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Full Length Research Paper

Analytic functions defined by a certain integral operator

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In this paper, some new classes of analytic functions, involving a certain integral operator, are introduced. Inclusion relationships, a radius problem and some other interesting properties are investigated. However, some applications of these results are also discussed.

Key words: Univalent, starlike, convex, integral operator, convolution.

INTRODUCTION

Let A be the class of functions f : given by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (1)$$

which are analytic in the open unit disc of $E = \{z : |z| < 1\}$. Let $P_k(\alpha)$ be the class of functions of $p(z)$ analytic in E satisfying the properties of $p(0)=1$ and

$$\int_0^{2\pi} \left| \frac{\operatorname{Re} p(z) - \alpha}{1-\alpha} \right| d\theta \le k\pi, \quad z = re^{i\theta},\ k \ge 2,\ 0 \le \alpha < 1. \quad (2)$$

For $\alpha = 0$, the class of P_k introduced in Pinchuk (1971) was obtained, and can also be written for $p \in P_k(\alpha)$ as:

$$p(z) = \left(\frac{k}{4} + \frac{1}{2}\right) p_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right) p_2(z), \quad p_i \in P_2(\alpha) = P(\alpha),\ i = 1, 2 \quad (3)$$

In this study, the following classes of analytic functions are defined as:

$$R_k(\alpha) = \left\{ f : f \in A \ \text{ and } \ \frac{zf'}{f} \in P_k(\alpha),\ 0 \le \alpha < 1 \right\},$$

$$V_k(\alpha) = \left\{ f : f \in A \ \text{ and } \ \frac{(zf')'}{f'} \in P_k(\alpha),\ \ 0 \le \alpha < 1 \right\},$$

$$T_k^*(\beta, \alpha) = \left\{ f : f \in A \ \text{ and } \ \frac{zf'}{g} \in P_k(\beta) \ \text{ for some } \ g \in R_2(\alpha),\ 0 \le \alpha, \beta < 1 \right\}$$

$$P_k'(\alpha) = \left\{ f : f \in A \ \text{ and } \ f' \in P_k(\alpha),\ \ 0 \le \alpha < 1 \right\}.$$

However, $f \in V_k(\alpha) \Leftrightarrow zf' \in R_k(\alpha)$ is noted and the following integral operators are considered.

$$L_\lambda^\mu : A \to A, \quad \text{for } \lambda > -1, \quad \mu > 0; \quad f \in A,$$

$$\begin{aligned} L_\lambda^\mu f(z) &= \binom{\lambda+\mu}{\lambda} \frac{\mu}{z^\lambda} \int_0^z t^{\lambda-1} \left(1 - \frac{t}{z}\right)^{\mu-1} f(t)\, dt \\ &= z + \frac{\Gamma(\lambda+\mu+1)}{\Gamma(\lambda+1)} \sum_{n=2}^{\infty} \frac{\Gamma(\lambda+n)}{\Gamma(\lambda+\mu+n)} a_n z^n, \end{aligned} \quad (4)$$

Where Γ denotes the gamma function and $f(z)$ is given by Equation (1). From Equation (4), the generalized Bernardi operator can be obtained as follows:

$$J_\mu f(z) = \frac{\mu+1}{z^\mu} \int_0^z t^{\mu-1} f(t)\, dt = \ = z + \sum_{n=2}^{\infty} \frac{\mu+1}{\mu+n} a_n z^n, \quad \mu > -1;\ f \in A.$$

From Equation (4), Equation 5 can be derived:

$$z\left(L_\lambda^{\mu+1} f(z)\right)' = (\lambda + \mu + 1)\, L_\lambda^\mu f(z) - (\lambda + \mu)\, L_\lambda^{\mu+1} f(z) \quad (5)$$

The class A is closed under the Hadamard product or convolution, and is defined by:

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$$(f_1 * f_2)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n,$$

where

$$f_1(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad f_2(z) = z + \sum_{n=2}^{\infty} b_n z^n.$$

Using the integral operators of L_λ^μ, we now introduce the following classes of analytic functions.

