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Cancellability and regularity of operator connections with applications to nonlinear operator equations involving means

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Abstract

An operator connection is a binary operation assigned to each pair of positive operators satisfying monotonicity, continuity from above, and the transformer inequality. A normalized operator connection is called an operator mean. In this paper, we introduce and characterize the concepts of cancellability and regularity of operator connections with respect to operator monotone functions, Borel measures, and certain nonlinear operator equations. As applications, we investigate the existence and the uniqueness of solutions for operator equations involving various kind of operator means.

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1 Introduction

A general theory of connections and means for positive operators was given by Kubo and Ando [1]. This theory is closely related to theory of operator inequalities and has deep applications in electrical network theory and mathematical physics. Let $B(\mathbb{H})$ be the algebra of bounded linear operators on a Hilbert space $\mathbb{H}$. The set of positive operators on $\mathbb{H}$ is denoted by $B(\mathbb{H})^+$. Denote the spectrum of an operator X by $\operatorname{Sp}(X)$. For Hermitian operators $A, B \in B(\mathbb{H})$, the partial order $A \le B$ indicates that $B - A \in B(\mathbb{H})^+$. The notation $A > 0$ suggests that A is a strictly positive operator. A *connection* is a binary operation σ on $B(\mathbb{H})^+$ such that for all positive operators A, B, C, D:

- (M1) *monotonicity*: $A \le C, B \le D \implies A \,\sigma\, B \le C \,\sigma\, D$;
- (M2) *transformer inequality*: $C(A \,\sigma\, B)C \le (CAC) \,\sigma\, (CBC)$;
- (M3) *continuity from above*: for $A_n, B_n \in B(\mathbb{H})^+$, if $A_n \downarrow A$ and $B_n \downarrow B$, then $A_n \,\sigma\, B_n \downarrow A \,\sigma\, B$. Here, $A_n \downarrow A$ indicates that (A_n) is a decreasing sequence converging strongly to A.

Two trivial examples are the left-trivial mean $\omega_l : (A, B) \mapsto A$ and the right-trivial mean $\omega_r : (A, B) \mapsto B$. The sum $(A, B) \mapsto A + B$ is clearly a connection. A connection was modeled from the notion of parallel sum, introduced by Anderson and Duffin [2],

$$A : B = \left(A^{-1} + B^{-1}\right)^{-1}, \quad A, B > 0.$$

This notion plays an important role in the analysis of electrical networks.

From the transformer inequality, every connection is invariant consider congruences in the sense that for each $A, B \geq 0$ and $C > 0$ we have

$$C(A \,\sigma\, B)C = (CAC)\,\sigma\,(CBC).$$

A *mean* is a connection σ with normalized condition $I \,\sigma\, I = I$ or, equivalently, fixed-point property $A \,\sigma\, A = A$ for all $A \geq 0$. The class of Kubo-Ando means cover many well-known operator means in practice, *e.g.*

- α-weighted arithmetic means: $A \,\triangledown_\alpha\, B = (1-\alpha)A + \alpha B$;
- α-weighted geometric means: $A \,\#_\alpha\, B = A^{1/2}(A^{-1/2}BA^{-1/2})^\alpha A^{1/2}$;
- α-weighted harmonic means: $A \,!_\alpha\, B = [(1-\alpha)A^{-1} + \alpha B^{-1}]^{-1}$;
- logarithmic mean: $(A,B) \mapsto A^{1/2}f(A^{-1/2}BA^{-1/2})A^{1/2}$ where $f : \mathbb{R}^+ \to \mathbb{R}^+$, $f(x) = (x-1)/\log x$, $f(0) \equiv 0$, and $f(1) \equiv 1$. Here, $\mathbb{R}^+ = [0,\infty)$.

See [3, 4], [5], Section 3, and [6], Chapter 5.

It is a fundamental that there are one-to-one correspondences between the following objects:

(1) operator connections on $B(\mathbb{H})^+$;
(2) operator monotone functions from $\mathbb{R}^+$ to $\mathbb{R}^+$;
(3) finite (positive) Borel measures on $[0,1]$;
(4) monotone (Riemannian) metrics on the smooth manifold of positive definite matrices.

