df(\theta_0)}{d\theta_0} \\ A_2 = \theta_2 \frac{df(\theta_0)}{d\theta_0} + \frac{\theta_1^2}{2!} \frac{d^2 f(\theta_0)}{d\theta_0^2} \\ A_3 = \theta_3 \frac{df(\theta_0)}{d\theta_0} + \theta_1 \theta_2 \frac{d^2 f(\theta_0)}{d\theta_0^2} + \frac{\theta_1^3}{3!} \frac{d^3 f(\theta_0)}{d\theta_0^3} \end{cases} \quad (19)$$

RESULTS

Small values

Due to existence of very long terms in the higher order solution, we continue the study by devoting the numerical values. The assumed numerical values are considered as follows:

$$a = 8 \times 10^6, P = 1 \times 10^5, n = 0.5, b = 0.01, M_0 = 1 \times 10^4, L = 1$$

$$EI(s) = ae^{-bs} = 8 \times 10^6 \times e^{-0.01s}$$

In this step, we can show the solution of every step and final solution.

Before obtaining the solution of 1st order, by devoting the numerical value, constants of integration c_1, c_2 can be defined as follows:

$$\theta_1 = \theta_{11} + \theta_{12} + c_1 s + c_2 \rightarrow \begin{cases} s=0 \rightarrow \theta_1(s)=0 \\ s=L \rightarrow \frac{\partial \theta_1}{\partial s} = \frac{M_0}{EI(s)} \end{cases} \rightarrow \begin{cases} c_1 = 1.219039 \\ c_2 = 118.430 \end{cases}$$

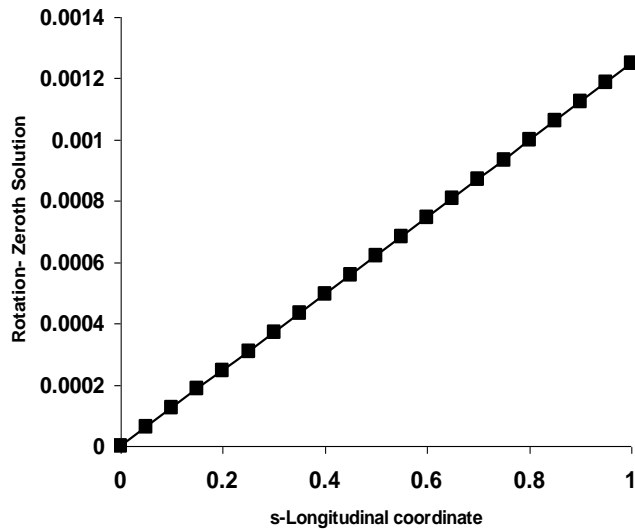
Figure 2a, b, c, d show the solution of zero th, 1st, 2nd and 3rd order, respectively. Due to convergence of solution, solution of 3rd order can be defined as final solution of the problem.

Maximum rotation of the beam as depicted in Figure (2) is equal to 0.0075 rads.

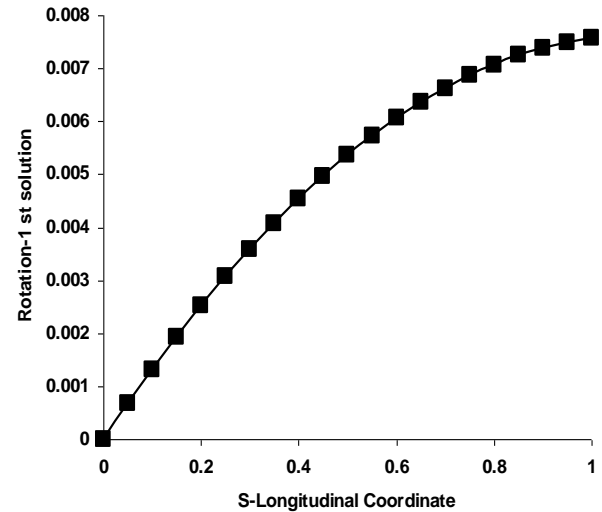
After finalizing the solution of the rotation of the beam, in the next step, we can calculate the deflection of the beam. Deflection of the beam can be calculated by the geometric relation of the beams.

$$\frac{\partial y}{\partial s} = \sin \theta \rightarrow y = \int \sin \theta(s) ds \quad (20)$$