Definition 1. Let $f \in A$, $\lambda > -1, \mu > 0$. Then $f \in R_k(\lambda, \mu, \alpha)$, if and only if

$L_\lambda^\mu f \in R_k(\alpha), \quad 0 \leq \alpha < 1 \quad$ and $\quad z \in E$.

Definition 2. Let $f \in A$, $\lambda > -1$, $\mu > 0$. Then $f \in V_k(\lambda, \mu, \alpha)$, if and only if

$L_\lambda^\mu f \in V_k(\alpha), \quad 0 \leq \alpha < 1$ and $z \in E$.

Definition 3. Let $f \in A$, $\lambda > -1, \mu > 0$. Then $f \in T_k^*(\lambda, \mu, \beta, \alpha)$, if and only if

$L_\lambda^\mu f \in T_k^*(\beta, \alpha)$, $0 \leq \alpha, \beta < 1$, $z \in E$.

Definition 4. Let $f \in A$, $\lambda > -1$, $\mu > 0$. Then $f \in P_k'(\lambda, \mu, \alpha)$, if and only if

$L_\lambda^\mu f \in P_k'(\alpha), \quad 0 \leq \alpha < 1, z \in E$.

Remark 1. It is noted that $f \in V_k(\lambda, \mu, \alpha)$ if and only if $zf' \in R_k(\lambda, \mu, \alpha)$ for $z \in E$.

PRELIMINARY RESULTS

Lemma 1 (Miller, 1975)

Let $u = u_1 + iu_2$ and $v = v_1 + iv_2$, and $\Psi(u, v)$ be a complex-valued function satisfying the following conditions:

(i) $\Psi(u, v)$ is continuous in the domain of $D \subset \mathbb{C}^2$,
(ii) $(1,0) \in D$ and $\Psi(1,0) > 0$,
(iii) $\operatorname{Re}\Psi(iu_2, v_1) \leq 0$, whenever $(iu_2, v_1) \in D$ and

$$v_1 \leq -\frac{1}{2}(1 + u_2^2).$$

If $h(z) = 1 + \sum_{n=2}^{\infty} c_n z^n$ plays an analytic function in E, such that $(h(z), zh'(z)) \in D$ and $\operatorname{Re}\Psi(h(z), zh'(z)) > 0$ for $z \in E$, then $\operatorname{Re} h(z) > 0$ in E.

Lemma 2

Let $p(z)$ be analytic in E with $p(0) = 1$ and $\operatorname{Re} p(z) > 0$, $z \in E$. Then, for $s > 0$ and $\eta \neq -1$ (complex), $\operatorname{Re}\left\{p(z) + \frac{s\,zp'(z)}{p(z) + \eta}\right\} > 0$, for $|z| < r_0$,

where r_0 is given by

$$r_0 = \frac{|1+\eta|}{\sqrt{m + (m^2 - |\eta^2 - 1|)^{\frac{1}{2}}}}, \quad m = 2(s+\eta)^2 + |\eta|^2 - 1, \quad (6).$$

As such, this radius is exact. For this result, Ruscheweyh and Singh (1976) can be referred to for clarity.

Lemma 3

Let ϕ and g be the convex and starlike in E., respectively Then, for F analytic in E, $F(0) = 1$, $\frac{\Psi * Fg}{\Psi * g}$ is contained in the convex hull of $F(E)$. Lemma 3 is due to Ruscheweyh and Shiel-Small (1973). The following result is an easy generalization of the one given in Ponnusamy (1995).

Lemma 4

If $p(z)$ is analytic in E with $p(0) = 1$ and if λ is a complex number satisfying $\operatorname{Re}\lambda \geq 0 (\lambda \neq 0)$, then $(p + \lambda zp') \in P_k(\beta), 0 \leq \beta < 1$, implies $p \in P_k(\beta_1)$, where $\beta_1 = \beta + (1 - \beta)(2\gamma - 1)$ and $\gamma = \int_0^1 (1 + t^{\operatorname{Re}\lambda})^{-1}\, dt$, are an increasing function of $\operatorname{Re}\lambda$ and $\frac{1}{2} \leq \gamma < 1$. This estimate is sharp in the sense that the bound cannot be improved.