Recall that a function $f : \mathbb{R}^+ \to \mathbb{R}^+$ is said to be *operator monotone* if

$$A \leq B \quad \Longrightarrow \quad f(A) \leq f(B)$$

for all positive operators $A, B \in B(\mathbb{H})$ and for all Hilbert spaces $\mathbb{H}$. This concept was introduced in [7]; see also [8], Chapter V, [5], Section 2, and [6], Chapter 4. Concrete examples of operator monotone functions are provided in [9]. A remarkable fact is that (see [10]) a function $f : \mathbb{R}^+ \to \mathbb{R}^+$ is operator monotone if and only if it is *operator concave*, *i.e.*

$$f\bigl((1-\alpha)A + \alpha B\bigr) \geq (1-\alpha)f(A) + \alpha f(B), \quad \alpha \in (0,1),$$

for all positive operators $A, B \in B(\mathbb{H})$ and for all Hilbert spaces $\mathbb{H}$.

A connection σ on $B(\mathbb{H})^+$ can be characterized via operator monotone functions as follows.

Theorem 1.1 ([1]) *Given a connection σ, there is a unique operator monotone function $f : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying*

$$f(x)I = I \,\sigma\, (xI), \quad x \geq 0.$$

Moreover, the map $\sigma \mapsto f$ is a bijection.

We call f the *representing function* of σ. A connection also has a canonical characterization with respect to a Borel measure via a meaningful integral representation as follows.

Theorem 1.2 ([11]) *Given a finite Borel measure μ on $[0,1]$, the binary operation*

$$A \,\sigma\, B = \int_{[0,1]} A \,!_t\, B \, d\mu(t), \quad A,B \geq 0, \tag{1.1}$$

is a connection on $B(\mathbb{H})^+$. Moreover, the map $\mu \mapsto \sigma$ is bijective, in which case the representing function of σ is given by

$$f(x) = \int_{[0,1]} (1 \,!_t\, x) \, d\mu(t), \quad x \geq 0. \tag{1.2}$$

We call μ the *associated measure* of σ. A connection is a mean if and only if $f(1) = 1$ or its associated measure is a probability measure. Hence every mean can be regarded as an average of weighted harmonic means. From (1.1) and (1.2), σ and f are related by

$$f(A) = I \,\sigma\, A, \quad A \geq 0. \tag{1.3}$$

A connection σ is said to be *symmetric* if $A \,\sigma\, B = B \,\sigma\, A$ for all $A,B \geq 0$.

The notion of monotone metrics arises naturally in quantum mechanics. A metric on a differentiable manifold of n-by-n positive definite matrices is a continuous family of positive definite sesquilinear forms assigned to each invertible density matrix in the manifold. A monotone metric is a metric with the contraction property under stochastic maps. It was shown in [12] that there is a one-to-one correspondence between operator connections and monotone metrics. Moreover, symmetric metrics correspond to symmetric means. In [13], the author defined a symmetric metric to be *nonregular* if $f(0) = 0$ where f is the associated operator monotone function. In [14], f is said to be *nonregular* if $f(0) = 0$, otherwise f is *regular*. It turns out that the regularity of the associated operator monotone function guarantees the extendability of this metric to the complex projective space generated by the pure states (see [15]).

In the present paper, we introduce the concept of cancellability for operator connections in a natural way. Various characterizations of cancellability with respect to operator monotone functions, Borel measures, and certain operator equations are provided. It is shown that a connection is cancellable if and only if it is not a scalar multiple of trivial means. Applications of this concept go to certain nonlinear operator equations involving operator means. It is shown that such equations are always solvable if and only if f is unbounded and $f(0) = 0$ where f is the associated operator monotone function. We also characterize the condition $f(0) = 0$ for arbitrary connections without assuming the symmetry. Such a connection is said to be nonregular.

This paper is organized as follows. In Section 2, the concept of cancellability of operator connections is defined and characterized. Applications of cancellability to certain nonlinear operator equations involving operators means are explained in Section 3. We investigate the regularity of operator connections in Section 4.

2 Cancellability of connections

The concept of cancellability for scalar means was considered in [16]. We generalize this concept to operator means or, more generally, operator connections as follows.

Definition 2.1 A connection σ is said to be

- *left cancellable* if for each $A > 0$, $B \geq 0$, and $C \geq 0$,

$$A \,\sigma\, B = A \,\sigma\, C \quad \Longrightarrow \quad B = C;$$

- *right cancellable* if for each $A > 0$, $B \geq 0$, and $C \geq 0$,

$$B \,\sigma\, A = C \,\sigma\, A \quad \Longrightarrow \quad B = C;$$

- *cancellable* if it is both left and right cancellable.

