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Imbedding inequalities for the composite operator in the Sobolev spaces of differential forms

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Abstract
We establish the imbedding inequalities for the composition of the homotopy operator and Green's operator in the weighted Sobolev spaces and Orlicz-Sobolev spaces of differential forms. First we prove both the local and global L^p estimates for the composite operator acting on differential forms and obtain the boundedness of the composite operator in the weighted L^p spaces. Then, we further study the local and global L^ϕ-norm inequalities for the composite operator. As a consequence we obtain the imbedding inequalities in the Orlicz-Sobolev spaces.

Keywords: differential forms; imbedding inequalities; the homotopy operator; Green's operator

1 Introduction

Recently the L^p-theory of differential forms on $\mathbb{R}^n$ has been widely applied in many fields, such as quasiconformal mappings and the nonlinear elasticity theory. In the meantime, the operator theory of differential forms and the applications also motivate interest in this subject; see [1–4]. Note that before these operators can be effectively exploited, their L^p theories must be developed. Specially, the study on the boundedness of operators and the descriptions to more geometric spaces of differential forms are crucial to the development of the L^p theory of differential forms. This paper is devoted to establishing the local and global imbedding inequalities for the composition of the homotopy operator and Green's operator in the weighted Sobolev space and Orlicz-Sobolev space of differential forms.

Let M be an open subset of $\mathbb{R}^n$, $n \geq 2$, and $B \subset \mathbb{R}^n$ be a ball. Here we do not distinguish the balls B and cubes Q. We use $|M|$ to denote the Lebesgue measure of a set $M \subset \mathbb{R}^n$. A k-form $\omega(x) = \sum_I \omega_I(x)\,dx_I = \sum \omega_{i_1,i_2,\ldots,i_k}(x)\,dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}$ with summation over all ordered k-tuples $I = (i_1, i_2, \ldots, i_k)$, $1 \leq i_1 < i_2 < \cdots < i_k \leq n$. $\wedge^k(M)$, $k = 0, 1, \ldots, n$, denote the linear space of all k-forms in M. If each coefficient $\omega_I(x)$ of k-form $\omega(x)$ is differential, we call $\omega(x)$ a differential k-form in M. $D'(M, \wedge^k)$ is the space of all differential k-forms in M. We use $\wedge = \wedge(M)$ to denote the graded algebra of differential forms in M constructed from the $\wedge^k(M)$. As usual, we still use $\star : \wedge \to \wedge$ to denote the Hodge star operator. Let $d : D'(M, \wedge^k) \to D'(M, \wedge^{k+1})$ denote the differential operator and $d^\star = (-1)^{nk+1} \star d \star : D'(M, \wedge^{k+1}) \to D'(M, \wedge^k)$, $k = 0, 1, \ldots, n-1$ be the Hodge codifferential operator.

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We are interested in the L^p spaces of differential forms. Let

$$L^p(M,\wedge^k) = \left\{\omega = \sum_I \omega_I(x)\,dx_I : \omega_I \in L^p(M)\right\}$$

be the Banach space with the norm

$$\|\omega\|_{p,M} = \left(\int_M |\omega(x)|^p\,dx\right)^{1/p} = \left(\int_M \left(\sum_I |\omega_I(x)|^2\right)^{p/2} dx\right)^{1/p}.$$

A weight $w(x)$ is a non-negative locally integrable function on $\mathbb{R}^n$. We use $L^p(M,\wedge^k,w)$ to denote the weighted L^p space with norm $\|\omega\|_{p,M,w} = (\int_M |\omega(x)|^p w(x)\,dx)^{1/p}$. The $(1,p)$-Sobolev space $W^{1,p}(M,\wedge^k)$ is the space of k-forms which equals $L^p(M,\wedge^k) \cap L_1^p(M,\wedge^k)$. For a bounded domain $M \subset \mathbb{R}^n$, the norm is

$$\|\omega\|_{W^{1,p}(M,\wedge^k)} = \operatorname{diam}(M)^{-1}\|\omega\|_{p,M} + \|\nabla\omega\|_{p,M},$$

where $\nabla\omega = (\frac{\partial\omega}{\partial x_1},\ldots,\frac{\partial\omega}{\partial x_n})$ is a vector-valued differential form. Similarly, we use $W^{1,p}(M,\wedge^k,w)$ to denote the weighted $(1,p)$-Sobolev space with the norm

$$\|\omega\|_{W^{1,p}(M,\wedge^k,w)} = \operatorname{diam}(M)^{-1}\|\omega\|_{p,M,w} + \|\nabla\omega\|_{p,M,w}.$$