MAIN RESULTS

Theorem 1

Let $f \in A$, $\lambda > -1$, $\mu > 0$ and $\lambda + \mu > -\alpha$. Then

$R_k(\lambda,\mu,\alpha) \subset R_k(\lambda,\mu+1,\beta)$, for $0 \le \alpha < 1$
and

$$\beta = \frac{2\ (2\alpha\lambda + 2\alpha\mu + 1)}{(2\lambda + 2\mu - 2\alpha + 1) + \sqrt{4\,(\lambda+\mu+\alpha)^2 + 4(\lambda+\mu-\alpha) + 9}}. \qquad (7)$$

Proof:

Let $f \in R_k(\lambda,\mu,\alpha)$ and

$$\frac{z(L_\lambda^{\mu+1} f(z))'}{L_\lambda^{\mu+1} f(z)} = h(z) = \left(\frac{k}{4}+\frac{1}{2}\right)h_1(z) - \left(\frac{k}{4}-\frac{1}{2}\right)h_2(z)\,, \qquad (8)$$

Thus, $h(z)$ is analytic in E and $h(0)=1$. From Equations (5) and (8), Equation 9 is obtained:

$$\frac{z\,L_\lambda^\mu\, f(z))'}{L_\lambda^\mu\, f(z)} = \left\{ h(z) + \frac{zh'(z)}{h(z)+\lambda+\mu} \right\} \in P_k(\alpha), \quad z \in E. \qquad (9)$$

and defined as:

$$\phi_{\lambda,\mu}(z) = \left(\frac{\lambda+\mu}{\lambda+\mu+1}\right)\frac{z}{(1-z)} + \left(\frac{1}{\lambda+\mu+1}\right)\frac{z}{(1-z)^2} = \sum_{j=1}^{\infty}\frac{(\lambda+\mu)+j}{(\lambda+\mu)+1} z^j.$$

Then, from Equation (8), Equation 10 is derived:

$$\begin{aligned} \left\{ h(z) + \frac{zh'(z)}{h(z)+\lambda+\mu} \right\} &= (\,h(z) * \phi_{\lambda,\mu}(z)\,) \\ &= \left(\frac{k}{4}+\frac{1}{2}\right)(h_1(z) * \phi_{\lambda,\mu}(z)\,) - \left(\frac{k}{4}-\frac{1}{2}\right)(\,h_2(z) * \phi_{\lambda,\mu}(z)\,) \qquad (10) \\ &= \left(\frac{k}{4}+\frac{1}{2}\right)\left\{ h_1(z) + \frac{zh_1'(z)}{h_1(z)+\lambda+\mu} \right\} - \left(\frac{k}{4}-\frac{1}{2}\right)\left\{ h_2(z) + \frac{zh_2'(z)}{h_2(z)+\lambda+\mu} \right\} \end{aligned}$$

From Equations (9) and (10), it follows that

$$\left\{ h_i(z) + \frac{zh_i'(z)}{h_i(z)+\lambda+\mu} \right\} \in P(\alpha), \quad i=1,2, \qquad z \in E. \qquad (11)$$

Moreover, it is shown that $h_i \in P(\beta), \beta$ is given by Equation (7).