Lemma 2.2 *Every nonconstant operator monotone function from* $\mathbb{R}^+$ *to* $\mathbb{R}^+$ *is injective.*

Proof Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be a nonconstant operator monotone function. Suppose there exist $b > a \geq 0$ such that $f(a) = f(b)$. Since f is monotone increasing (in the usual sense), $f(x) = f(a)$ for all $a \leq x \leq b$ and $f(y) \geq f(b)$ for all $y \geq b$. Since f is operator concave, f is concave in the usual sense and hence $f(x) = f(b)$ for all $x \geq b$. The case $a = 0$ contradicts the fact that f is nonconstant. For the case $a > 0$, suppose that there is a point $c \in (0, a)$ such that $0 \leq f(c) < f(a)$. The convexity of the function $g(x) = xf(x)$ (see [1], Lemma 5.1) yields a contradiction. □

A similar result for this lemma under the restriction that $f(0) = 0$ was obtained in [17]. The left cancellability of connections is now characterized as follows.

Theorem 2.3 *Let* σ *be a connection with representing function* f *and associated measure* μ. *Then the following statements are equivalent*:

(1) σ *is left cancellable*;
(2) *for each* $A \geq 0$ *and* $B \geq 0$, $I \,\sigma\, A = I \,\sigma\, B \Longrightarrow A = B$;
(3) σ *is not a scalar multiple of the left-trivial mean*;
(4) f *is injective, i.e.,* f *is left cancellable in the sense that*

$$f \circ g = f \circ h \quad \Longrightarrow \quad g = h;$$

(5) f *is a nonconstant function*;
(6) μ *is not a scalar multiple of the Dirac measure* δ_0 *at* 0.

Proof Clearly, (1) ⇒ (2) ⇒ (3) and (4) ⇒ (5). For each $k \geq 0$, it is straightforward to show that the representing function of the connection

$$k\omega_l : (A, B) \mapsto kA$$

is the constant function $f \equiv k$ and its associated measure is given by $k\delta_0$. Hence, we have the implications (3) ⇔ (5) ⇔ (6). By Lemma 2.2, we have (5) ⇒ (4).

(4) ⇒ (2): Assume that f is injective. Consider $A \geq 0$ and $B \geq 0$ such that $I \,\sigma\, A = I \,\sigma\, B$. Then $f(A) = f(B)$ by (1.3). Since $f^{-1} \circ f(x) = x$ for all $x \in \mathbb{R}^+$, we have $A = B$.

(2) ⇒ (1): Let $A > 0$, $B \geq 0$, and $C \geq 0$ be such that $A\,\sigma\, B = A\,\sigma\, C$. By the congruence invariance of σ, we have

$$A^{\frac{1}{2}}\big(I\,\sigma\, A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\big)A^{\frac{1}{2}} = A^{\frac{1}{2}}\big(I\,\sigma\, A^{-\frac{1}{2}}CA^{-\frac{1}{2}}\big)A^{\frac{1}{2}}$$

and thus $I\,\sigma\, A^{-\frac{1}{2}}BA^{-\frac{1}{2}} = I\,\sigma\, A^{-\frac{1}{2}}CA^{-\frac{1}{2}}$. The assumption (2) implies $B = C$. □

Recall that the *transpose* of a connection σ is the connection

$$(A,B) \mapsto B\,\sigma\, A.$$

If f is the representing function of σ, then the representing function of the transpose of σ is given by the *transpose* of f (see [1], Corollary 4.2), defined by

$$x \mapsto xf(1/x), \quad x > 0.$$

A connection is *symmetric* if it coincides with its transpose.

Theorem 2.4 *Let σ be a connection with representing function f and associated measure μ. Then the following statements are equivalent*:

(1) *σ is right cancellable*;
(2) *for each $A \geq 0$ and $B \geq 0$, $A\,\sigma\, I = B\,\sigma\, I \implies A = B$*;
(3) *σ is not a scalar multiple of the right-trivial mean*;
(4) *the transpose of f is injective*;
(5) *f is not a scalar multiple of the identity function $x \mapsto x$*;
(6) *μ is not a scalar multiple of the Dirac measure δ_1 at* 1.

Proof It is straightforward to see that, for each $k \geq 0$, the representing function of the connection

$$k\omega_r : (A,B) \mapsto kB$$

is the function $x \mapsto kx$ and its associated measure is given by $k\delta_1$. The proof is done by replacing σ with its transpose in Theorem 2.3. □

Remark 2.5 The injectivity of the transpose of f does not imply the surjectivity of f. To see that, take $f(x) = (1 + x)/2$. Then the transpose of f is f itself.