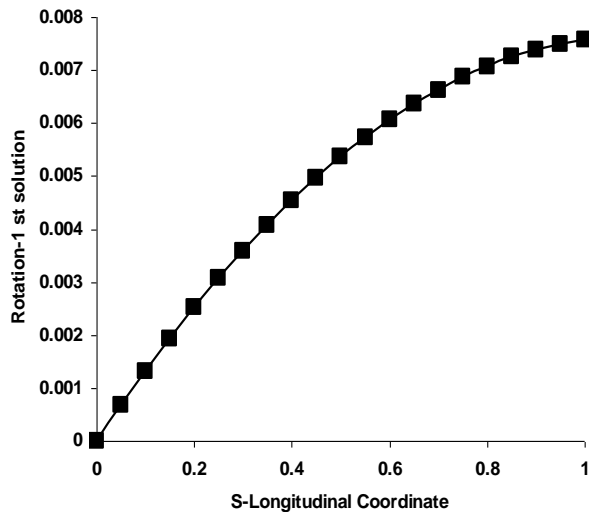
By substitution of rotation expression (Equation (9)) in



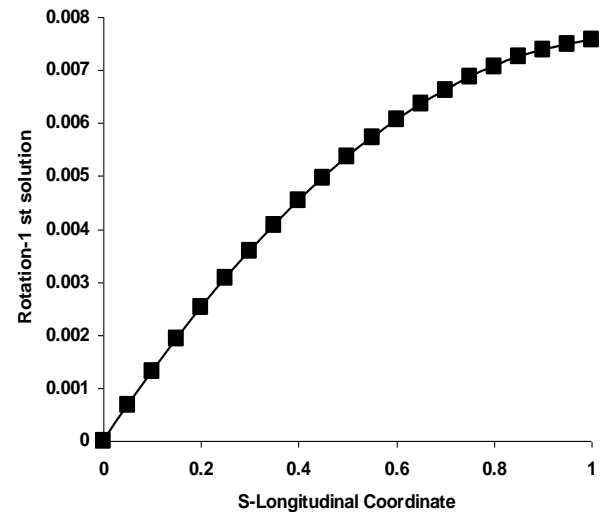
a) Zero th step solution



b) 1st step solution



c) 2nd step solution



d) Final step solution

Figure 2. Procedure of convergence of solution for rotation of beam (four steps).

Equation (20), deflection of the beam can be obtained. Figure 3 shows the deflection of the beam.

Large values

In this step, rotation and deflection of the beam for large value of force and moment can be evaluated. Only external load and moment changes as follows:

$$P = 1 \times 10^7$$

$$M_0 = 1 \times 10^6$$

Figure 4a, b show the final results of the rotation and deflection, respectively.

Comparison of the obtained results with numerical results (FEM)

Here, previous obtained results can be compared with results of finite element method. For simulation of the beam in software of finite element (ANSYS), we devote for every elements, modulus of elasticity and moment of inertia that differs from another element. Figure 5a, b