Let $h_i(z) = (1-\beta)\,p_i(z) + \beta$ in Equation (11). Then, for $i=1,2$ and $z \in E$, we have:

$$\left\{ (1-\beta)\,p_i(z) + (\beta-\alpha) + \frac{(1-\beta)\,z\,p_i'(z)}{(1-\beta)\,p_i(z) + \beta + \lambda + \mu} \right\} \in P.$$

At this instant, a functional $\Psi(u,v)$ is constructed by taking $u = p_i(z)$ and $v = zp_i'(z)$. Thus,

$$\Psi(u,v) = (1-\beta)\,u + \beta - \alpha + \frac{(1-\beta)\,v}{(1-\beta)\,u + \beta + \lambda + \mu}.$$

The first two conditions of Lemma 1 are clearly satisfied as $\Psi(u,v)$ is continuous

in $D = \left(\frac{\boxed{\le}}{\left\{ -\frac{\beta+\lambda+\mu}{1-\beta} \right\}} \right) \times \boxed{\le},\ (1,0) \in D$ and $\mathrm{Re}\,\{\,\Psi(1,0)\,\} > 0.$

Then, condition (iii) in Lemma 1 is verified as follows:

$$\begin{aligned} \mathrm{Re}\,\Psi(iu_2, v_1) &= \beta - \alpha + \frac{(\beta+\lambda+\mu)\,(1-\beta)\,v_1}{(\beta+\lambda+\mu)^2 - (1-\beta)^2\,u_2^2} \\ &\le \beta - \alpha - \frac{1}{2}\,\frac{(\beta+\lambda+\mu)\,(1-\beta)(1+u_2^2)}{(\beta+\lambda+\mu)^2 + (1-\beta)^2\,u_2^2}, \ \text{with}\ v_1 \le -\frac{1}{2}(1+u_2^2) \\ &= \frac{A + Bu_2^2}{2C}, \end{aligned}$$

Where,

$$\begin{aligned} A &= (\beta+\lambda+\mu)\,\{\,2(\beta-\alpha)(\beta+\lambda+\mu) - (1-\beta) \\ B &= (1-\beta)\,\{\,2\,(\beta-\alpha)(1-\beta) - (\beta+\lambda+\mu)\,\}, \\ C &= (\beta+\lambda+\mu)^2 + (1-\beta)^2\,u_2^2 \ > 0. \end{aligned}$$

It is noted that $\mathrm{Re}\,\Psi(iu_2, v_1) \le 0$ if and only if, $A \le 0$ and $B \le 0$. From $A \le 0$, β is obtained as given by Equation (7) and $B \le 0$ is realized as $0 \le \beta < 1$. Therefore, Lemma 1 is applied to conclude that $\mathrm{Re}\ P_i(z) > 0$ in E and this implies $h_i \in P(\beta)$.

Consequently, $h \in P_k(\beta)$ and hence $f \in R_k(\lambda,\mu+1,\beta)$ for $z \in E$. With $\alpha = 0$, it is noted that the result obtained is proven in Noor (2006) and the case for $k=2$, and $\alpha = \beta$, has been studied in Gao et al. (2005) and Noor (2006a).

Theorem 2

For $\lambda > -1, \mu > 0$ and $\lambda + \mu > -\alpha$, $V_k(\lambda,\mu,\alpha) \subset V_k(\lambda,\mu+1,\beta)$,

Where $0 \le \alpha < 1$ and β are given by Equation (7).

Proof

Applying Remark 1 and Theorem 1, the following are observed:

$$f \in V_k(\lambda,\mu,\alpha) \Leftrightarrow L_\lambda^\mu f \in V_k(\alpha) \Leftrightarrow z(L_\lambda^\mu f)' \in R_k(\alpha)$$
$$\Leftrightarrow L_\lambda^\mu (zf') \in R_k(\alpha \Rightarrow zf' \in R_k(\lambda,\mu+1,\beta)$$
$$\Leftrightarrow L_\lambda^{\mu+1}(zf') \in R_k(\beta) \Leftrightarrow z(L_\lambda^{\mu+1} f)' \in R_k(\beta)$$
$$\Leftrightarrow L_\lambda^{\mu+1} f \in V_k(\beta) \Leftrightarrow f \in V_k(\lambda,\mu+1,\beta).$$

This completes the proof.