Furthermore, if $M \subset \mathbb{R}^n$ is a bounded convex domain, Iwaniec and Lutoborski in [5] defined the linear operator $K_y : C^\infty(M,\wedge^k) \to C^\infty(M,\wedge^{k-1})$ as

$$(K_y\omega)(x;\xi_1,\xi_2,\ldots,\xi_{k-1}) = \int_0^1 t^{k-1}\omega(tx+y-ty;x-y,\xi_1,\xi_2,\ldots,\xi_{k-1})\,dt.$$

The homotopy operator $T : C^\infty(M,\wedge^k) \to C^\infty(M,\wedge^{k-1})$ is defined by $T\omega = \int_D \varphi(y)K_y\omega\,dy$, where $\varphi \in C_0^\infty(M)$ is normalized so that $\int \varphi(y)\,dy = 1$. In [5], they also proved that there exists an operator $T : L^1_{\mathrm{loc}}(M,\wedge^k) \to L^1_{\mathrm{loc}}(M,\wedge^{k-1})$, $k = 1,2,\ldots,n$, such that

$$T(d\omega) + dT\omega = \omega, \tag{1.1}$$

$$\|T\omega\|_{p,B} \le C\|\omega\|_{p,B}. \tag{1.2}$$

The k-form $\omega_M \in D'(M,\wedge^k)$ is defined by $\omega_M = \frac{1}{|M|}\int_M \omega(y)\,dy$ if $k=0$ and $\omega_M = d(T\omega)$ if $k = 1,2,\ldots,n$. It is easy to see that ω_M is a closed form. Recently, the definition of the homotopy operator is further generalized to any domain Ω in $\mathbb{R}^n$. See [6] for the details.

Let $\wedge^k E$ denote the kth exterior power of the cotangent bundle. $\mathcal{W}(\wedge^k E)$ is the space of all $\omega \in L^1_{\mathrm{loc}}(\wedge^k E)$ which has generalized gradient, and the harmonic l-field $\mathcal{H}(\wedge^k E)$ is the subspace of $\mathcal{W}(\wedge^k E)$ where the element ω satisfies $d\omega = d^\star\omega = 0$, $\omega \in L^p$ for some $1 < p < \infty$. The orthogonal complement of $\mathcal{H}$ in L^1 is denoted as $\mathcal{H}^\perp = \{\omega \in L^1 : (\omega,h) = 0 \text{ for all } h \in \mathcal{H}\}$. Green's operator $G : C^\infty(\wedge^k E) \to \mathcal{H}^\perp \cap C^\infty(\wedge^k E)$ is defined by assigning $G(\omega)$ as the unique element of $\mathcal{H}^\perp \cap C^\infty(\wedge^k E)$ satisfying Poisson's equation $\Delta G(\omega) = \omega - H(\omega)$. In [7], Scott developed the definition of Green's operator to L^p spaces, $2 \le p < \infty$, as $G : L^p(\wedge^k E) \to \mathcal{W}^{1,p} \cap \mathcal{H}^\perp$ by $G(\omega) = \Omega(\omega - H(\omega))$. This implies that $\nabla G(\omega) = \omega - H(\omega)$.

Here $\Omega(\omega)$ satisfies $\nabla\Omega(\omega) = \omega$, for $\omega \in \mathcal{H}^{\perp} \cap L^p$ $(p \geq 2)$. Further, for $p \geq 2$, Scott obtained the estimate

$$\begin{aligned} &\|dd^*G(\omega)\|_{p,E} + \|d^*dG(\omega)\|_{p,E} + \|dG(\omega)\|_{p,E} \\ &\quad + \|d^*G(\omega)\|_{p,E} + \|G(\omega)\|_{p,E} \leq C\|\omega\|_{p,E} \end{aligned} \tag{1.3}$$

for all $\omega \in L^p$, where $C = C(p)$ is a constant.

2 The local imbedding inequalities for the composite operator in the weighted Sobolev space

In this section, we prove a local imbedding inequality for the composition of Green's operator and the homotopy operator in the weighted $(1,p)$-Sobolev space when $n < p < \infty$. It should be pointed out that the weighted inequality holds not only for the solutions of the A-harmonic equation but also for all differential form $u \in L^p(M, \wedge^l)$.