The following results are characterizations of cancellability for connections.

Corollary 2.6 *Let σ be a connection with representing function f and associated measure μ. Then the following statements are equivalent*:

(1) *σ is cancellable*;
(2) *σ is not a scalar multiple of the left/right-trivial mean*;
(3) *f and its transpose are injective*;
(4) *f is neither a constant function nor a scalar multiple of the identity function*;
(5) *μ is not a scalar multiple of δ_0 or δ_1.*

In particular, every nontrivial mean is cancellable.

Remark 2.7 The 'order cancellability' does not hold for general connections, even if we restrict them to the class of means. For each $A, B > 0$, it is not true that the condition $I \sigma A \leq I \sigma B$ or the condition $A \sigma I \leq B \sigma I$ implies $A \leq B$. To see this, take σ to be the geometric mean. It is not true that $A^{1/2} \leq B^{1/2}$ implies $A \leq B$ in general.

3 Applications to certain nonlinear operator equations involving means

Cancellability of connections can be restated in terms of the uniqueness of certain operator equations as follows. A connection σ is left cancellable if and only if

> for each given $A > 0$ and $B \geq 0$, if the equation $A \sigma X = B$ has a solution X, then it has a unique solution.

The similar statement for right cancellability holds. In this section, we characterize the existence and the uniqueness of a solution of the operator equation $A \sigma X = B$. The equations of this type with specific operator means σ are also considered.

Theorem 3.1 *Let σ be a connection which is not a scalar multiple of the left-trivial mean. Let f be its representing function. Given $A > 0$ and $B \geq 0$, the operator equation*

$$A \sigma X = B$$

has a (positive) solution if and only if $\operatorname{Sp}(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}) \subseteq \operatorname{Range}(f)$. *In fact, such a solution is unique and given by*

$$X = A^{\frac{1}{2}} f^{-1}\big(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}\big) A^{\frac{1}{2}}.$$

Proof Suppose that there is a positive operator X such that $A \sigma X = B$. The congruent invariance of σ yields

$$A^{\frac{1}{2}} \big(I \sigma A^{-\frac{1}{2}} X A^{-\frac{1}{2}}\big) A^{\frac{1}{2}} = B.$$

The property (1.3) now implies

$$f\big(A^{-\frac{1}{2}} X A^{-\frac{1}{2}}\big) = I \sigma A^{-\frac{1}{2}} X A^{-\frac{1}{2}} = A^{-\frac{1}{2}} B A^{-\frac{1}{2}}.$$

By the spectral mapping theorem,

$$\operatorname{Sp}\big(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}\big) = \operatorname{Sp}\big(f\big(A^{-\frac{1}{2}} X A^{-\frac{1}{2}}\big)\big) = f\big(\operatorname{Sp}\big(A^{-\frac{1}{2}} X A^{-\frac{1}{2}}\big)\big) \subseteq \operatorname{Range}(f).$$

Conversely, suppose that $\operatorname{Sp}(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}) \subseteq \operatorname{Range}(f)$. Since $\sigma \neq k\omega_l$ for all $k \geq 0$, we see that f is nonconstant by Theorem 2.3. It follows that f is injective by Lemma 2.2. The assumption yields the existence of the operator $X \equiv A^{\frac{1}{2}} f^{-1}(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}) A^{\frac{1}{2}}$. We obtain from the property (1.3) $A \sigma X = B$. The uniqueness of a solution follows from the left cancellability of σ. □

Similarly, we have the following theorem.

Theorem 3.2 *Let σ be a connection which is not a scalar multiple of the right-trivial mean. Given $A > 0$ and $B \geq 0$, the operator equation*

$$X \sigma A = B$$

has a (positive) solution if and only if $\mathrm{Sp}(A^{-1/2}BA^{-1/2}) \subseteq \mathrm{Range}(g)$, *here g is the representing function of the transpose of σ. In fact, such a solution is unique and given by*

$$X = A^{1/2}g^{-1}\left(A^{-1/2}BA^{-1/2}\right)A^{1/2}.$$

Theorem 3.3 *Let σ be a connection with representing function f. Then the following statements are equivalent*:

(1) *the operator equation*

$$A \sigma X = B \tag{3.1}$$

has a unique solution for any given $A > 0$ and $B \geq 0$;

(2) *f is unbounded and $f(0) = 0$;*

(3) *f is surjective, i.e., f is right cancellable in the sense that*

$$g \circ f = h \circ f \quad \Longrightarrow \quad g = h.$$

Moreover, if (1) *holds, then the solution of* (3.1) *varies continuously in each given $A > 0$ and $B \geq 0$, i.e. the map $(A,B) \mapsto X$ is separately continuous with respect to the strong-operator topology.*

Proof (2) $\Rightarrow$ (3): This follows directly from the intermediate value theorem.