Theorem 3

Let $\lambda > -1, \mu > 0$ and $\lambda + \mu > -\alpha$. Then $T_k^*(\lambda,\mu,\beta,\alpha) \subset T_k^*(\lambda,\mu+1,\beta_1,\alpha_1),\ 0 \le \alpha, \beta < 1,$

Where α_1 is given by Equation (7) and β_1 is as given by Equation (15).

Proof

Let $f \in T_k^*(\lambda,\mu,\beta,\alpha)$. Then, $g \in R_2(\lambda,\mu,\alpha)$ exist such that

$$\frac{z\,(L_\lambda^\mu f(z))'}{L_\lambda^\mu g(z)} \in P_k(\beta), \quad z \in E. \tag{11}$$

We set,

$$\frac{z(L_\lambda^{\mu+1} f(z))'}{L_\lambda^{\mu+1} g(z)} = H(z) \tag{12}$$
$$= \left(\frac{k}{4}+\frac{1}{2}\right)\{(1-\beta_1)H_1(z)+\beta_1\} - \left(\frac{k}{4}-\frac{1}{2}\right)\{(1-\beta_1)H(z)+\beta_1\},$$

where H is analytic in E and $H(0)=1$.

Since $g \in R_2(\lambda,\mu,\alpha)$,, Theorem 1 was used with $k=2$ and $\beta = \alpha_1$, so that $g \in R_2(\lambda,\mu+1,\alpha_1)$. was realized. Therefore, Equation 13 can be written as:

$$\frac{z(L_\lambda^{\mu+1} g(z))'}{L_\lambda^{\mu+1} g(z)} = H_0(z), \quad H_0 \in P(\alpha_1), \quad z \in E. \tag{13}$$

Using Equations (5), (11), (12) and (13) and some simple computation, Equation 14 is realized:

$$\frac{z(L_\lambda^\mu f(z))'}{L_\lambda^\mu g(z)} = \left\{H(z) + \frac{zH'(z)}{H_0(z)+\lambda+\mu}\right\} \tag{14}$$
$$= \{(\frac{k}{4}+\frac{1}{2})\{(1-\beta_1)H_1(z)+\beta_1 + \frac{(1-\beta_1)zH_1'(z)}{H_0(z)+\lambda+\mu}\}$$
$$-(\frac{k}{4}-\frac{1}{2})\{(1-\beta_1)H_2(z)+\beta_1 + \frac{(1-\beta_1)zH_2'(z)}{H_0(z)+\lambda+\mu}\}\} \in P_k(\beta),\ z \in E.$$

and this implies that

$$\left\{(1-\beta_1)\,H_i(z)+\beta_1-\beta+\frac{(1-\beta_1)\,zH_i'(z)}{H_0(z)+\lambda+\mu}\right\} \in P,\ H_0 \in P(\alpha_1)$$

for $z \in E,\ i=1,2$.

The functional $\Psi(u,v)$ is formed by choosing $u = H_i(z),\ v = zH_i'(z)$. Thus,

$$\Psi(u,v) = (1-\beta_1)u + (\beta_1-\beta) + \frac{(1-\beta_1)\,v}{H_0(z)+\lambda+\mu}.$$

The first two conditions of Lemma 1 are clearly satisfied. As such, condition (iii) is verified as follows:

$$\operatorname{Re}\Psi(iu_2,v_1) = (\beta_1-\beta) + \frac{(1-\beta_1)\,v_1\,(\lambda+\mu+\operatorname{Re}H_0)}{|H_0(z)+\lambda+\mu|^2}$$
$$\le (\beta_1-\beta) - \frac{(1-\beta_1)(\lambda+\mu+\operatorname{Re}H_0)(1+u_2^2)}{2|H_0(z)+\lambda+\mu|^2}$$
$$= \frac{A+Bu_2^2}{2C},$$

Where,

$$A = 2|H_0(z)+\lambda+\mu|^2(\beta_1-\beta) - (1-\beta_1)(\lambda+\mu+\operatorname{Re}H_0(z))$$
$$B = -(1-\beta_1)(\lambda+\mu+\operatorname{Re}H_0) \le 0$$
$$C = |H_0(z)+\lambda+\mu|^2 > 0.$$