The A_r weight was first introduced by Muckenhoupt in [8].

Definition 2.1 We say a weight $w(x)$ satisfies the $A_r(M)$ condition for $r > 1$, write $w(x) \in A_r(M)$ if $w(x) > 0$ a.e. and

$$\sup_{B\subset M}\left(\frac{1}{|B|}\int_B w\,dx\right)\left(\frac{1}{|B|}\int_B\left(\frac{1}{w}\right)^{1/(r-1)}dx\right)^{r-1} = [c_{r,w}] < \infty.$$

Here $[c_{r,w}]$ is called the A_r constant. The A_r weight satisfies the following lemma, which appears in [9].

Lemma 2.2 *If $w \in A_r(M)$, then there exist constants $\beta > 1$ and C, independent of w, such that*

$$\|w\|_{\beta,B} \leq C|B|^{(1-\beta)/\beta}\|w\|_{1,B}$$

for all balls $B \subset M$.

The following result appears in [10].

Lemma 2.3 *Let $M \subset \mathbb{R}^n$ be a bounded convex domain. The operator T maps $L^p(M, \wedge^k)$ continuously to $L^q(M, \wedge^{k-1})$ in the following cases*:

$$\begin{aligned} &\textit{Either } 1 \leq p, q \leq \infty \textit{ and } 1/p - 1/q < 1/n, \\ &\quad \textit{or } 1 < p, q \leq \infty \textit{ and } 1/p - 1/q \leq 1/n. \end{aligned}$$

We now prove the following local weighted norm inequality for the composite operator $G \circ T$ acting on differential forms.

Theorem 2.4 *Let $M \subset \mathbb{R}^n$ be a bounded convex domain, $n < p < \infty$. $T : L^p(M, \wedge^l) \to L^p(M, \wedge^{l-1})$, $l = 1, 2, \ldots, n$, is the homotopy operator and G is Green's operator. If w satisfies the $A_r(M)$ condition with $1 < r < p/n$, then we have the following inequality*:

$$\|G(T(u))\|_{p,B,w} \leq C(r,p,M)[C_{r,w}]^{\frac{1}{p}}\|u\|_{p,B,w}$$

for all balls $B \subset M$.

Proof Note that $w(x) \in A_r(M)$. From Lemma 2.2, there exist constants $\beta > 1$ and C_r such that

$$\|w(x)\|_{\beta,B} \le C_r |B|^{(1-\beta)/\beta} \|w(x)\|_{1,B} \tag{2.1}$$

for all balls $B \subset M$. Take $k = \beta p/(\beta - 1)$. It is easy to see that $k > 1$. From the Hölder inequality with $\frac{1}{k} + \frac{1}{\beta p} = \frac{1}{p}$, we have

$$\begin{aligned} \|G(T(u))\|_{p,B,w} &= \left(\int_B |G(T(u))|^p w(x)\,dx\right)^{1/p} \\ &\le \left(\int_B |G(T(u))|^k\,dx\right)^{1/k} \left(\int_B w^\beta\,dx\right)^{1/\beta p} \\ &= \|G(T(u))\|_{k,B} \|w\|_{\beta,B}^{1/p}. \end{aligned} \tag{2.2}$$

Substituting (2.1) into (2.2), we obtain

$$\|G(T(u))\|_{p,B,w} \le C_r^{\frac{1}{p}} |B|^{(1-\beta)/\beta p} \|G(T(u))\|_{k,B} \|w\|_{1,B}^{1/p}. \tag{2.3}$$

Take $s = p/r$. It is easy to check that $s > 1$ and $\frac{1}{s} - \frac{1}{k} < \frac{1}{n}$. Thus, from Lemma 2.3 and (1.3), we immediately have

$$\|G(T(u))\|_{k,B} \le C_1(p)\|T(u)\|_{k,B} \le C_2(p)\|u\|_{s,B}. \tag{2.4}$$

Combining (2.3) and (2.4), we have

$$\|G(T(u))\|_{p,B,w} \le C(r,p)|B|^{(1-\beta)/\beta p} \|u\|_{s,B} \|w\|_{1,B}^{1/p}. \tag{2.5}$$