(3) $\Rightarrow$ (1): It is immediate from Theorem 3.1.

(1) $\Rightarrow$ (3): Assume (1). The uniqueness of solution for the equation $A \sigma X = B$ implies the left cancellability of σ. By Theorem 2.3, f is injective. The assumption (1) implies the existence of a positive operator X such that

$$f(X) = I \sigma X = 0.$$

The spectral mapping theorem implies that $f(\lambda) = 0$ for all $\lambda \in \mathrm{Sp}(X)$. Since f is injective, we have $\mathrm{Sp}(X) = \{\lambda\}$ for some $\lambda \in \mathbb{R}^+$. Since X is not invertible (otherwise, $I \sigma X > 0$), we have $\lambda = 0$ and hence $f(0) = 0$.

Now, let $k > 0$. The assumption (1) implies the existence of $X \geq 0$ such that $I \sigma X = kI$. Since $f(X) = kI$, we have $f(\lambda) = k$ for all $\lambda \in \mathrm{Sp}(X)$. Since $\mathrm{Sp}(X)$ is nonempty, there is $\lambda \in \mathrm{Sp}(X)$ such that $f(\lambda) = k$. Therefore, f is unbounded.

Assume that (1) holds. Then the map $(A,B) \mapsto X$ is well defined. Recall that if $A_n \in B(\mathbb{H})^+$ converges strongly to A, then $\phi(A_n)$ converges strongly to $\phi(A)$ for any continuous function ϕ. It follows that the map

$$(A,B) \mapsto X = A^{\frac{1}{2}} f^{-1}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)A^{\frac{1}{2}}$$

is separately continuous in each variable. □

Next, we investigate certain nonlinear operator equations involving operator means. First, consider a class of parametrized means, namely, the *quasi-arithmetic power mean* $\#_{p,\alpha}$ with exponent $p \in [-1,1]$ and weight $\alpha \in (0,1)$, defined by

$$A \#_{p,\alpha} B = \left[(1-\alpha)A^p + \alpha B^p\right]^{1/p}.$$

Its representing function of this mean is given by

$$f_{p,\alpha}(x) = \left(1-\alpha+\alpha x^p\right)^{1/p}.$$

The special cases $p = 1$ and $p = -1$ are the α-weighted arithmetic mean and the α-weighted harmonic mean, respectively. The case $p = 0$ is defined by continuity and, in fact, $\#_{0,\alpha} = \#_\alpha$ and $f_{0,\alpha}(x) = x^\alpha$.

Example 3.4 Let $p \in [-1,1]$ and $\alpha \in (0,1)$. Given $A > 0$ and $B \geq 0$, consider the operator equation

$$A \#_{p,\alpha} X = B. \tag{3.2}$$

The case $p = 0$: Since the range of $f_{0,\alpha}(x) = x^\alpha$ is $\mathbb{R}^+$, equation (3.2) always has a unique solution given by

$$X = A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^{1/\alpha}A^{1/2} \equiv A \#_{1/\alpha} B.$$

The case $0 < p \leq 1$: The range of $f_{p,\alpha}$ is the interval $[(1-\alpha)^{1/p},\infty)$. Hence, equation (3.2) is solvable if and only if $\mathrm{Sp}(A^{-1/2}BA^{-1/2}) \subseteq [(1-\alpha)^{1/p},\infty)$, *i.e.*, $B \geq (1-\alpha)^{1/p}A$.

The case $-1 \leq p < 0$: The range of $f_{p,\alpha}$ is the interval $[0,(1-\alpha)^{1/p})$. Hence, equation (3.2) is solvable if and only if $\mathrm{Sp}(A^{-1/2}BA^{-1/2}) \subseteq [0,(1-\alpha)^{1/p})$, *i.e.*, $B < (1-\alpha)^{1/p}A$.