Thus, $\operatorname{Re}\Psi(i\mu_2,v_1) \le 0$ if $A \le 0$ and Equation 15 is realized:

$$\beta_1 = \frac{(\lambda+\mu+\operatorname{Re}H_0) + 2\beta\,|\lambda+\mu+H_0(z)|^2}{(\lambda+\mu+\operatorname{Re}H_0)+2\,|\lambda+\mu+H_0(z)|^2} \tag{15}$$

At this instant, if Lemma 1 is applied, $H_i \in P$ will be obtained in E and therefore in $H \in P_k(\beta_1)$. Consequently, $f \in T_k^*(\lambda,\mu+1,\beta_1,\alpha_1)$ for $z \in E$.

In this study, it is noted that, for special choices of k, λ and μ, several known results, as well as new results, are obtained as special cases.

Theorem 3

Let $z \in E$, $f \in R_k(\lambda,\mu+1,0)$. Then, $f \in R_k(\lambda,\mu,0)$ for $|z| < r_0$, where r_0 is given by Lemma 1 with $\eta = \lambda+\mu$, $m = 7+(\lambda+\mu)^2$ and $s=1$. This radius is exact.

Proof

Let,

$$\frac{z(L_\lambda^{\mu+1} f(z))'}{L_\lambda^{\mu+1} f(z)} = H(z) = (\frac{k}{4}+\frac{1}{2}) h_1(z) - (\frac{k}{4}-\frac{1}{2}) h_2(z).$$

Then $H \in P_k$ in E and consequently $h_i \in P$ in $E, i=1,2.$ Using Definition 3 with similar argument in Theorem 1, we have:

$$\frac{z(L_\lambda^\mu f(z))'}{L_\lambda^\mu f(z)} = H(z) + \frac{zH'(z)}{H(z)+\lambda+\mu}$$
$$= \left(\frac{k}{4}+\frac{1}{2}\right)\left(h_1(z)+\frac{zh_1'(z)}{h_1(z)+\lambda+\mu}\right) - \left(\frac{k}{4}-\frac{1}{2}\right)\left(h_2(z)+\frac{zh_2'(z)}{h_2(z)+\lambda+\mu}\right).$$

Using Lemma 2 with $s=1,$ in $\eta=\lambda+\mu,\ m = 7+(\lambda+\mu)^2,$ it can be seen that $\left\{h_i(z) + \frac{zh_i'(z)}{h_i(z)+\lambda+\mu}\right\} \in P$ for $|z| < r_0,\ r_0$ is given by Equation (6). This implies that,

$$\left\{H(z) + \frac{zH'(z)}{H(z)+\lambda+\mu}\right\} \in P_k, \text{ for } |z|<r_0,$$

and consequently $f \in R_k(\lambda,\mu,0)$ for $|z|<r_0$.

As a special case, it is noted that $f \in R_2(1,2,0)$ implies that $f \in R_2(1,1,0)$ for $|z| < 0.8514$. That is, $f \in R_2(1,2,0)$ implies that $J_1 f$ is starlike for $|z| < 0.8514$.

Theorem 5

Let $\lambda > -1, \mu > 0.$ Then $P_k'(\lambda,\mu,\alpha) \subset P_k'(\lambda,\mu+1,\delta),$

Where,

$$\delta = \alpha+(1-\alpha)(2\gamma-1),\ \gamma = \int_0^1 \left(1+t^{\frac{1}{\lambda+\mu+1}}\right)^{-1} dt \tag{16}$$

which is an increasing function of $\frac{1}{\lambda+\mu+1}$ and $\frac{1}{2} \le \gamma < 1.$

Proof

We set $(L_\lambda^{\mu+1} f(z))' = H(z) = (1-\delta)\, h(z)+\delta,$

Where H is analytic in E with $H(0)=1$. Using Equation (5) with some computations, we have

$$(L_\lambda^\mu f(z))' = H(z) + \lambda_1\, zH'(z).$$

At this instant, using Lemma 4, the required result is obtained.