Using the Hölder inequality with $\frac{1}{p} + \frac{r-1}{p} = \frac{1}{s}$, we have

$$\begin{aligned} \|u\|_{s,B} &= \left(\int_B |u|^s\,dx\right)^{\frac{1}{s}} \le \left(\int_B \left(|u|w^{1/p}\right)^p\,dx\right)^{1/p} \left(\int_B \left(\frac{1}{w}\right)^{\frac{1}{r-1}}\,dx\right)^{\frac{r-1}{p}} \\ &= \|u\|_{p,B,w} \left(\int_B \left(\frac{1}{w}\right)^{\frac{1}{r-1}}\,dx\right)^{\frac{r-1}{p}}. \end{aligned} \tag{2.6}$$

Note that $w \in A_r(M)$. Hence, for all balls $B \subset M$,

$$\left(\frac{1}{|B|}\int_B w\,dx\right)^{1/p} \left(\frac{1}{|B|}\int_B \left(\frac{1}{w}\right)^{1/(r-1)}\,dx\right)^{(r-1)/p} < [C_{r,w}]^{\frac{1}{p}} < \infty.$$

Thus, combining (2.5) and (2.6), we have

$$\begin{aligned} \|G(T(u))\|_{p,B,w} &\le C(r,p)|B|^{\frac{1-\beta}{p\beta}} \|u\|_{p,B,w} \left(\int_B w\,dx\right)^{\frac{1}{p}} \left(\int_B \left(\frac{1}{w}\right)^{\frac{1}{r-1}}\,dx\right)^{\frac{r-1}{p}} \\ &\le C(r,p)|B|^{\frac{(1-\beta)}{p\beta}} |B|^{\frac{1}{p}} |B|^{\frac{r-1}{p}} [C_{r,w}]^{\frac{1}{p}} \|u\|_{p,B,w} \\ &\le C(r,p,M)[C_{r,w}]^{\frac{1}{p}} \|u\|_{p,B,w}. \end{aligned} \tag{2.7}$$

We complete the proof of Theorem 2.4. □

Theorem 2.5 *Let $M \subset \mathbb{R}^n$ be a bounded convex domain, $n < p < \infty$. $T : L^p(M, \wedge^l) \to L^p(M, \wedge^{l-1})$, $l = 1, 2, \ldots, n$, is the homotopy operator and G is Green's operator. If w satisfies the $A_r(M)$ condition with $1 < r < p/n$, then we have*

$$\|\nabla G(T(u))\|_{p,B,w} \le C(r,p,M)[C_{r,w}]^{\frac{1}{p}} \|u\|_{p,B,w}$$

for all balls $B \subset M$.

Proof Using a similar method and notation from (2.1) and (2.2) as we did in the proof of Theorem 2.4, we have

$$\|\nabla G(T(u))\|_{p,B,w} \le C_r^{\frac{1}{p}} |B|^{\frac{1-\beta}{p\beta}} \|\nabla G(T(u))\|_{k,B} \|w\|_{1,B}^{\frac{1}{p}}. \tag{2.8}$$

We still take $s = p/r$. From (1.3), it is easy to see that there exists a constant $C(p)$ such that $\|\nabla G(u)\|_{k,B} \le C(p)\|u\|_{k,B}$. Thus, from Lemma 2.3 and (1.3), we obtain

$$\|\nabla G(T(u))\|_{k,B} \le C_1(p)\|T(u)\|_{k,B} \le C_2(p)\|u\|_{s,B}. \tag{2.9}$$

Thus, combining (2.8) and (2.9), we have

$$\begin{aligned} \|\nabla G(T(u))\|_{p,B,w} &\le C_r^{\frac{1}{p}} |B|^{\frac{1-\beta}{p\beta}} C_2(p)\|u\|_{s,B}\|w\|_{1,B}^{\frac{1}{p}} \\ &= C(r,p)|B|^{\frac{1-\beta}{p\beta}} \|u\|_{s,B}\|w\|_{1,B}^{\frac{1}{p}}. \end{aligned} \tag{2.10}$$

Using the Hölder inequality with $\frac{1}{p} + \frac{r-1}{p} = \frac{1}{s}$, we have

$$\|u\|_{s,B} \le \|u\|_{p,B,w} \left(\int_B \left(\frac{1}{w}\right)^{\frac{1}{r-1}} dx \right)^{\frac{r-1}{p}}. \tag{2.11}$$