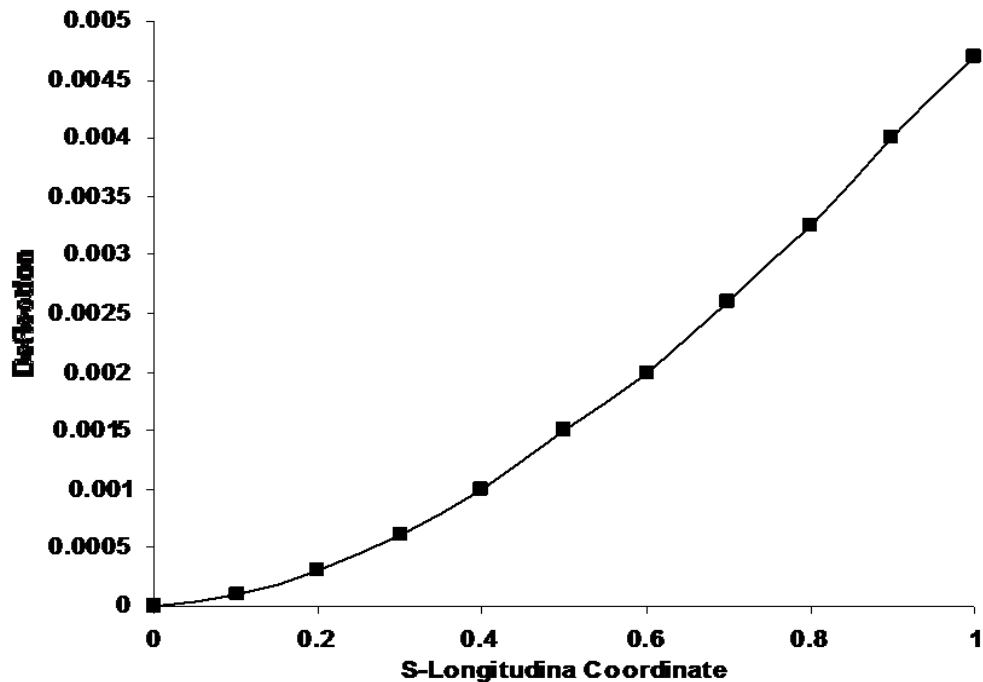
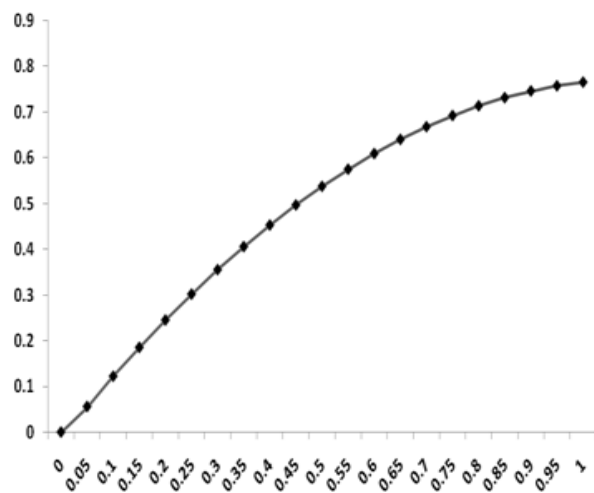
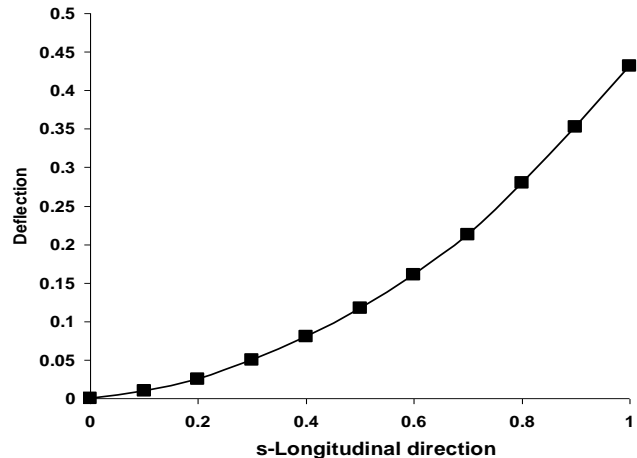


Figure 3. Deflection of beam.



a) Rotation of beam



b) Deflection of beam

Figure 4. Rotation and deflection of beam for large value of loads.

shows comparison between results that are obtained by finite element method with those results obtained by ADM. This comparison shows that difference between two methods is ignorable.

Evaluation of the results for different values of b

Here, deflection and rotation of a FG beam for different

values of the non-homogenous index of the beam (b) are plotted. Figure 6a, b show the distribution of the rotation and deflection for different values of b in the range of $-0.1 \dots 0.1$.