Theorem 6

Let ϕ be a convex function and let $f \in R_2(\lambda,\mu,\alpha)$. Then $\phi * f \in R_2(\lambda,\mu,\alpha)$.

Proof:

Let $G = \phi * f$. First, the study shows that $L_\lambda^\mu G = \phi * L_\lambda^\mu f$.

For this, let $\phi(z) = z + \sum_{n=2}^{\infty} b_n z^n$ and $f(z)$ be given by Equation (1). Then Equation 17 will be realized as:

$$L_\lambda^\mu(\phi * f)(z) = z + \frac{\Gamma(\lambda+\mu+1)}{\Gamma(\lambda+1)} \sum_{n=2}^{\infty} \frac{\Gamma(\lambda+n)}{\Gamma(\lambda+\mu+n)} a_n b_n z^n \tag{17}$$
$$= (\phi * L_\lambda^\mu f)(z)$$

Also, since $f \in R_2(\lambda,\mu,\alpha)$, it follows that $L_\lambda^\mu f \in R_2(\alpha) \equiv S^*(\alpha)$, when $S^*(\alpha)$ is the class of starlike functions of order α. Now, by logarithmic differentiation of Equation (17), the following is realized:

$$\frac{z(L_\lambda^\mu G(z))'}{L_\lambda^\mu G(z)} = \frac{z((\phi * L_\lambda^\mu f)'(z))}{(\phi * L_\lambda^\mu f)(z)}$$
$$= \frac{\phi(z) * \frac{z(L_\lambda^\mu f(z))'}{L_\lambda^\mu f(z)} \cdot L_\lambda^\mu f(z)}{\phi * L_\lambda^\mu f(z)} = \frac{\phi * F\, L_\lambda^\mu f}{\phi * L_\lambda^\mu f},$$

Where $F = \frac{z(L_\lambda^\mu f(z))'}{L_\lambda^\mu f(z)}$ is analytic in E and $F(0) = 1$.

As such, Lemma 3 is used to obtain this result $(\phi * f) \in R_2(\lambda,\mu,\alpha)$.
When Theorem 6 was applied to the study, Theorem 7 was realized.

Theorem 7

The class $R_2(\lambda,\mu,\alpha)$ is invariant under the following integral operators.

(i)

$$f_1(z) = \int_0^z \frac{f(t)}{t}\,dt = (\phi_1 * f)(z),\ \ \phi_1(z) = -\log(1-z)$$

(ii)

$$f_2(z) = \frac{2}{z}\int_0^z f(t)\,dt = (\phi_2 * f)(z),\ \ \phi_2(z) = -2\left[\frac{z+\log(1-z)}{z}\right]$$

(iii)

$$f_3(z) = \int_0^z \frac{f(t)-f(xt)}{t-xt}\,dt, \qquad (|x| \le 1, x \ne 1),$$

$$= (\phi_3 * f)(z),\ \phi_3(z) \quad = \frac{1}{1-x}\log\left(\frac{1-xz}{1-z}\right),$$

(iv)

$$f_4(z) = \frac{1+c}{z^c}\int_0^z t^{c-1} f(t)\,dt, \qquad \operatorname{Re} c > 0,$$

$$= (\phi_4 * f)(z),\ \ \phi_4(z) = \sum_{n=1}^{\infty} \frac{1+c}{n+c} z^n,\ \ \operatorname{Re} c > 0$$

The proof follows immediately, since $\phi_i \in c$ is for $i = 1, 2, 3, 4$ in E.

Conclusion

In this paper, some new classes of analytic function have been defined by using the linear integral operator. These new classes are general and they include various known classes of analytic functions as special cases. Thus, the Miller-Mocanu Lemma has been used to obtain several new and interesting results. The results obtained in this paper may be viewed as a refinement and improvement of the previously known results.

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