Combining (2.10), (2.11) and the $A_r(M)$ condition, we obtain

$$\begin{aligned} \|\nabla G(T(u))\|_{p,B,w} &\le C(r,p)|B|^{\frac{1-\beta}{p\beta}} \|u\|_{p,B,w} \left(\int_B w\,dx \right)^{\frac{1}{p}} \left(\int_B \left(\frac{1}{w}\right)^{\frac{1}{r-1}} dx \right)^{\frac{r-1}{p}} \\ &\le C(r,p)|B|^{\frac{(1-\beta)}{p\beta}} |B|^{\frac{1}{p}} |B|^{\frac{r-1}{p}} [C_{r,w}]^{\frac{1}{p}} \|u\|_{p,B,w} \\ &\le C(r,p,M)[C_{r,w}]^{\frac{1}{p}} \|u\|_{p,B,w}. \end{aligned}$$

We complete the proof of Theorem 2.5. □

Note that the past weighted inequalities with A_r weights only hold for the solutions of the A-harmonic equation when $r > 1$ and $1 < p < \infty$. However, the results in Theorems 2.4 and 2.5 show that the A_r weight inequalities for the composite operator also apply to all the differential forms in $L^p(M, \wedge^l)$ under the condition of $n < p < \infty$ and $1 < r < p/n$. This is to say, on the one hand, we extend the scope of the operator; on the other hand, the results limit the spaces where the operator inequalities hold.

3 The global imbedding inequalities for the composite operator in the weighted Sobolev space

The problem of proving sharp one or two-weight norm inequalities for the classical operators of harmonic analysis has a long history. It usually needs to find the best value for the exponent $\alpha(p)$ to prove the sharp dependence on the A_r constant $[C_{r,w}]$. Therefore, it is attractive to estimate the sharp value of $\alpha(p)$. In the previous section, we have obtained the local imbedding inequalities for the composite operator. In this section, we prove the global results in the weighted Sobolev space by means of the modified Whitney cover and obtain the power $\alpha(p) = \frac{1}{p}$. In order to obtain the main theorem, we need the following lemma, which appears in [11].

Lemma 3.1 *Each open subset $M \subset \mathbb{R}^n$ has a modified Whitney cover of cubes $Q = \{Q_i\}$ which satisfies*

$$\bigcup_i Q_i = M,$$

$$\sum_{Q_i \in Q} \chi_{\sqrt{5/4}Q_i}(x) \le K \cdot \chi_M(x)$$

for all $x \in \mathbb{R}^n$ and some $K > 1$, where χ_M is the characteristic function for the set M.

Theorem 3.2 *Let $M \subset \mathbb{R}^n$ be a bounded convex domain, $n < p < \infty$. $T : L^p(M, \wedge^l) \to L^p(M, \wedge^{l-1})$, $l = 1, 2, \dots, n$, is the homotopy operator and G is Green's operator. If w satisfies the $A_r(M)$ condition with $1 < r < p/n$, then we have the following inequality*:

$$\|G(T(u))\|_{p,M,w} \le C(r,p,M)[C_{r,w}]^{\frac{1}{p}} \|u\|_{p,M,w}.$$

Proof From Lemma 3.1 and the properties of the modified Whitney cover, we know that there exists a sequence of cubes $\{Q_i\}$ such that $\bigcup Q_i = M$ and $\sum_{i=1}^{\infty} \chi_{\sqrt{\frac{5}{4}}Q_i}(x) \le K \cdot \chi_M(x)$ for all $x \in M$, where $K > 1$ is some constant. Thus, we have

$$\begin{aligned}
&\|G(T(u))\|_{p,M,w}^p \\
&\quad = \int_M |G(T(u))|^p w(x)\,dx \\
&\quad \le \sum_{i=1}^{\infty} \int_{Q_i} |G(T(u))|^p w(x)\,dx \\
&\quad \le \sum_{i=1}^{\infty} C(r,p,M)[C_{r,w}]^{\frac{1}{p}} \int_{Q_i} |u|^p w(x)\,dx \\
&\quad = C(r,p,M)[C_{r,w}]^{\frac{1}{p}} \sum_{i=1}^{\infty} \int_M |u|^p \chi_{Q_i}(x) w(x)\,dx \\
&\quad = C(r,p,M)[C_{r,w}]^{\frac{1}{p}} \lim_{n\to\infty} \int_M \sum_{i=1}^{n} |u|^p \chi_{Q_i}(x) w(x)\,dx,
\end{aligned} \tag{3.1}$$