For each $p \in [-1,0) \cup (0,1]$ and $\alpha \in (0,1)$, we have

$$f_{p,\alpha}^{-1}(x) = \left(1-\frac{1}{\alpha}+\frac{1}{\alpha}x^p\right)^{1/p}.$$

Hence, the solution of (3.2) is given by

$$X = \left[\left(1-\frac{1}{\alpha}\right)A^p + \frac{1}{\alpha}B^p\right]^{1/p} \equiv A \#_{p,\frac{1}{\alpha}} B.$$

Example 3.5 Let σ be the logarithmic mean with representing function

$$f(x) = \frac{x-1}{\log x}, \quad x > 0.$$

Here, $f(0) \equiv 0$ and $f(1) \equiv 1$ by continuity. We see that f is unbounded. Thus, the operator equation $A \,\sigma\, X = B$ is solvable for all $A > 0$ and $B \geq 0$.

Example 3.6 Let η be the dual of the logarithmic mean, *i.e.*,

$$\eta : (A,B) \mapsto \left(A^{-1} \,\sigma\, B^{-1}\right)^{-1},$$

where σ denotes the logarithmic mean. The representing function of η is given by

$$f(x) = \frac{x}{x-1}\log x, \quad x > 0.$$

Since $f(0) \equiv 0$ and f is unbounded, the operator equation $A\eta X = B$ is solvable for all $A > 0$ and $B \geq 0$.

Next, we will consider a parametrized symmetric mean. For each $r \in [-1,1]$, recall that the function

$$g_r(x) = \left(\frac{3r-1}{3r+1}\right)\frac{x^{\frac{3r+1}{2}}-1}{x^{\frac{3r-1}{2}}-1}, \quad x \geq 0,$$

is operator monotone (see [4]). This function satisfies $g_r(1) = 1$ and $g_r(x) = xg_r(1/x)$. Thus it associates to a unique symmetric operator mean, denoted by $\diamondsuit_r$. In particular,

$$\diamondsuit_1 = \triangledown, \qquad \diamondsuit_0 = \#, \qquad \diamondsuit_{-1} = !.$$

The operator means $\diamondsuit_{1/3}$ and $\diamondsuit_{-1/3}$ are the logarithmic mean and its dual.

Example 3.7 Let $A > 0$ and $B \geq 0$. Consider the operator equation

$$A \diamondsuit_r X = B \tag{3.3}$$

for each given $r \in [-1,1]$.

The case $1/3 < r \leq 1$: Observe that

$$\lim_{x\to\infty}\left(\frac{3r-1}{3r+1}\right)\frac{x^{(3r+1)/2}-1}{x^{(3r-1)/2}-1} = \infty,$$

meaning that g_r is unbounded. By the intermediate value theorem,

$$\operatorname{Range} F_r = \big[g_r(0),\infty\big) = \left[\frac{3r-1}{3r+1},\infty\right).$$

By Theorem 3.1, the operator equation (3.3) has a (unique) solution if and only if

$$B \geq \left(\frac{3r-1}{3r+1}\right)A.$$

The case $0 < r < 1/3$: Observe that

$$\lim_{x\to\infty}\left(\frac{3r-1}{3r+1}\right)\frac{x^{(3r+1)/2}-1}{x^{(3r-1)/2}-1} = \infty,$$

so that g_r is unbounded. By L'Hôspital's rule, we have

$$g_r(0) = \left(\frac{3r-1}{3r+1}\right)(-1)\lim_{x\to 0}\frac{x^{(-1/2)(3r-1)}}{1-x^{(-1/2)(3r-1)}} = 0.$$

It follows that Range $g_r = [0, \infty)$. Theorem 3.1 guarantees the existence and uniqueness for the solution of the equation (3.3).

The case $-(1/3) < r < 0$: It is similar to the case $0 < r < 1/3$. The operator equation (3.3) always has a unique solution.

The case $-1 < r < -(1/3)$: We have

$$\lim_{x\to\infty}\left(\frac{3r-1}{3r+1}\right)\frac{x^{(3r+1)/2}-1}{x^{(3r-1)/2}-1} = \frac{3r-1}{3r+1}.$$

By L'Hôspital's rule, we have $g_r(0) = 0$. By continuity,

$$\operatorname{Range} g_r = \left[0, \frac{3r-1}{3r+1}\right).$$

By Theorem 3.1, the operator equation (3.3) has a (unique) solution if and only if

$$B < \left(\frac{3r-1}{3r+1}\right)A.$$

The cases $r = 1/3$ and $r = -1/3$ are already done in Examples 3.5 and 3.6.

4 Regularity of connections

In this section, we give various characterizations for the non-regularity of an operator connection.