Conclusion

In this study, the comprehensive formulation of a FG

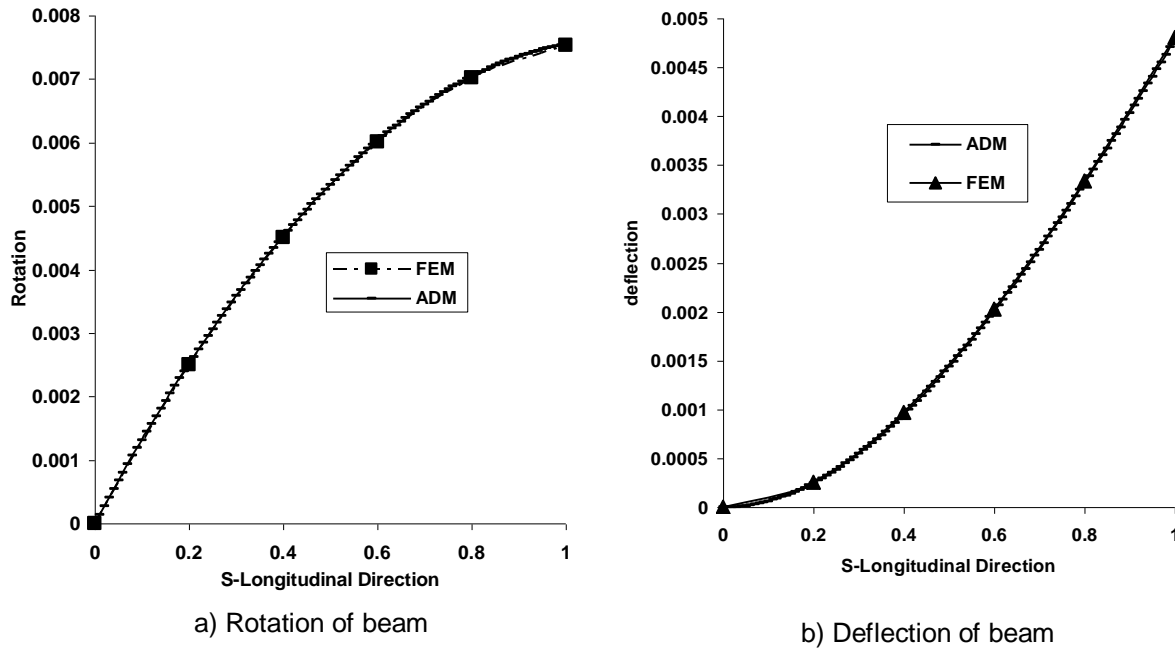


Figure 5. Comparison between analytical (ADM) and numerical results (FEM).

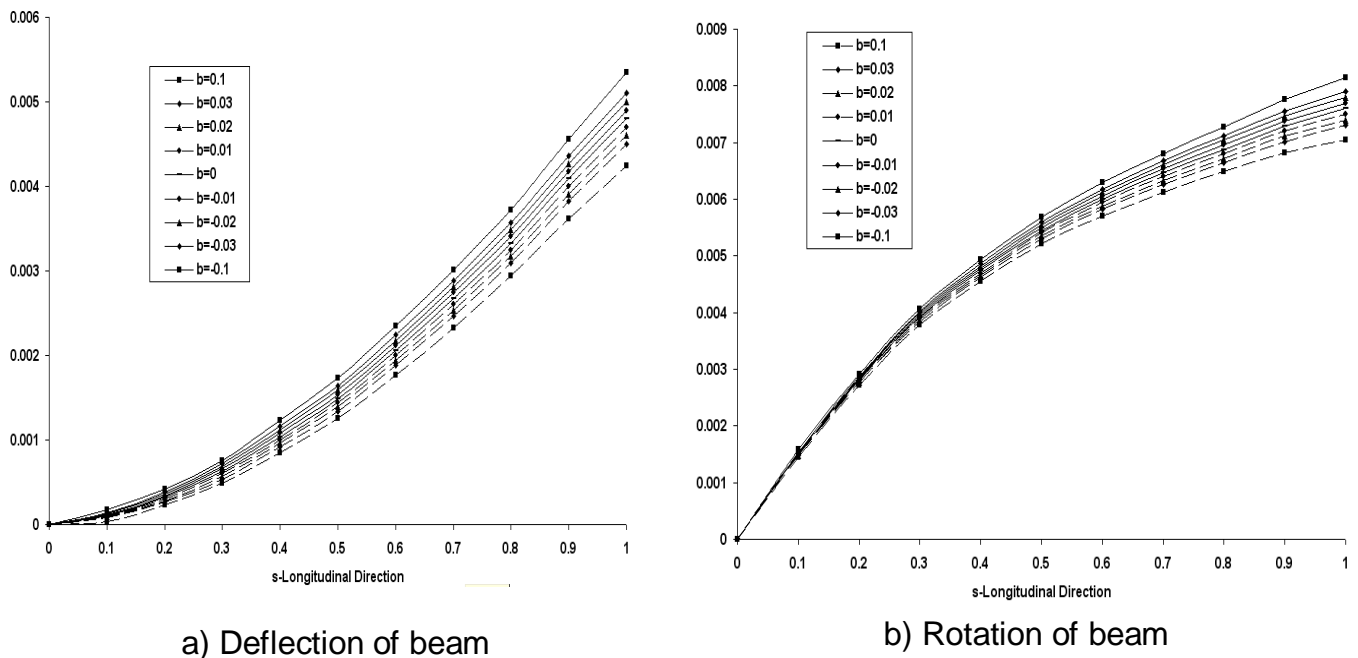


Figure 6. comparison between rotation and deflection of beam for different value of b .

beam exposed to point load and moment that is graded in the longitudinal direction is presented. Obtained nonlinear differential equation is solved by ADM. The numerical results are compared with analytical results and therefore verified. The important results that were obtained in this paper can be classified as follows:

1. As a novel subject, general governing differential equation of a FG beam that is graded in the longitudinal direction, exposed to point loads and moment is derived. This nonlinear differential equation can be solved by using an analytical or numerical method.
2. Obtained governing nonlinear differential equation of

FG beam is solved by analytical method (ADM). This method evaluates the results of rotation and deflection by using the successive approximation method. Selecting the appropriate zero th order solution can facilitate the convergence of solution procedure.

3. Numerical results are obtained by finite element method. This method presented the results that are comparable to analytical results. Therefore, numerical results are verified by analytical results (ADM). The present paper suggests an analytical method for consideration of the beams that is graded in longitudinal direction and variable thickness.

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