where the constant $C(r,p,M)$ is independent of u and each Q_i. If we write $b_n(x) = \sum_{i=1}^{n} |u|^p \chi_{Q_i}(x)$ for $x \in M$, then it is easy to find that $\{b_n(x)\}$ is an increasing sequence of functions in M and $b_n(x) \le K|u|^p \chi_M(x)$ for all $x \in M$. Thus, from the monotone convergence theorem, we have

$$\begin{aligned}
&\lim_{n\to\infty} \int_M \sum_{i=1}^{n} |u|^p \chi_{Q_i}(x) w(x)\,dx \\
&\quad = \int_M \lim_{n\to\infty} \sum_{i=1}^{n} |u|^p \chi_{Q_i}(x) w(x)\,dx \\
&\quad \le K \int_M |u|^p \chi_M(x) w(x)\,dx \\
&\quad = K\|u\|_{p,M,w}^p. \qquad (3.2)
\end{aligned}$$

Combining (3.1) and (3.2), we obtain the boundedness of the composite operator $G \circ T$ in the weighted L^p space. □

Using the same method, we have Theorem 3.3.

Theorem 3.3 *Let $M \subset \mathbb{R}^n$ be a bounded convex domain, $n < p < \infty$. $T : L^p(M, \wedge^l) \to L^p(M, \wedge^{l-1})$, $l = 1, 2, \dots, n$, is the homotopy operator and G is Green's operator. If w satisfies the $A_r(M)$ condition with $1 < r < p/n$, then we have the following inequality*:

$$\|\nabla G(T(u))\|_{p,M,w} \le C(r,p,M)[C_{r,w}]^{\frac{1}{p}} \|u\|_{p,M,w}.$$

Combining Theorems 3.2 and 3.3, we have the following imbedding inequality.

Theorem 3.4 *Let $M \subset \mathbb{R}^n$ be a bounded convex domain, $n < p < \infty$. $T : L^p(M, \wedge^l) \to L^p(M, \wedge^{l-1})$, $l = 1, 2, \dots, n$, is the homotopy operator and G is Green's operator. If w satisfies the $A_r(M)$ condition with $1 < r < p/n$, then we have*

$$\|G(T(u))\|_{W^{1,p}(M,\wedge^l,w)} \le C(r,p,M)[C_{r,w}]^{\frac{1}{p}} \|u\|_{p,M,w}.$$

Proof Applying the results of Theorems 3.2 and 3.3, we have

$$\begin{aligned}
&\|G(T(u))\|_{W^{1,p}(M,\wedge^l,w)} \\
&\quad = \operatorname{diam}(M)^{-1} \|G(T(u))\|_{p,M,w} + \|\nabla G(T(u))\|_{p,M,w} \\
&\quad \le C_1(r,p,M)[C_{r,w}]^{\frac{1}{p}} \|u\|_{p,M,w} + C_2(r,p,M)[C_{r,w}]^{\frac{1}{p}} \|u\|_{p,M,w} \\
&\quad = C(r,p,M)[C_{r,w}]^{\frac{1}{p}} \|u\|_{p,M,w}.
\end{aligned}$$

□

4 The imbedding inequality for the composite operator in the Orlicz-Sobolev space

In this section, we prove the imbedding inequality for the composite operator in the Orlicz-Sobolev spaces. Precisely, for the Young functions in the class $G(p,q,C)$, we establish the local L^φ-norm estimates and the subsequent global version by the modified Whitney cover. To do this, we need some definitions and notation.

We call a continuously increasing function $\phi : [0,\infty) \to [0,\infty)$ with $\phi(0) = 0$ an Orlicz function. A convex Orlicz function is further called a Young function. $L^\phi(M, \wedge^l)$ is the space of all l-forms ω on M such that $\int_M \phi(|\omega|/\lambda)\,dx < \infty$ for some $\lambda = \lambda(\omega) > 0$. $L^\phi(M, \wedge^l)$ is equipped with the nonlinear Luxemburg functional $\|\omega\|_{\phi,M} = \inf\{\lambda > 0 : \int_M \phi(\frac{|\omega|}{\lambda})\,dx \le 1\}$. We use $W^{1,\phi}(M, \wedge^l) = L^\phi(M, \wedge^l) \cap L_1^\phi(M, \wedge^l)$ to denote the Orlicz-Sobolev space of l-forms, which is equipped with the norm

$$\|\omega\|_{W^{1,\phi}(M,\wedge^l)} = \operatorname{diam}(M)^{-1}\|\omega\|_{\phi,M} + \|\nabla\omega\|_{\phi,M}.$$