Theorem 4.1 *Let σ be a connection with representing function f and associated measure μ. Then the following statements are equivalent.*

(1) $f(0) = 0$;

(2) $\mu(\{0\}) = 0$;

(3) $I\,\sigma\,0 = 0$;

(4) $A\,\sigma\,0 = 0$ *for all* $A \geq 0$;

(5) *for each* $A \geq 0$, *the condition* $0 \in \operatorname{Sp}(A)$ *implies* $0 \in \operatorname{Sp}(I\,\sigma\,A)$;

(6) *for each* $A, X \geq 0$, *the condition* $0 \in \operatorname{Sp}(A)$ *implies* $0 \in \operatorname{Sp}(X\,\sigma\,A)$.

Proof From the integral representation (1.2), we have

$$f(x) = \mu\big(\{0\}\big) + \mu\big(\{1\}\big)x + \int_{(0,1)} (1\,!_t\,x)\,d\mu(t), \quad x \geq 0, \tag{4.1}$$

i.e. $f(0) = \mu(\{0\})$. From the property (1.3), we have $I\,\sigma\,0 = f(0)I$. Hence, (1)-(3) are equivalent. It is clear that (4) $\Rightarrow$ (3) and (6) $\Rightarrow$ (5).

(3) $\Rightarrow$ (4): Assume that $I\,\sigma\,0 = 0$. For any $A > 0$, we have by the congruence invariance

$$A\,\sigma\,0 = A^{\frac{1}{2}}(I\,\sigma\,0)A^{\frac{1}{2}} = 0.$$

For general $A \geq 0$, we have $(A + \epsilon I)\,\sigma\,0 = 0$ for all $\epsilon > 0$ by the previous claim and hence $A\,\sigma\,0 = 0$ by the continuity from above.

(5) $\Rightarrow$ (1): We have $0 \in \operatorname{Sp}(I\,\sigma\,0) = \operatorname{Sp}(f(0)I) = \{f(0)\}$, *i.e.* $f(0) = 0$.

(1) $\Rightarrow$ (6): Assume $f(0) = 0$. Consider $A \geq 0$ such that $0 \in \mathrm{Sp}(A)$, *i.e.* A is not invertible. Assume first that $X > 0$. Then

$$X \,\sigma\, A = X^{\frac{1}{2}}\big(I \,\sigma\, X^{-\frac{1}{2}}AX^{-\frac{1}{2}}\big)X^{\frac{1}{2}} = X^{\frac{1}{2}}f\big(X^{-\frac{1}{2}}AX^{-\frac{1}{2}}\big)X^{\frac{1}{2}}.$$

Since $X^{-\frac{1}{2}}AX^{-\frac{1}{2}}$ is not invertible, we have $0 \in \mathrm{Sp}(X^{-\frac{1}{2}}AX^{-\frac{1}{2}})$ and hence by the spectral mapping theorem

$$0 = f(0) \in f\big(\mathrm{Sp}\big(X^{-\frac{1}{2}}AX^{-\frac{1}{2}}\big)\big) = \mathrm{Sp}\big(f\big(X^{-\frac{1}{2}}AX^{-\frac{1}{2}}\big)\big).$$

This implies that $X \,\sigma\, A$ is not invertible. Now, consider $X \geq 0$. The previous claim shows that $(X + I) \,\sigma\, A$ is not invertible. Since $X \,\sigma\, A \leq (X + I) \,\sigma\, A$, we conclude that $X \,\sigma\, A$ is not invertible. □

We say that a connection σ is *nonregular* if one of the conditions in Theorem 4.1 holds (and thus they all do), otherwise σ is *regular*. Hence, regular connections correspond to regular operator monotone functions and regular monotone metrics.

Remark 4.2 Let σ be a connection with representing function f and associated measure μ. Let g be the representing function of the transpose of σ. From (4.1),

$$g(0) = \lim_{x\to 0^+} xf\left(\frac{1}{x}\right) = \lim_{x\to\infty} \frac{f(x)}{x} = \mu\big(\{1\}\big).$$

Thus, the transpose of σ is nonregular if and only if $\mu(\{1\}) = 0$.

Theorem 4.3 *The following statements are equivalent for a mean σ.*

(1) *σ is nonregular*;
(2) *$I \,\sigma\, P = P$ for each projection P.*

Proof (1) $\Rightarrow$ (2): Assume that $f(0) = 0$ and consider a projection P. Since $f(1) = 1$, we have $f(x) = x$ for all $x \in \{0,1\} \supseteq \mathrm{Sp}(P)$. Thus $I \,\sigma\, P = f(P) = P$.

(2) $\Rightarrow$ (1): We have $0 = I \,\sigma\, 0 = f(0)I$, *i.e.* $f(0) = 0$. □

To prove the next result, recall the following lemma.