Definition 4.1 We say a Young function ϕ lies in the class $G(p,q,C)$, $1 \le p < q < \infty$, $C \ge 1$, if

(i) $\dfrac{1}{C} \le \dfrac{\phi(t^{1/p})}{g(t)} \le C,$

(ii) $\dfrac{1}{C} \le \dfrac{\phi(t^{1/q})}{h(t)} \le C$

for all $t > 0$, where g is a convex increasing function and h is a concave increasing function on $[0,\infty)$.

From [12], each of ϕ, g and h mentioned in Definition 4.1 is doubling, from which it is easy to know that

$$C_1 t^q \le h^{-1}\bigl(\phi(t)\bigr) \le C_2 t^q, \qquad C_1 t^p \le g^{-1}\bigl(\phi(t)\bigr) \le C_2 t^p \tag{4.1}$$

for all $t > 0$, where C_1 and C_2 are constants.

Theorem 4.2 *Let ϕ be a Young function in the class $G(p,q,C)$, $1 \le p < q < \infty$, $C \ge 1$ and M be a bounded convex domain. Assume that $T : C^\infty(M, \wedge^l) \to C^\infty(M, \wedge^{l-1})$, $l = 1, 2, \ldots, n$, is the homotopy operator and G is Green's operator. If $\phi(|u|) \in L^1_{\mathrm{loc}}(M)$ for all $u \in C^\infty(M, \wedge^l)$ and $\frac{1}{p} - \frac{1}{q} < \frac{1}{n}$, then we have*

$$\bigl\|G\bigl(T(u)\bigr)\bigr\|_{\phi,B} \le C\|u\|_{\phi,B}$$

for all balls $B \subset M$.

Proof Note that $\frac{1}{p} - \frac{1}{q} < \frac{1}{n}$. Thus, from (1.3) and Lemma 2.3, we have

$$\bigl\|G\bigl(T(u)\bigr)\bigr\|_{q,B} \le C(q)\bigl\|T(u)\bigr\|_{q,B} \le C(q)\|u\|_{p,B} \tag{4.2}$$

for all $u \in C^\infty(M, \wedge^l)$. Thus, by Jensen's inequality for h^{-1} and (4.1), we have

$$\begin{aligned}
\int_B \phi\bigl(\bigl|G\bigl(T(u)\bigr)\bigr|\bigr)\,dx &= h\biggl(h^{-1}\biggl(\int_B \phi\bigl(\bigl|G\bigl(T(u)\bigr)\bigr|\bigr)\,dx\biggr)\biggr) \\
&\le h\biggl(\int_B h^{-1}\bigl(\phi\bigl(\bigl|G\bigl(T(u)\bigr)\bigr|\bigr)\bigr)\,dx\biggr) \\
&\le h\biggl(C_1 \int_B \bigl|G\bigl(T(u)\bigr)\bigr|^q\,dx\biggr).
\end{aligned} \tag{4.3}$$

Note that $\phi \in G(p,q,C)$. From Definition 4.1, (4.2) and (4.3), we obtain

$$\begin{aligned} &h\left(C_1 \int_B \left|G\bigl(T(u)\bigr)\right|^q dx\right) \\ &\quad \le C_2\phi\left(C_1^{1/q}\left(\int_B \left|G\bigl(T(u)\bigr)\right|^q dx\right)^{1/q}\right) \\ &\quad \le C_3\phi\left(C_4\left(\int_B |u|^p dx\right)^{1/p}\right) \\ &\quad = C_3\phi\left(\left(C_4^p\left(\int_B |u|^p dx\right)\right)^{1/p}\right). \end{aligned} \tag{4.4}$$

Furthermore, (4.4) implies that

$$\begin{aligned} &C_3\phi\left(\left(C_4^p\left(\int_B |u|^p dx\right)\right)^{1/p}\right) \\ &\quad \le C_5 g\left(C_4^p \int_B |u|^p dx\right) \\ &\quad = C_5 g\left(\int_B C_4^p |u|^p dx\right) \\ &\quad \le C_6 \int_B g\bigl(C_4^p |u|^p\bigr) dx \\ &\quad \le C_7 \int_B \phi\bigl(C_4 |u|\bigr) dx. \end{aligned} \tag{4.5}$$

Noting that ϕ is doubling, we have

$$\int_B \phi\bigl(C_4|u|\bigr) dx \le C_8 \int_B \phi\bigl(|u|\bigr) dx. \tag{4.6}$$

Combining (4.3)-(4.6), we have

$$\int_B \phi\bigl(\left|G\bigl(T(u)\bigr)\right|\bigr) dx \le C_9 \int_B \phi\bigl(|u|\bigr) dx. \tag{4.7}$$

Replacing $|G(T(u))|$ by $\frac{1}{\lambda}|G(T(u))|$, we immediately obtain

$$\int_B \phi\left(\frac{|G(T(u))|}{\lambda}\right) dx \le C_{10} \int_B \phi\left(\frac{|u|}{\lambda}\right) dx. \tag{4.8}$$

Finally, (4.8) implies that

$$\left\|G\bigl(T(u)\bigr)\right\|_{\phi,B} \le C_{10}\|u\|_{\phi,B}.$$

We complete the proof of Theorem 4.2. □

Using the same program as we did in the proof of Theorem 4.2 and replacing (4.2) by

$$\left\|\nabla G\bigl(T(u)\bigr)\right\|_{q,B} \le C\left\|T(u)\right\|_{q,B} \le C\|u\|_{p,B}, \tag{4.9}$$

we have the following theorem.

Theorem 4.3 *Let ϕ be a Young function in the class $G(p,q,C)$, $1 \leq p < q < \infty$, $C \geq 1$ and M be a bounded convex domain. Let $T : C^\infty(M, \wedge^l) \to C^\infty(M, \wedge^{l-1})$, $l = 1, 2, \ldots, n$, be the homotopy operator and G be Green's operator. If $\phi(|u|) \in L^1_{\mathrm{loc}}(M)$ for all $u \in C^\infty(M, \wedge^l)$ and $\frac{1}{p} - \frac{1}{q} < \frac{1}{n}$, then we have*

$$\|\nabla G(T(u))\|_{\phi,B} \leq C\|u\|_{\phi,B}$$

for all balls $B \subset M$.

Using the method analogous to the proof of Theorem 3.2, we have the following global estimates.

Theorem 4.4 *Let ϕ be a Young function in the class $G(p,q,C)$, $1 \leq p < q < \infty$, $C \geq 1$ and M be a bounded convex domain. Let $T : C^\infty(M, \wedge^l) \to C^\infty(M, \wedge^{l-1})$, $l = 1, 2, \ldots, n$, be the homotopy operator and G be Green's operator. If $\phi(|u|) \in L^1_{\mathrm{loc}}(M)$ for all $u \in C^\infty(M, \wedge^l)$ and $\frac{1}{p} - \frac{1}{q} < \frac{1}{n}$, then we have*

$$\|G(T(u))\|_{\phi,M} \leq C_1\|u\|_{\phi,M}, \tag{4.10}$$

$$\|\nabla G(T(u))\|_{\phi,M} \leq C_2\|u\|_{\phi,M}. \tag{4.11}$$

Here the constants C_1 and C_2 are independent of u.

Finally, we have the following imbedding inequality in the Orlicz-Sobolev spaces.

Theorem 4.5 *Let ϕ be a Young function in the class $G(p,q,C)$, $1 \leq p < q < \infty$, $C \geq 1$ and M be a bounded convex domain. Let $T : C^\infty(M, \wedge^l) \to C^\infty(M, \wedge^{l-1})$, $l = 1, 2, \ldots, n$, be the homotopy operator and G be Green's operator. If $\phi(|u|) \in L^1_{\mathrm{loc}}(M)$ for all $u \in C^\infty(M, \wedge^l)$ and $\frac{1}{p} - \frac{1}{q} < \frac{1}{n}$, then we have*

$$\|G(T(u))\|_{W^{1,\phi}(M,\wedge^l)} \leq C\|u\|_{\phi,M}.$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
HB and SY jointly contributed to the main results and HB drafted the manuscript. All authors read and approved the final manuscript.

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