Lemma 4.4 ([1]) *If $f : \mathbb{R}^+ \to \mathbb{R}^+$ is an operator monotone function such that $f(1) = 1$ and f is neither the constant function 1 nor the identity function, then*

(1) $0 < x < 1 \implies x < f(x) < 1$;
(2) $1 < x \implies 1 < f(x) < x$.

Theorem 4.5 *Let σ be a nontrivial mean. For each $A \geq 0$, if $I \,\sigma\, A = A$, then A is a projection. Hence, the following statements are equivalent*:

(1) *σ is nonregular*;
(2) *for each $A \geq 0$, A is a projection if and only if $I \,\sigma\, A = A$.*

Proof Since $I \,\sigma\, A = A$, we have $f(A) = A$ by (1.3). Hence $f(x) = x$ for all $x \in \mathrm{Sp}(A)$ by the injectivity of the continuous functional calculus. Since σ is a nontrivial mean, Lemma 4.4 implies that $\mathrm{Sp}(A) \subseteq \{0,1\}$, *i.e.* A is a projection. □

Theorem 4.6 *Under the condition that σ is a left-cancellable connection with representing function f, the following statements are equivalent*:

(1) *σ is nonregular.*
(2) *The equation $f(x) = 0$ has a solution x.*
(3) *The equation $f(x) = 0$ has a unique solution x.*
(4) *The only solution to $f(x) = 0$ is $x = 0$.*
(5) *For each $A > 0$, the equation $A \,\sigma\, X = 0$ has a solution X.*
(6) *For each $A > 0$, the equation $A \,\sigma\, X = 0$ has a unique solution X.*
(7) *For each $A > 0$, the only solution to the equation $A \,\sigma\, X = 0$ is $X = 0$.*
(8) *The equation $I \,\sigma\, X = 0$ has a solution X.*
(9) *The equation $I \,\sigma\, X = 0$ has a unique solution X.*
(10) *The only solution to the equation $I \,\sigma\, X = 0$ is $X = 0$.*

Similar results for the case of right cancellability hold.

Proof It is clear that (1) ⇒ (2), (6) ⇒ (5) and (10) ⇒ (8). Since f is injective by Theorem 2.3, we have (2) ⇒ (3) ⇒ (4).

(4) ⇒ (10): Let $X \geq 0$ be such that $I \,\sigma\, X = 0$. Then $f(X) = 0$ by (1.3). By spectral mapping theorem, $f(\mathrm{Sp}(X)) = \{0\}$. Hence, $\mathrm{Sp}(X) = \{0\}$, *i.e.* $X = 0$.

(8) ⇒ (9): Consider $X \geq 0$ such that $I \,\sigma\, X = 0$. Then $f(X) = 0$. Since f is injective with continuous inverse, we have $X = f^{-1}(0)$.

(9) ⇒ (6): Use congruence invariance.

(5) ⇒ (7): Let $A > 0$ and consider $X \geq 0$ such that $A \,\sigma\, X = 0$. Then $A^{\frac{1}{2}}(I \,\sigma\, A^{-\frac{1}{2}}XA^{-\frac{1}{2}})A^{\frac{1}{2}} = 0$, *i.e.* $f(A^{-\frac{1}{2}}XA^{-\frac{1}{2}}) = I \,\sigma\, A^{-\frac{1}{2}}XA^{-\frac{1}{2}} = 0$. Hence,

$$f\bigl(\mathrm{Sp}\bigl(A^{-\frac{1}{2}}XA^{-\frac{1}{2}}\bigr)\bigr) = \mathrm{Sp}\bigl(f\bigl(A^{-\frac{1}{2}}XA^{-\frac{1}{2}}\bigr)\bigr) = \{0\}.$$

Suppose there exists $\lambda \in \mathrm{Sp}(A^{-\frac{1}{2}}XA^{-\frac{1}{2}})$ such that $\lambda > 0$. Then $f(0) < f(\lambda) = 0$, a contradiction. Hence, $\mathrm{Sp}(A^{-\frac{1}{2}}XA^{-\frac{1}{2}}) = \{0\}$, *i.e.* $A^{-\frac{1}{2}}XA^{-\frac{1}{2}} = 0$ or $X = 0$.

(7) ⇒ (1): We have $f(0)I = I \,\sigma\, 0 = 0$, *i.e.* $f(0) = 0$. □

Competing interests
The author declares that he has no competing interests.

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